Transitivity and Topological Entropy on Fuzzy Dynamical Systems Through Fuzzy Observation

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Abstract: Any physical or geometrical variation on a natural dynamical system should be identified by an observer. Also a method is required to compare different observers and evaluate their perspectives. Moreover complexity and/or uncertainty of the system should be measured through viewpoint of observers. In the approach presented in this paper an observer is identified mathematically by a function $\mu : X \to [0, 1]$, where X denotes the base space of the system; the μ-Fuzzy Topology is defined as a description of the topological notion on X by the eyes of the observer μ. This idea will be applied to the other physical and geometrical notions such as minimality, transitivity and topological entropy for a fuzzy dynamical system on X. It will suggest a rational description of uncertainty in natural systems.

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1 Introduction

Any scientific approach towards studying dynamics on natural systems relies on modeling (analytical, numerical, or observational). In this case, a mathematical model is an appropriate representation of the natural system under consideration; it allows us to make predictions about specific values or provides explanations by showing that certain things are followed necessarily from others. A model is accepted or validated by evaluating its accuracy [1], i.e., how well the formal system describes the natural system? This can be done by matching experimental observations and/or measurements with the theory. Regarding the system theory, we would summarize the process of mathematical modeling as follows:
1. Beginning with observations, we start with a question or hypothesis, which is investigated within a conceptual framework (the model).
2. testing and validating the model with experimental data.
However all data are not crisp, also getting the facts through observational process depends on the idea of observer. So we should add the evaluation of the ”thought of observer” to the above two main points and extend the fuzzy version of such mathematical model.

In order to develop a mathematical model underlying uncertainty and fuzziness in a dynamical system, we will apply the above notions. In this case, any variation and/or approximation on a system should be identified by an observer. Moreover, we need a method to compare between perspective of the observers, also to measure the complexity and/or the uncertainty of the system through viewpoint of the observers. So first, we should mathematically identify the observer. In our approach, there is a one to one correspondence between $[0, 1]^X$, all functions $\mu : X \longrightarrow [0, 1]$, and the observers where X denotes the base space of the system. We should indicate any structure or dynamics on X in terms of μ-qualify or μ-relative which means from the viewpoint of μ. For example: μ-fuzzy topology is the description of topological notion on X by eyes of observer μ.

An extension of a fuzzy dynamical system [2,4,9], which is called relative semi-dynamical system has been introduced in [6] to explain the dynamics on the system related to the observer’s perspective. In this paper first the concept of transitivity from the viewpoint of the observer and the extended notion of minimality are considered in section 2. Then topological entropy, as an invariant object under the conjugate relation [6], for classifying some relative semi-dynamical systems is presented in section 3. Finally, the computational example in semi-definite programming is illustrated in section 4.

let us recall some basic notations of relative struc-

tures [3,7,6,8]. We assume that X is a non-empty set, and μ is a fuzzy subset of X, i.e. $\mu \in [0,1]^X$. Moreover we assume that τ_μ is a μ-fuzzy topology on X which means a collection of members of $[0,1]^X$ with the following properties [8]:
i) μ, $\chi_\phi \in \tau_\mu$ where χ is the characteristic function;
ii) If $\lambda \in \tau_\mu$ then $\lambda \subseteq \mu$, ie. $\lambda(x) \leq \mu(x)$ on X;
iii) If $\lambda_1, \lambda_2 \in \tau_\mu$ then $\lambda_1 \cap \lambda_2 \in \tau_\mu$;
iv) If $\{\lambda_i : i \in \Gamma\} \subset \tau_\mu$ then $\bigcup_{i\in\Gamma} \lambda_i \in \tau_\mu$.
In some sense τ_μ is a fuzzy model of the topology on X from the viewpoint of the observer μ. However, if we denote $\lambda_\alpha = \{x \in X : \lambda(x) > \alpha\}$ and $(\tau_\mu)_\alpha = \{\lambda_\alpha : \lambda \in \tau_\mu\}$ for the given $\alpha \in (0,1]$, them $(\mu_\alpha, (\tau_\mu)_\alpha)$ can be consider as a crisp topological space.
With the above notations, let (X, τ_μ) denotes a μ-fuzzy topological space; a mapping $f : X \to X$ is called (μ,μ)-fuzzy continuous if $f^{-1}(\eta) \cap \mu \in \tau_\mu$ for all $\eta \in \tau_\mu$, where $f^{-1}(\eta)(x) = \eta(f(x))$. Moreover the triple (f, X, τ_μ) is called relative semi-dynamical system or briefly RSD-system.

2 Minimality and Transitivity on RSD-Systems

Definition 1 *An RSD-system (f, X, τ_μ) is called a minimal relative semi-dynamical system on μ_α or briefly "μ_α-minimal" if:*
i) $f(\mu_\alpha) \subset \mu_\alpha$;
ii) For all $x \in \mu_\alpha$, the set $\{f^n(x) : n = 0,1,2,...\}$ is a dense subset of μ_α, where the topology of μ_α is $(\tau_\mu)_\alpha$.

Theorem 2 *let $\alpha \in [0,1]$. If (f, X, τ_μ) is an RSD-system, then the following statements are equivalent.*
i) f is μ_α-minimal.
ii) Let $f(\mu_\alpha) \subset \mu_\alpha$. If C is a closed subset of the topological space $(\mu_\alpha, (\tau_\mu)_\alpha)$ such that $f(C) \subset C$, then $C = \mu_\alpha$ or $C = \emptyset$.
iii) Let $f(\mu_\alpha) \subset \mu_\alpha$. If $O \in (\tau_\mu)_\alpha$ is a nonempty open set, then $\mu_\alpha = \bigcup_{n=-\infty}^{0} f^n(O)$, where $f^0(O) = O$.

Proof:
i)$\Longrightarrow$ii) Let $f(\mu_\alpha) \subset \mu_\alpha$. Suppose that C is a nonempty closed subset of μ_α and $f(C) \subset C$. so there exists $x \in C$ such that $\mu_\alpha = \overline{\{f^n(x) : n = 1,2,...\}} \subset C$. Therefore $C = \mu_\alpha$.
ii)$\Longrightarrow$iii) If $f(\mu_\alpha) \subset \mu_\alpha$ and $O \in (\tau_\mu)_\alpha$, then there exists $\lambda \in \tau_\mu$ such that $O = \lambda_\alpha$. The straightforward calculation shows that $f^{-1}(O) = (f^{-1}(\lambda))_\alpha$. Therefore $C = \mu_\alpha - \bigcup_{n=-\infty}^{0} f^n(O)$ is a closed subset of μ_α. Moreover $f(C) \subset C$ and C is a nonempty set.
iii)$\Longrightarrow$i) Suppose that $x \in \mu_\alpha$ be given and O is a nonempty open subset of μ_α; then $x \in f^{-n}(O)$ for some $n \in \{0\} \cup N$. Therefore $f^n(x) \in O$. Thus we can see that $\overline{\{f^n(x) : n = 1,2,...\}} = \mu_\alpha$. $\square$

We recall [6], that a subset D of X is called invariant for the RSD-system (f, X, τ_μ) if $f(D) \subset D$. An invariant subset D of μ_α is called μ_α-minimal if $f : D \longrightarrow D$ is D-minimal.

Theorem 3 *Let (f, X, τ_μ) be an RSD-systems. Also let $\alpha \in [0,1]$ be given such that $f(\mu_\alpha) \subset \mu_\alpha$ and $\mu_\alpha \neq \emptyset$. Then f has a μ_α-minimal set.*

Proof: If M denotes the set of all nonempty closed invariant subsets of f, then Cantor's intersection property and Zorn's lemma imply that M as an ordered set under inclusion has the minimal set which is $\mu_\alpha-$minimal. $\square$

Theorem 4 *let $f : X \longrightarrow X$ be a $\mu_\alpha-$minimal and $g : \mu_\alpha \longrightarrow \mathbf{R}$ be a continuous function such that $gof = f$, where the topology of μ_α is $(\tau_\mu)_\alpha$. Then g should be constant.*

Proof: The condition $gof = g$ implies that $gof^n = g$ for all $n \in N$. This means, if $q \in \mu_\alpha$ then $g(O_\mu^f(q)) = \{g(q)\}$, where $O_\mu^f(q)$ is the orbit of q. Also the continuity of g implies that $g(\mu_\alpha) = g(\overline{O_\mu^f(q)}) = \overline{g(O_\mu^f(q))} = \{g(q)\}$. $\square$

Definition 5 *f is called μ_α-transitive if there exists an orbit in X, such that its intersection with μ_α is a dense subset of μ_α. Such orbit is called μ_α-transitive one.*

Theorem 6 *Let f be a (μ,μ)-homeomorphism. Then the following statements are equivalent*
i) f is μ_α-transitive.
ii) If U be a nonempty open subset of μ_α, where $f(U) = U$, then U is dense in μ_α.
iii) Let V and W be two nonempty open sets in μ_α. Then there exists $n \in \mathbf{Z}$ such that $f^n(V) \cap W \neq \emptyset$.
iv) If $O_\alpha(x) = O_\mu^f(x) \cap \mu_\alpha$, then the following set

$$\{x \in X : O_\alpha(x) \text{ is dense in } \mu_\alpha\} \cap \mu_\alpha$$

can be written as an intersection of countable collection of the open dense subsets in μ_α.

Proof:
i)$\Longrightarrow$ii) Since f is μ_α-transitive, there exist $x \in X$ such that $O_\alpha(x) = \{x, f(x), f^2(x), \cdots\} \cap \mu_\alpha$ is dense in μ_α. So $f^n(x) \in U$ for some $n \in \mathbf{Z}$. Let $f^k(x) \in$

$O_\alpha(x)$, then $f^k(x) \in f^k(f^{-n}(U)) \subset U$. Therefore $O_\alpha(x) \subset U$. Thus U is dense in μ_α.
ii)$\Longrightarrow$iii) The set $U = \bigcup_{n\in \mathbf{Z}} f^n(V)$ is an open subset of μ_α and $f(U) = U$. So U is dense in μ_α. Thus $U \cap W \neq \emptyset$. Hence there exists an $n \in \mathbf{Z}$ such that $f^n(V) \cap W \neq \emptyset$.
iii)$\Longrightarrow$iv) Let $\{D_n\}_{n\in N}$ be a family of dense subsets in μ_α. Then we can see that:

$$O_\alpha(x) \bigcap \mu_\alpha = \bigcap_{n=0}^{\infty} \bigcap_{k=1}^{\infty} \Big(\bigcup_{l=-\infty}^{\infty} f^l(B(D_n, 1/k))\Big),$$

where $B(D_n, 1/k) = \{y \in X \cap \mu_\alpha : d(y, D_n) < \frac{1}{k}\}$.
iv)$\Longrightarrow$i) Since $\{x \in X : O_\alpha(x)$ is dense in $\mu_\alpha\} \cap \mu_\alpha$ is the dense subset of μ_α, then it is nonempty. So there exist $O_\alpha(x)$ such that $O_\alpha(x)$ is dense in μ_α. $\square$

3 Relative Topological Entropy

This section is presenting the notion of topological entropy from the viewpoint of different observers which describe a relative perspective of complexity and uncertainty in fuzzy systems.

Suppose that τ_μ is a μ-fuzzy topology on X. Let $\alpha \in (0, 1)$ be given such that (X, μ) is a compact (α, μ)-Hausdorff space [6]. Moreover, assume that $\Theta = \{\lambda^i_\alpha : \lambda^i \in \tau_\mu\ , i = 1, \cdots, n\}$ is an open cover for μ_α. By the above notations, the open cover Σ is called subcover of Θ if $\Sigma \subset \Theta$.

Definition 7 *The relative topological entropy of the open cover Θ with the level α is $H_\alpha(\Theta) = logN(\Theta)$, where $N(\Theta)$ is the smallest number of sets which can be used in any subcover of Θ.*

Let $\{\Theta^r = \{(\lambda^1_r)_\alpha, \cdots, (\lambda^{Nr}_r)_\alpha\} : r = 1, \cdots, k\}$ be a family of open covers for μ_α, then an α-refinement of this family is the open cover $\bigvee_{r=1}^k \Theta^r$, which is defined by:

$$\Big\{(\lambda^{i_1}_1)_\alpha \cap (\lambda^{i_2}_2)_\alpha \cap \cdots \cap (\lambda^{i_n}_k)_\alpha : (\lambda^{i_j}_j)_\alpha \in \Theta^j, j \leq k\Big\}$$

Lemma 8 *Let X be a compact (α, μ)-Hausdorff space and $f : X \longrightarrow X$ be an RSD-system. Moreover, let $\Theta = \{\lambda^1_\alpha, \cdots, \lambda^n_\alpha\}$ be an open cover for μ_α. Then*

$$f^{-1}\Theta = \{(\mu \cap f^{-1}\lambda^1)_\alpha, \cdots, (\mu \cap f^{-1}\lambda^n)_\alpha\}$$

would be an open cover for $(\mu \cap f^{-1}\mu)_\alpha$ too.

Proof: Since Θ is an open cover for μ_α, then $\mu_\alpha \subset \bigcup_{i=1}^n \lambda^i_\alpha$. Now if $x \in ((f^{-1}\mu) \cap \mu)_\alpha$, then $min\{\mu(f(x)), \mu(x)\} > \alpha$. So $f(x) \in \mu_\alpha$. Hence there is $1 \leq m \leq n$ such that $f(x) \in \lambda^m_\alpha$. Thus $\lambda^m(f(x)) > \alpha$. So $(f^{-1}\lambda^m)(x) > \alpha$. Moreover we have $\mu(x) > \alpha$. Therefore $x \in (\mu \cap f^{-1}\lambda^m)_\alpha$. Hence $f^{-1}(\Theta)$ is a cover for $(\mu \cap f^{-1}\mu)_\alpha$. Since f is (μ, μ)-continuous, $\mu \cap f^{-1}\lambda^i, i = 1, \cdots, n$ are open sets in τ_μ. So $f^{-1}(\Theta)$ is an open cover for $(\mu \cap f^{-1}(\mu)_\alpha$. $\square$

Theorem 9 *Let Θ and Σ be two finite open covers for μ_α. Then the following inequality is true.*

$$H_\alpha(\Theta \vee \Sigma) \leq H_\alpha(\Theta) + H_\alpha(\Sigma).$$

Proof: Let $\Theta \supseteq \Theta' = \{\lambda^1_\alpha, \cdots, \lambda^n_\alpha\}$ and $\Sigma \supseteq \Sigma' = \{\gamma^1_\alpha, \cdots, \gamma^m_\alpha\}$ be subcovers of Θ, and Σ such that $H_\alpha(\Theta) = \log n$, and $H_\alpha(\Sigma) = \log m$. Now $\Theta' \vee \Sigma'$ is a subcover of $\Theta \vee \Sigma$. So $H_\alpha(\Theta \vee \Sigma) \leq \log(nm) = \log n + \log m = H_\alpha(\Theta) + H_\alpha(\Sigma)$. $\square$

Theorem 10 *If $f : X \longrightarrow X$ be an RSD-system and Θ be a finite open cover for μ_α, then*

$$H_\alpha(\Theta) \geq H_\alpha(f^{-1}\Theta)$$

Proof: If $\Theta' = \{\lambda^1_\alpha, \cdots, \lambda^n_\alpha\} \subseteq \Theta$ such that $H_\alpha(\Theta) = \log n$, then

$$f^{-1}\Theta' = \{(\mu \cap f^{-1}\lambda^1)_\alpha, \cdots, (\mu \cap f^{-1}\lambda^n)_\alpha\}$$

is an open cover for $(\mu \cap f^{-1}\mu)_\alpha$. Therefor

$$N(f^{-1}\Theta) \leq N(f^{-1}\Theta') \leq n = N(\Theta).$$

Thus $H_\alpha(\Theta) \geq H_\alpha(f^{-1}\Theta)$. $\square$

Theorem 11 *Let $f : X \longrightarrow X$ be an RSD-system, where X is a compact (α, μ)-Hausdorff space. Moreover, let Θ be an open cover for μ_α, where $\alpha \in (0, 1)$. Then the following limit exists.*

$$\limsup_{n\to\infty} \frac{1}{n} H_\alpha(\bigvee_{i=0}^{n-1} f^{-i}(\Theta))$$

Proof: Let consider $x_n = H_\alpha(\bigvee_{i=0}^{n-1} f^{-i}(\Theta))$. Then for all $n, m \in N$ we have:

$$\begin{aligned} x_{n+m} = H_\alpha\Big(\bigvee_{i=0}^{n+m-1} f^{-i}(\Theta)\Big) & \\ \leq \quad & H_\alpha\Big(\bigvee_{i=0}^{n-1} f^{-i}(\Theta)\Big) + H_\alpha\Big(f^{-n}(\bigvee_{j=0}^{m-1} f^{-j}(\Theta))\Big) \\ \leq \quad & H_\alpha\Big(\bigvee_{i=0}^{n-1} f^{-i}(\Theta)\Big) + H_\alpha\Big(\bigvee_{j=0}^{m-1} f^{-j}(\Theta)\Big) \\ = \quad & x_n + x_m. \end{aligned}$$

Thus $\lim_{n\to\infty} \frac{x_n}{n}$ exists, since $\{x_n\}_{n\in N}$ is a subadditive sequence. $\square$

Regarding the above theorem, the α-level relative topological entropy for the RSD-system $f : X \longrightarrow X$ associated to the open cover Θ is define by:

$$h_\alpha(f,\Theta) = \lim_{n\to\infty} \frac{1}{n} H_\alpha(\bigvee_{i=0}^{n-1} f^{-i}\Theta)\cdot$$

when X is a compact (α,μ)-Hausdorff space.

Definition 12 *The α-level relative topological entropy for f is defied by:*

$$h_\alpha(f) = \sup\{h_\alpha(f,\Theta) : \Theta \textit{ is a finite cover of } \mu_\alpha\}$$

By recalling [6], two RSD-systems (f, X, τ_μ) and (g, X, τ_μ) are called μ-conjugate if there exists a μ-homeomorphism $\varphi : X \longrightarrow X$ such that $\varphi o f = g o \varphi$. Next theorem shows that the relative topological entropy is invariant under μ-conjugate relation.

Theorem 13 *If $f : X \longrightarrow X$ and $g : X \longrightarrow X$ are μ-conjugate then $h_\alpha(f) = h_\alpha(g)$ for all $\alpha \in (0,1)$.*

Proof: Because of the μ-conjugate relation, there exists μ-homeomorphism $\varphi : X \longrightarrow X$ such that $\varphi \circ f = g \circ \varphi$. In this regard, let $\alpha \in (0,1)$ and Θ be a finite open cover for μ_α, then:

$$\begin{aligned} h_\alpha(g,\Theta) &= \limsup_{n\to\infty} \frac{1}{n} H_\alpha\Big(\bigvee_{i=0}^{n-1} g^{-i}(\Theta)\Big) \\ &= \limsup_{n\to\infty} \frac{1}{n} H_\alpha\Big(\bigvee_{i=0}^{n-1} g^{-i}(\varphi^{-1}(\Theta))\Big) \\ &= h_\alpha(f,\varphi^{-1}\Theta). \end{aligned}$$

So $h_\alpha(g) = h_\alpha(f)$ Since φ is μ-homeomorphism. $\square$

4 Computational Example

Recently progress has been made in the development of algorithm for optimizing polynomials. The main idea being stressed is that reducing problem to an easier problem involving semi-definite programming. Lesserre, in [5], describes an extension of the method to minimizing a polynomial on an arbitrary semi-algebraic set, which is the set defined by Boolean combination of polynomial equations and inequalities. However, the study of semi-algebraic sets is based mainly on the slicing technique, which makes it possible to decompose them into the finite number of subsets semi-algebraically homomorphic to an open hypercubes. Using this composition, allows us to investigate semi-algebraically connected component for every semi-algebraic set, with finite cover. That is just one of the reasons to care the notion of connectedness and compactness for polynomial function space.

Our approach to above problem is to develop the μ-relative semi-dynamical system over one variable polynomial function space $\mathbf{R}[x]$ based on derivative operator. we are going to use orbits as the μ-open sets to decompose $\mathbf{R}[x]$. This topic may be interesting for further independent research subject on semi-definite programming. But here, we have just looked over it as an illustration example.

In order to present an RSD-system over one variable polynomial function space $\mathbf{R}[x]$ based on derivative function, let $X = \mathbf{R}[x]$ and $\mu : X \longrightarrow [0,1]$ defined by:

$$\mu(f) = \begin{cases} \frac{1}{\deg(f)} & \text{if } \deg(f) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Also let $\lambda^i : X \longrightarrow [0,1]$ defined by:

$$\lambda^i(f) = \begin{cases} \frac{1}{i} & \text{if } \deg(f) = i \\ 0 & \text{otherwise} \end{cases}$$

Since $\lambda^i \cap \lambda^j = \chi_\emptyset$ for $i \neq j$ and $\bigcup_{i\in\mathbf{N}} \lambda^i = \mu$, then we can consider τ_μ as the μ-topology generated by $\{\lambda^i : i \in \mathbf{N}\}$, also we have the following results:
$\alpha = 1 \Rightarrow \mu_1 = \chi_\emptyset$
$\alpha = 0 \Rightarrow \mu_0 = \{f \in \mathbf{R}[x] : \deg(f) \geq 1\}$
If $\alpha \in (0,1)$ then

$$\mu_\alpha = \begin{cases} \{f \in \mathbf{R}[x] : \deg(f) \leq [\frac{1}{\alpha}]\} & \text{if } \frac{1}{\alpha} \notin \mathbf{N} \\ \{f \in \mathbf{R}[x] : \deg(f) \leq [\frac{1}{\alpha}] - 1\} & \text{if } \frac{1}{\alpha} \in \mathbf{N} \end{cases}$$

Suppose that $F : X \longrightarrow X$ is the derivation map, i.e. $F(f) = f'$. Then F is (μ,μ)-continuous since:

$$\begin{aligned} F^{-1}(\lambda^i)(f) &= \lambda^i(F(f)) \\ &= \lambda^i(f') = \begin{cases} \frac{1}{i} \text{ if } \deg(f') = i \\ 0 \text{ otherwise} \end{cases} \\ (\mu \cap F^{-1}(\lambda^i))(f) &= \min\{\mu(f), F^{-1}(\lambda^i)(f)\} \\ &= \begin{cases} \frac{1}{i+1} & \text{if } \deg(f) = i+1 \\ 0 & \text{otherwise} \end{cases} \\ &= \lambda^{i+1}(f). \end{aligned}$$

So $\mu \cap F^{-1}(\lambda^i) = \lambda^{i+1} \in \tau_\mu$. Thus (F, X, τ_μ) is a relative semi-dynamical system.
Now for $k \in \mathbf{Z}$ we consider the orbit of the element $f(x) = x^k$ in X as follows:

$$\begin{aligned} O(f) &= \{F^n(f) : n \in \{0,1,2,\ldots\}\} \\ &= \{x^k, kx^{k-1}, k(k-1)x^{k-2}, \ldots, k!, 0\} \end{aligned}$$

The structure of μ-topology on X implies that $O(f)$ is dense in μ_α. Moreover, μ_α is compact and (F, X, τ_μ) is μ_α-minimal for all $\alpha \in (0,1)$. In fact, the derivation map F is μ_α-transitive when $\alpha \neq 0$.
It is easy to see that $h_\alpha(F) = 0$ for all $\alpha \neq 0$, but

the presented computational method seems complicated for calculating $h_0(F)$. So an alternative method for calculation of the relative topological entropy is needed; that would be our next research goal.

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