Euler's Difference Equation

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Abstract: - It is well known that Euler's differential equation is the unique one which can be translated into difference equation with constant coefficients by substitution of argument. We tried to solve the corresponding difference equation (we called it Euler's difference equation). It turned out that Euler's difference equation couldn't be reduced to difference equation with constant coefficients in general case. It is possible only in the case when sum of coefficients in Euler's difference equation is equal to zero. This case was investigated in detail and two linear independent particular solutions are found.

Key-Words: - difference equation, linear independence, coefficients of Euler's equation

1 Introduction

Introduction of the new argument t with help of substitution $x = \varphi(t)$, where x is the actual argument, t is new argument and φ arbitrary function is often used in the theory of differential equations. The main goal of this substitution is simplifying of the initial equation. It can be achieved by convenient choice of the function φ. As a typical example of successful application of the mentioned method is solving of Euler's difference equation.

$$\frac{d^2 y}{dx^2} + \frac{a}{x}\frac{dy}{dx} + \frac{b}{x^2} y = 0 \qquad (1)$$

The substitution $x = \mathrm{e}^t$ reduces Euler's equation to second order differential equation with constant coefficients

$$\frac{d^2 y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0 \qquad (2)$$

The general solution of (2) is given by

$$y(t) = \mathrm{e}^{\frac{1-a}{2}t}\left(C_1 \mathrm{e}^{t\sqrt{\left(\frac{1-a}{2}\right)^2 - b}} + \right.$$

$$\left. + C_2 \mathrm{e}^{-t\sqrt{\left(\frac{1-a}{2}\right)^2 - b}} \right) \qquad (3)$$

wherefrom for $t = \ln x$ can be obtained the general solution of Euler's differential equation in the form:

$$y(t) = x^{\frac{1-a}{2}t}\left(C x^{t\sqrt{\left(\frac{1-a}{2}\right)^2 - b}} + \right.$$

$$\left. + C_2\, x^{-t\sqrt{\left(\frac{1-a}{2}\right)^2 - b}} \right) \qquad (4)$$

The method described can be applied to the difference equation

$$\frac{\Delta^2}{\Delta n^2} Y_n + \frac{A}{n}\frac{\Delta}{\Delta n} Y_n + \frac{B}{n^2} Y_n = 0 \qquad (5)$$

where $\frac{\Delta^2}{\Delta n^2} \equiv (T_1 - 1)^2$ and $\frac{\Delta}{\Delta n} \equiv T_1 - 1$, T – being translational operator with the properties:

$$T_k \equiv (T_1)^k ; \quad T_k f_n = f_{n+k} ; \quad T_k^{-1} \equiv T_{-k}$$

$$k = 0, 1, 2, \ldots \qquad (6)$$

The equation (5) will be called Euler's difference equation.

As it will be seen in further going over to new argument in difference equation (5) not allways leads to solvable form of this equation.

2. Introduction of the new argument into Euler's difference equation

In the equation (5) we are going over from discrete varible *n* to the new discrete variable *m* using general substitution

$$n = \varphi_m . \qquad (7)$$

From the substitution (6) it follows

$$\frac{\Delta m}{\Delta n} = \frac{1}{\Phi_m} ; \quad \Phi_m = \varphi_{m+1} - \varphi_m \qquad (8)$$

$$\frac{\Delta Y}{\Delta n} = \frac{\Delta m}{\Delta m}\frac{\Delta Y}{\Delta n} = \frac{\Delta m}{\Delta n}\frac{\Delta Y}{\Delta m} = \frac{1}{\Phi_m}\frac{\Delta Y}{\Delta m} \qquad (9)$$

and

$$\frac{\Delta^2 Y}{\Delta n^2} = \frac{\Delta m}{\Delta m}\frac{\Delta}{\Delta n}\frac{\Delta Y}{\Delta n} = \frac{\Delta m}{\Delta n}\frac{\Delta}{\Delta m}\left(\frac{\Delta Y}{\Delta m}\frac{1}{\Phi_m}\right) =$$

$$= \frac{1}{\Phi_m}\left(\frac{\Delta^2 Y}{\Delta m^2}\frac{1}{\Phi_m} - \frac{\frac{\Delta}{\Delta m}\Phi_m}{\Phi_m \Phi_{m+1}}\frac{\Delta Y}{\Delta m} - \frac{\frac{\Delta}{\Delta m}\Phi_m}{\Phi_m \Phi_{m+1}}\frac{\Delta^2 Y}{\Delta m^2} \right) =$$

$$= \frac{1}{\Phi_m^2}\left(1 - \frac{\frac{\Delta\Phi_m}{\Delta m}}{\Phi_{m+1}} \right)\frac{\Delta^2 Y}{\Delta m^2} - \frac{1}{\Phi_m^2}\frac{\frac{\Delta\Phi_m}{\Delta m}}{\Phi_{m+1}}\frac{\Delta Y}{\Delta m}$$

(10)

Substituting (9) and (10) into (5) we reduce it to the form:

$$\frac{\Delta^2 Y}{\Delta m^2} + \frac{A\left(\frac{\Phi_m}{\varphi_m}\right) - \frac{1}{\Phi_{m+1}}\frac{\Delta\Phi_m}{\Delta m}}{1 - \frac{1}{\Phi_{m+1}}\frac{\Delta\Phi_m}{\Delta m}}\frac{\Delta Y}{\Delta m} +$$

$$+ B\frac{\left(\frac{\Phi_m}{\varphi_m}\right)^2}{1 - \frac{1}{\Phi_{m+1}}\frac{\Delta\Phi_m}{\Delta m}} Y = 0$$

(11)

The arbitrary function φ_m will be determined in such a way which leads to constant coefficients in (11). We hall take $\frac{\Phi_m}{\varphi_m} = C - 1$, where C is arbitrary constant. This condition gives the difference equation $\varphi_{m+1} = C\varphi_m$ for determining of the function φ_m. The solution of this equation for $\varphi_0 = 1$ is

$$\varphi_m = C^m = n . \qquad (12)$$

wherefrom it follows:

$$\Phi_m = \varphi_{m+1} - \varphi_m = (C-1)C^m ;$$

$$\frac{\Delta\Phi_m}{\Delta m} = (C-1)^2 C^m ; \qquad (13)$$

$$\Phi_{m+1} = (C-1)C^{m+1} .$$

After substituion (12) and (13) into (10), we obtain:

$$\frac{\Delta^2 Y}{\Delta m^2} + (C-1)(AC-1)\frac{\Delta Y}{\Delta m} +$$

$$+ BC(C-1)^2 Y = 0 . \qquad (14)$$

This equation will be written in developed form, i.e. we shall take that $\frac{\Delta Y}{\Delta m} = Y_{m+1} - Y_m$ and

$\frac{\Delta^2 Y}{\Delta m^2} = Y_{m+2} - 2Y_{m+1} + Y_m$. In such a way (14) becomes:

$$Y_{m+2} + pY_{m+1} + qY_m = 0, \tag{15}$$

where

$$p = (C-1)(AC-1) - 2$$
$$q = BC(C-1)^2 - (C-1)(AC-1) + 1. \tag{16}$$

The solution of (15) will be looked for in the form:

$$Y_m = x^m. \tag{17}$$

After this substitution (15) becomes

$$x^m\left(x^2 + px + q\right) = 0. \tag{18}$$

Trivial solution $x = 0$ will be rejected and the solutions:

$$x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}. \tag{19}$$

will be used in further analyses.

We require that one of two solution (18), say x_1, be equal *C*. This requirement significantly simplifies solving of the initial equation (5). If $x_1 = C$, it follows:

$$Y_m = C^m \equiv n = Y_n. \tag{20}$$

On the other hand the requirement $x_1 = C$, with respect to (12) reduces to:

$$(C-1)^2 C(A+B) = 0. \tag{21}$$

After rejection of two trivial solutions $C = 1$ and $C = 0$ of the equation (21) the condition

$$A + B = 0. \tag{22}$$

remains.

Taking into account (22) we conclude that Euler's difference equation (5) can be solved by introduction of the new argument only in the case when the coefficients *A* and *B* in (5) satisfies the condition (22). In this case the particular solution of (5) is

$$Y_n = n. \tag{23}$$

In all other cases going over to new argument in (1.5) does not give the solvable form of this equation.

3. Illustrative example

As an illustrative example we shall solve the Euler's difference equation

$$\frac{\Delta^2 Y_n}{\Delta n^2} - \frac{4}{n}\frac{\Delta Y_n}{\Delta n} + \frac{4}{n^2} Y_n = 0. \tag{24}$$

In this equation the condition $A + B = 0$ is satisfied and consequently the particular solution of this equation is

$$Y_n = n. \tag{24}$$

In order to find the second particular solution of (24) we shall use the analogy between differential ad difference equations. It means that the second particular solution of (24) will be looked for in the form:

$$Y_n = nZ_n. \tag{25}$$

From (25) we find

$$\frac{\Delta Y_n}{\Delta n} = (n+1)\frac{\Delta Z_n}{\Delta n} + Z_n$$
$$\frac{\Delta^2 Y_n}{\Delta n^2} = (n+2)\frac{\Delta^2 Z_n}{\Delta n^2} + 2\frac{\Delta Z_n}{\Delta n} \tag{26}$$

After substitution (26) into (24) we obtain

$$\frac{\Delta^2 Z_n}{\Delta n^2} = \frac{2}{n}\frac{\Delta Z_n}{\Delta n}. \tag{27}$$

Taking

$$\frac{\Delta Z_n}{\Delta n} = u_n. \tag{28}$$

we lower the order of the difference equation (27) and obtain the first order difference equation

$$u_{n+1} = \frac{n+2}{n} u_n. \tag{29}$$

From (29) it follows:

$$u_2 = \frac{3}{1}u_1;\ u_3 = \frac{4}{2}u_2 = \frac{3\cdot 4}{2\cdot 1}u_1,$$
$$u_4 = \frac{5!}{2\cdot 3!}u_1, \ldots,$$

wherefrom for $u_1 = 2$ we obtain:

$$u_n = \frac{(n+1)!}{(n-1)!}u_1 = n(n+1). \tag{30}$$

Substituting (30) into (28) we have

$$\frac{\Delta Z_n}{\Delta n} = n(n+1), \tag{31}$$

which with the use of the translational operator $\hat{T}_1$ can be written:

$$\left(\hat{T}_1 - 1\right)Z_n = n(n+1). \tag{32}$$

After application the inverse operator $\left(\hat{T}_1 - 1\right)^{-1}$ taking in the form:

$$\left(\hat{T}_1 - 1\right)^{-1} = \sum_{k=0}^{\infty} \hat{T}_{-k-1}. \tag{33}$$

we find that:

$$Z_n = \left(\hat{T}_1 - 1\right)^{-1} n(n+1) = \sum_{k=0}^{\infty} \hat{T}_{-k-1} n(n+1) \cdot \tag{34}$$

The infinite series cut off for $k = n$ and (34) becomes:

$$Z_n = \sum_{k=0}^{n} (n-k)^2 + \sum_{k=0}^{n} (n-k) = \\ = \sum_{\nu=1}^{n} \nu^2 + \sum_{\nu=1}^{n} \nu \tag{35}$$

Using the formulae for finite sums from [12-13], we finally obtain:

$$Z_n = \frac{1}{3} n\left(n^2 - 1\right). \tag{36}$$

wherefrom, taking into account (25), we finally obtain the second particular solution of Euler's difference equation (24):

$$Y_n = \frac{1}{3} n^2\left(n^2 - 1\right). \tag{37}$$

The general solution of (24) is given by:

$$Y_n = C_1 n + C_2 n^2\left(n^2 - 1\right). \tag{38}$$

4 Conclusion

The results of the analyses exposed can be summarized as follows:

1. Introduction of the new discrete argument into the Euler's difference equation is successful only in the case when coefficients A and B of this equation satisfy the condition $A + B = 0$. If this condition is satisfied the particular solution $Y = n$ can be found immediately.
2. The general solution requires linearly independent second particular solution which can be found by the substitution $Y = nZ$. The procedure of findig the function Z requires a series of operations, such as lowering of the order of the difference equation, application of the inverse operator $\left(\hat{T}_1 - 1\right)^{-1}$, and so on.

It should be noticed that the illustrative example chosen enables finding of the second perticular integral after simple calculations. In general case the procedure of determining of second particular integral is significantly complicated.

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