A mathematical analysis of real options interactions

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Abstract. The main goal of this paper is to deepen the discussion on real option interactions in multiple investment projects, focusing in the case of option to expand and/or to contract. Combining an analytical methodology with Mathematica experiments, we are able to perform sensitivity analysis and to study the nonadditivity of the interactions. The interesting effects pointed out in this paper were find out in [5] for the first time.

Key-words: Real option analysis, compound options, multinormal distribution.

1. Introduction

In [5] the nature of options interactions was investigated through a generic investment project with multiple operating options. [5] demonstrated that "interactions among real options present in combination generally make their individual values nonadditive". In other words, considering each investment opportunity separately and summing up these individual option values might substantially over(under)estimate the overall value of a project. In this paper we focus on the options to expand or to contract the project scale and explore the extent to which option values are not additive. Thus [6] is revisited following a different mathematical method. Indeed, an analytical solution for the combined value of two options to expand and/or to contract is provided, building on a proper adaptation of Geske's methodology [3]. As a by-product, a generalization of Geske's formula for compound call options is obtained. By means of our valuation formula the super/subadditivity is proved, depending on whether the prior option is a call or a put. Moreover the degree of interaction and (non)additivity is investigated and is related to the separation of the exercise times of the two options, their being of the same or of opposite type, their order in the sequence. As a consequence, we are able to confirm the interesting effects found out in [5] building on a different mathematical methodology. Then we develop a programme tailored to our application employing *Mathematica*. Indeed, *Mathematica* has built-in-routines allowing us to implement our closed-form solution and to get numbers out. Another useful feature of the analytical treatment within *Mathematica* is the ability to work out sensitivity parameters. Thus the sensitivity analysis performed in [1] can be completed, dealing with some issues which are hard to handle within a purely analytical approach and therefore remain unanswered in [1]. Section 2 is based on [1] and presents the analytical result; the numerical programme is exposed in Section 3. Section 4 is devoted to the investigation of (non)additivity, while sensitivity analysis is performed in Section 5 combining analytical and numerical considerations. Section 6 concludes.

2. The analytical formula

This section is devoted to recall the problem and the analytical results of [1]. We consider an investment opportunity allowing management to expand (respectively, to contract) the project's scale by a fraction α_1 at time T_1 by making an investment outlay (respectively, by reducing the investment outlay) of A_1. The first option is followed by a subsequent option to expand (respectively, to contract) by a fraction α_2 at time $T_2 \geq T_1$, if the cost is increased (respectively, reduced) of A_2 . Let X_i denote A_i / α_i , i =1,2. As usual in ROA, it is assumed that the gross project value V follows a geometric Brownian motion and its instantaneous standard deviation is denoted by σ. As standard in option pricing theory, any contingent claim is priced as if the world were risk neutral, by adjusting the expected growth rate and employing a certainty-equivalent rate r. In [1] it is proved that the present value added to the base-scale project by the options to expand and/or to contract is given by Ψ , whose analytical expression is determined in the following:

Proposition 1. *In the case of contraction at time T_2 , we make the additional assumption $\alpha_2 \leq 1$ and $\alpha_2 X_2 e^{-rT_2} \leq X_1 e^{-rT_1}$. Then, the following valuation formula holds for the project at the initial time t=0 :*

$$\Psi(V;T_1,X_1,\alpha_1,\omega_1;T_2,X_2,\alpha_2,\omega_2) =$$
$$= \alpha_2\omega_2\left\{VN(\omega_2\widetilde{h}_{X_2,T_2}) - X_2e^{-rT_2}N(\omega_2 h_{X_2,T_2})\right\} +$$
$$+\alpha_1\omega_1\left\{VN(\omega_1\widetilde{h}_{V*,T_1}) - X_1e^{-rT_1}N(\omega_1 h_{V*,T_1})\right\} +$$
$$+\alpha_1\alpha_2\omega_1\omega_2(VN_2(\omega_1\widetilde{h}_{V*,T_1},\omega_2\widetilde{h}_{X_2,T_2};\omega_1\omega_2\rho) -$$
$$-X_2e^{-rT_2}N_2(\omega_1 h_{V*,T_1},\omega_2 h_{X_2,T_2};\omega_1\omega_2\rho))$$

Here the parameter ω_i takes on the value +1 (respectively -1) in the case of an option to expand (respectively, to contract) at time T_i, N is the univariate cumulative normal distribution function and N_2 is the bivariate cumulative normal distribution,

$$h_{X,T} = (\ln\frac{V}{X} + (r - \frac{\sigma^2}{2})T)/(\sigma\sqrt{T}),$$

$\widetilde{h}_{X,T} = h_{X,T} + \sigma\sqrt{T}$, $\rho = \sqrt{T_1/T_2}$ *and* V^* *is such that*

$$\alpha_2 F^{(1)}(V*,T_2 - T_1,X_2,\omega_2) + V* \quad = X_1$$

where $F^{(1)}(V,T,X,\omega)$ *denotes the Black-Scholes value of a European call (ω=1) or put (ω=-1) option with X as exercise price and T as expiration date.*

The method used in [1] to prove this result is based on Geske's approach to the pricing of compound options. However our result is not a straightforward application of the classical formula, because we consider compound options whose value is contingent on a combination of an option with its underlying asset, that is on $F^{(1)}(V,T_2,Y,\omega_2) + \gamma V,$ for some real γ. We have called such a contingent claim a "generalized" compound option. In [1] an explicit valuation formula is proved, at current time t, for a compound option whose value $F^{(2)}$ is contingent on the value $F^{(1)}(V,T_2,Y,\omega_2) + \gamma V$, precisely, for a compound option with T_1 as the maturity date and X as the exercise price and whose underlying is a combination of a European option (with maturity date T_2 , $T_1 \leq T_2$, and exercise price Y) with a proportion γ of its underlying. Denoting

$$F^{(2)}(F^{(1)}(V,T_2,Y,\omega_2) + \gamma V,T_1,X,\omega_1)$$

by $\Phi(V,t;T_1,X,\omega_1;T_2,Y,\omega_2;\gamma)$, the following formula is obtained:

$$\Phi(V,t;T_1,X,\omega_1;T_2,Y,\omega_2;\gamma) =$$
$$= \omega_1\omega_2 VN_2(\omega_1 h(t) + \sigma\sqrt{T_1 - t},\omega_2 k(t) +$$
$$+ \sigma\sqrt{T_2 - t};\omega_1\omega_2\rho(t)) -$$
$$- \omega_1\omega_2 Ye^{-r(T_2-t)}N_2(\omega_1 h(t),\omega_2 k(t);\omega_1\omega_2\rho(t)) -$$
$$- \omega_1 Xe^{-r(T_1-t)}N(\omega_1 h(t)) +$$

$+\omega_1 \gamma V N(\omega_1 h(t) + \sigma\sqrt{T_1 - t})$

where $\rho(t) = \sqrt{\frac{T_1 - t}{T_2 - t}}$,

$k(t) = (\ln\frac{V}{Y} + (r - \frac{\sigma^2}{2})(T_2 - t))/(\sigma\sqrt{T_2 - t})$,

$h(t) = (\ln\frac{V}{V*} + (r - \frac{\sigma^2}{2})(T_1 - t))/(\sigma\sqrt{T_1 - t})$,

$F^{(1)}(V*, T_2 - T_1, Y, \omega_2) + \gamma V* = X$.

Finally, a Lemma is given that will be useful in the sequel:

Lemma *The following identities hold:*
$\frac{\partial\Phi}{\partial h} = 0$, $\frac{\partial\Phi}{\partial k} = 0$ *and* $\frac{\partial\Phi}{\partial \rho} = 0$.

3. Implementation with *MATHEMATICA*

In this section our model is implemented in *Mathematica*, making use of its built-in-routines to perform the numerical calculation with our formula. In the following section our *Mathematica* programme will be used to investigate the (non)additivity of option interaction and sensitivity analysis, using *Mathematica*'s symbolic calculus capabilities or producing plots to support intuition. Our programme goes through three steps. At first the Black-Scholes formulas for European call and put options are defined. This step is obvious and the functions BlackScholesCall and BlackScholesPut have been made into a standard *Mathematica* Package. Then the critical value $V*$ is determined as the solution of

$$\alpha_2 F^{(1)}(V*, T_2 - T_1, X_2, \omega_2) + V* = X_1.$$

To the purpose we make use of the numerical solver FindRoot.

```
Norm[x_] := (1 + Erf[x/Sqrt[2]])/2;
kbs[w_, σ _, z_, t_, r_] :=
 (r*t + Log[w/z])/( σ*Sqrt[t]) + (σ*Sqrt[t])/2;
hbs[w_, σ _, z_, t_, r_] :=
 (r*t + Log[w/z])/( σ*Sqrt[t]) - (σ*Sqrt[t])/2;
BlackScholes[v_, y_, s_, r_, t_, ω_] :=
v*ω*Norm[ω*kbs[v, s, y, t, r]] − y*ω*Exp[-r*t]*Norm[ω*hbs[v, s, y, t, r]];
CriticalPoint[Xone_, Xtwo_, σ _, r_, Tone_, Ttwo_, Atwo_, Φtwo_] :=
V /. FindRoot [Xone == Atwo* BlackScholes [V, Xtwo, σ, r, Ttwo - Tone, Ωtwo] +V, {V, {0.01, Xtwo}}];
```

In the second step some routines are loaded to model the bivariate normal distribution:

```
Needs["Statistics`MultinormalDistribution`"];
mu = {0, 0};
sigma[rho_] := {{1, rho}, {rho, 1}};
Mfunc[h_, b_, rho_] :=
CDF[MultinormalDistribution[mu, sigma[rho]], {h, b}];
```

Finally, we write down our formula, that will be called "ValFormula", as follows:

```
kone[V_, Cp_, σ _, r_, Tone_] :=(Log[V/Cp] + (r + σ^2/2)*Tone)/( σ*Sqrt[Tone]);
hone[V_, Cp_, σ _, r_, Tone_] :=(Log[V/Cp] + (r - σ ^2/2)*Tone)/( σ*Sqrt[Tone]);
ktwo[V_,Xtwo_, σ_,r_,Ttwo_]:= (Log[V/Xtwo] + (r + σ ^2/2)*Ttwo)/( σ *Sqrt[Ttwo]);
htwo[V_,Xtwo_, σ_,r_,Ttwo_]:= (Log[V/Xtwo] + (r - σ ^2/2)*Ttwo)/( σ *Sqrt[Ttwo]);
ValFormula[V_, Xo_, Xt_, σ_, r_,To_, Tt_, Ao_, At_, Ω o_, Ω t_] :=
Module[{Cp,ko,ho,kt,ht,rho=Sqrt[To/Tt]},
Cp = CriticalPoint[Xo, Xt, σ, r, To, Tt, At, Wt];
ko = kone[V, Cp, σ, r, To];
ho = hone[V, Cp, σ, r, To];
kt = ktwo[V, Xt, σ , r, Tt];
 ht = htwo[V, Xt, σ , r, Tt];
Ao*At* Ωo* Ωt*(V*Mfunc[Ωo*ko,Ωt*kt, Ωo* Ωt*rho]-Xt*Exp[-r*Tt]*Mfunc[Ωo*ho, Ωt*ht, Ωo* Ω t*rho])+Ao* Ωo* (V* Norm[Ωo*ko]-Xo*Exp[r*To]*Norm[Ωo*ho])+ At* Ωt*(V* Norm[Ωt*kt] - Xt*Exp[-r*Tt]* Norm[Ωt*ht])]
```

4. Are option interactions additive?

In [5] Trigeorgis showed by numerical valuation that the combined value of two real options may differ greatly from the sum of their individual values. In other words, option interactions are generally nonadditive. Here we focus on the combination of options to expand and/or to contract. The kind of interaction between the two options depends on their being of the same or of different types. At first we employ *Mathematica* to illustrate the effect in the case of options of the same type. In the following figures the combined value of two options to expand (Figure 1) and, respectively, to contract (Figure 2) is plotted against V, using a thick line. The same figures display the graphs of the sum of the individual options' values (thin line). The conclusion is that the combination of two options to expand exhibits superadditivity, while the combination of two options to contracts exhibits subadditivity. This result was obtained in [1] by analytical methods. We recall it here for readers' facility.

Proposition 2. *The combined value of two options to expand is greater than the sum of the individual options' values; the combined value of two options to contract is smaller than the sum of the individual options' values.*

Proof. Let D(V) denote

$$\Psi(V;T_1,X_1,\alpha_1,\omega_1;T_2,X_2,\alpha_2,\omega_2) -\alpha_1 F^{(1)}(V,T_1,X_1,\omega_1) -\alpha_2 F^{(1)}(V,T_2,X_2,\omega_2).$$

Note that $D(0+)=0$ *if* $\omega_1 = \omega_2 = 1$ and $D(0+) < 0$ if $\omega_1 = \omega_2 = -1$.

In view of Lemma 1 we have:

$$\partial_V D(V) = \alpha_1\omega_1\left\{N(\omega_1\tilde{h}_{V*,T_1}) - N(\omega_1\tilde{h}_{X_1,T_1})\right\} + \\ + \alpha_1\alpha_2\omega_1\omega_2 N_2(\omega_1\tilde{h}_{V*,T_1},\omega_2\tilde{h}_{X_2,T_2};\omega_1\omega_2\rho)$$

From

$$\alpha_2 F^{(1)}(V*,T_2-T_1,X_2,\omega_2) = X_1 - V*$$

we get $V* \le X_1$, whence

$$\omega_1\left\{N(\omega_1\tilde{h}_{V*,T_1}) - N(\omega_1\tilde{h}_{X_1,T_1})\right\} \ge 0$$

and thus $\partial_V D(V) > 0$. Therefore $D(V) \ge 0$ if $\omega_1 = \omega_2 = 1$. On the other hand, when $\omega_1 = \omega_2 = -1$, $D(V) \to 0$ as $V \to +\infty$ and therefore $D(V) \le 0$.

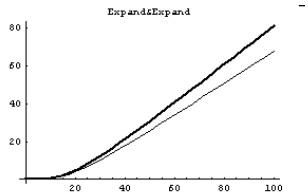

Fig. 1

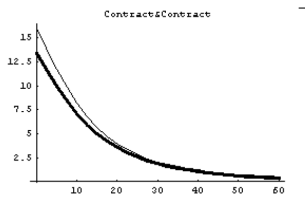

Fig. 2

Let us consider now the combination between two options of opposite type. In this case the analytic proof we quoted above is hard to apply. Thus a graphic investigation is very helpful. In Figures 3 and 4 we compare the value of the combined options (thick line) with the sum of their separate values (thin line). In the former case the parameter values are $X_1 = 25$, $X_2 = 30$, $T_1 = 2$, $T_2 = 14$, in the latter they are $X_1 = 20$, $X_2 = 30$, $T_1 = 4$, $T_2 = 23$

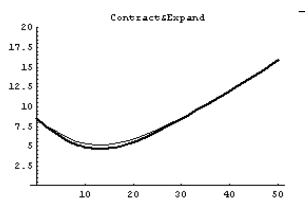

Fig. 3

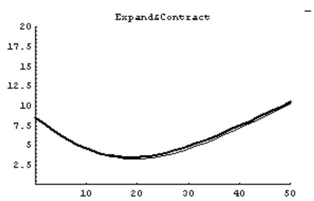

Fig. 4

Note that in both situations the two graphs nearly overlap, especially when V is far apart from X_1 and X_2. Further graphic investigation would show that the two graphs overlap more and more as $T_2 \to T_1$. This can be confirmed by analytical calculation (see [1]). Moreover, our plots show that the combined value is greater (respectively, smaller) than the sum of the individual values, depending on the prior option being a call (respectively, put). Let us now confirm our last guess analytically. We confine ourselves to the case of an option to contract followed by an option to expand.

Proposition 3 *The combined value of an option to contract followed by an option to expand is smaller than the sum of the individual options' values.*

Proof. Let $D(T_2)$ denote $\Psi(V;T_1,X_1,\alpha_1,\omega_1;T_2,X_2,\alpha_2,\omega_2)$- $\alpha_1 F^{(1)}(V,T_1,X_1,\omega_1)$ with $\omega_1 = -1$ and $\omega_2 = 1$. We proved in [1] that $D(T_2) \to 0$ when $T_2 \to T_1$. In view of Lemma 1 we have:

$$\partial_{T_2} D =$$
$$-\alpha_1\alpha_2 V \partial_{h_2} N_2(-\tilde{h}_{V*,T_1}, \tilde{h}_{X_2,T_2}; -\rho)\sigma/(2\sqrt{T_2}) +$$
$$-r e^{-rT_2}\alpha_1\alpha_2 X_2 N_2(-h_{V*,T_1}, h_{X_2,T_2}; -\rho)$$

where ∂_{h_2} denotes differentiation with respect to the second variable. Clearly $\partial_{T_2} D < 0$, which yields $D(T_2) \le 0$ for $T_2 \ge T_1$.

Let us summarize the arguments above as follows:

Proposition 4. *If the two interacting options are of opposite type, i.e. to expand and to contract, then their interaction is small. It is purely additive for* $T_2 \to T_1$. *Furthermore, the interaction is positive if the prior option is an expansion and negative if it is a contraction.*

5. Sensitivity analysis

In this section we perform sensitivity analysis making use of *Mathematica* whenever an analytic treatment is hard to apply. In what follows differentiation is obtained by means of Lemma 1, which greatly simplifies calculation.

$$\partial_V \Psi(V;T_1,X_1,\alpha_1,\omega_1;T_2,X_2,\alpha_2,\omega_2) =$$
$$= \alpha_2\omega_2 N(\omega_2 \tilde{h}_{X_2,T_2}) + \alpha_1\omega_1 N(\omega_1 \tilde{h}_{V*,T1}) +$$
$$+ \alpha_1\alpha_2\omega_1\omega_2 N_2(\omega_1\tilde{h}_{V*,T_1}, \omega_2\tilde{h}_{X_2,T_2}; \omega_1\omega_2\rho)$$

If $\omega_1 = \omega_2 = 1$, then $\partial_V \Psi > 0$, whilst $\partial_V \Psi < 0$ if $\omega_1 = \omega_2 = -1$. (See [1]). When $\omega_1\omega_2 < 0$ the sign of $\partial_V \Psi$ seems ambiguous. Indeed the graphs in Figure 3 and Figure 4 show that Ψ exhibits a minimum, that is, the sign of $\partial_V \Psi$ is not unchanged in the case of options of opposite type. We conclude that:

if the two options are of the same type, then the combined value of the options to expand (contract) increases (decreases) with the (gross) project value; if the two options are of opposite type, then the combined value is decreasing for small V *and increasing for large* V.

Let us now differentiate Ψ with respect to the parameters. Differentiation with respect to X_1 yields:

$$\partial_{X_1} \Psi(V;T_1,X_1,\alpha_1,\omega_1;T_2,X_2,\alpha_2,\omega_2) =$$
$$= -\alpha_1\omega_1 e^{-rT_1} N(\omega_1 h_{V*,T_1})$$

and

$$\partial_{X_2} \Psi(V;T_1,X_1,\alpha_1,\omega_1;T_2,X_2,\alpha_2,\omega_2) =$$
$$-\alpha_1\alpha_2\omega_1\omega_2 e^{-rT_2} N_2(\omega_1 h_{V*,T_1}, \omega_2 h_{X_2,T_2}; \omega_1\omega_2\rho) -$$
$$-\alpha_2\omega_2 e^{-rT_2} N(\omega_2 h_{X_2,T_2})$$

Clearly, $\partial_{X_i} \Psi \gtreqless 0$ whenever $\omega_i \lesseqgtr 0$, that is, the less expensive (respectively, the more cost-saving) is the option to expand (respectively, to contract), the more valuable is the project.
Moreover, it is straightforward to realize that

$$\partial_\sigma \Psi(V; T_1, X_1, \alpha_1, \omega_1; T_2, X_2, \alpha_2, \omega_2) \geq 0$$

that is, the higher is the volatility, the more valuable is the project.
A more controversial issue is the behaviour with respect to the exercise time. Let us write down the partial derivative with respect to T_2 :

$$\partial_{T_2} \Psi =$$
$$= \alpha_1 \alpha_2 \omega_1 [\mathrm{V}\,\partial_{h_2}\, N_2(\omega_1 \tilde{h}_{V*,T_1}, \omega_2 \tilde{h}_{X_2,T_2}; \omega_1 \omega_2 \rho) .$$
$$\sigma / (2\sqrt{T_2}) +$$
$$+ \omega_2 r e^{-rT_2} X_2\, N_2(\omega_1 h_{V*,T_1}, \omega_2 h_{X_2,T_2}; \omega_1 \omega_2 \rho)] +$$
$$+ \alpha_2 [\mathrm{V}\,\partial_{h_2}\, N(\omega_2 \tilde{h}_{X_2,T_2})\, \sigma / (2\sqrt{T_2}) +$$
$$+ \omega_2 r e^{-rT_2} X_2\, N(\omega_2 h_{X_2,T_2})]$$

If $\omega_2 = 1$ then $\partial_{T_2} \Psi > 0$. The case $\omega_2 = -1$ seems dubious. In Figure 5 we plot the project value against T_2 for an option to contract twice (with $T_1 = 4$) and in Figure 6 for an option to expand followed by an option to contract (with T_1 =1). Even in these cases the project value seems to increase with T_2.

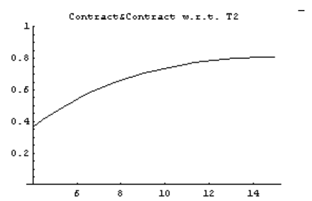

Fig. 5

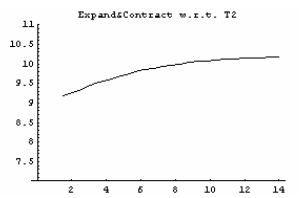

Fig. 6

6. Conclusion

In this paper a valuation formula is obtained for the intrinsic value of a project allowing to expand and/or to scale down operations. The theoretical base behind it is the real options approach. Our analysis is complemented by a programme in *Mathematica*.
The main result allows us to confirm some results contained in [5], [6], following a different mathematical method. For example, we have shown that the combined value of two options is generally different from the sum of the values of the single options embedded in the project. The conclusion is that in a complex investment project, the options that go to make it do interact and cannot be regarded as isolated from each other. In this paper the nature of such interactions has been analyzed with a special emphasis on their sign, which depends on the type of the options involved in the project. Moreover it is shown that, despite interaction, some familiar properties of simple options are preserved.

References:

[1] Agliardi R, Options to expand and to contract in combination, *Applied Mathematics Letters* , 20, 2007, 790-794
[2] Brennan, M.J., and L. Trigeorgis (Eds.), *Project Flexibility, Agency, and Competition,* Oxford University Press, 2000
[3] Geske, R., The valuation of compound options, *Journal of Financial Economics*, 7, 1979, 63-81
[4] Shaw W., *Modelling Financial Derivatives with Mathematica*, C.U.P., 1999
[5] Trigeorgis L., The nature of option interactions and the valuation of investments with multiple real options, *Journal of Financial and Quantitative Analysis*, 28, 1993, 1-20
[6] Trigeorgis L., *Real Options: Managerial flexibility and Strategy in Resource Allocation*, Cambridge, MA, MIT Press, 1996