Minimizing Total Standard Deviation of Counting Rates For n Radioactive Materials In A Nuclear Detector

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***Abstract*:**
In nuclear counting detection experiments, only small fraction of nuclei are picked up and recorded by the detector. This is depicted as a sampling process done by the detector. The more samples picks up by the detector, the better will be the accuracy of the detector's readings. Therefore, the accuracy in measuring radioactivity of a material is directly proportional to the time allocated for measurement. Laboratories are faced with huge number of materials to be analyzed in a limited span of time.
The objective in this research is to find the optimum testing time for these materials to minimize the inaccuracy. Since a smaller value for the standard deviation is used as a measure of accuracy, the problem translates into minimizing the standard deviations under time constraint. Analytical solution was devised for validation purposes. Results found using this formulation were compared with those using the analytical solution; both produced identical results.

***Key Words*:** Nuclear, Radiation Counts, Detectors, Minimization, Standard Deviation.

1 Mathematical Background

Laboratories are sometimes overwhelmed with large number of radioactive samples for measurements and constrained by time limits. Such a situation was encountered during the Chernobyl accident when many nuclear laboratories were flooded by foodstuffs and other samples to be analyzed [1-5]. The most recent work related to finding the optimal allocation of nuclear detector's time for a given number of samples, under time constraints was done by Aljohani [1]. His methodology was based on minimizing the sum of associated standard deviations of the net counting rate of samples. The assumption in Aljohani's [1] was based on having a constant radiation background for all samples which may not be exactly true if samples are measured by a spectrometer or if samples are measured in different locations.

The case of counting nuclear radiation events can best be modeled by a binomial distribution. Knoll [5] states that in the case of a trial that consists of observing a given radioactive nucleus for a period of time t_i, the number of trials is equal to the number of nuclei in the material under observation, and the measurement consists of counting those nuclei that undergo decay. The probability of success is identified as the proportion of nuclei that undergo decay, which is:

$$p = 1 - e^{-\lambda t} \qquad (1)$$

where λ is the decay constant of the radioactive material.

It is well known that the mean μ in a binomial distribution is equal to np and the variance σ^2 is equal to $n\,p\,(1-p)$, where n is the number of trials and p is the probability of success.

Binary processes with low probability of success for each individual trial can best be estimated by a Poisson distribution. In nuclear counting experiments large numbers of nuclei (in the order of Avogadro's number 10^{23}) make up the number of trials, whereas a relatively small fraction of these give rise to recorded counts. This small fraction of nuclei to be recorded by the detector is the primary focus in this study. Under these conditions, where $p \ll 1$ and x is the reading from a radiation counter for a time interval, the binomial distribution can be mathematically simplified by a Poisson distribution, as follows:

$$p(x) = \frac{(pn)^x e^{-pn}}{x!} \qquad (2)$$

Since $np = \bar{x}$ holds, then equation (2) becomes:

$$p(x) = \frac{(\bar{x})^x e^{-\bar{x}}}{x!} \qquad (3)$$

where $\bar{x}$ is the average of successive readings from a radiation counter for repeated time intervals of equal length. If $\bar{x}$ is large (traditionally > 20) then more simplification can be achieved as follows:

$$p(x) = \frac{1}{\sqrt{2\bar{x}\pi}} \exp\left[\frac{(x-\bar{x})^2}{2\bar{x}}\right] \qquad (4)$$

Under the aforementioned conditions this distribution is characterized by a single parameter $\bar{x}$, which is equal to np which is also equal to the predicted variance σ^2. From now on the main focus will be this single parameter σ^2 or $\sigma = \sqrt{\bar{x}}$.

2 Problem Formulation

The large number of radioactive materials coming for measurement is constrained by time limits. This time is usually dictated by the supplier of these materials and/or by the laboratory conditions. Since radioactivity usually differs from one material to another, the measurement time needed for each material has to be different. Thus, challenge is to optimize the allocation of measurement time among these materials. The longer the time allocated for measurement the higher the accuracy of results. However, with scarcity of time one cannot measure indefinitely.

In this paper, (n) materials are used; each material has a different radioactivity level. The objective is to minimize the sum of associated standard deviation of counting rates for (n) materials. This objective function is nonlinear one, and thus, requiring a nonlinear programming algorithm to solve it. The solution found using the proposed nonlinear model was verified by analytically solving the system of equations under the assumption that the background radiation is negligible. Two examples are introduced; the results found in the examples for the verification stage are identical to the results found using the nonlinear model.

3 Definition of Variables

M_i = Counts due to both radioactive material i and background.

B_i = Counts due to background only while testing material i.

r_{M_i} = Counting rate due to both radioactive material i and background.

r_i = Counting rate due to the radioactive material i without background.

b_i = Counting rate due to background while testing material i.

t_i = Measurement time of radioactive material i with background.

t_{bi} = Measurement time for background only while testing material i.

T = Total time given to test all (n) materials.

σ_i = Associated standard deviation of net counting rate for material i.

4 Theory

Consider the measurement of the net counting rate from a long-lived radioactive material in the presence of background. The net counting rate due to the radioactive material must be corrected by subtracting the background counting rate[2] as:

$$r_i = r_{M_i} - b_i \text{ , Or } \quad r_i = \frac{M_i}{t_i} - \frac{B_i}{t_{bi}} \qquad (5)$$

Applying error propagation formula[1] results in:

$$\sigma_i^2 = \sigma_{M_i}^2 \left(\frac{\partial r_i}{\partial M_i}\right)^2 + \sigma_{B_i}^2 \left(\frac{\partial r_i}{\partial B_i}\right)^2 \qquad (6)$$

Or

$$\sigma_i = \left[\left(\frac{\sigma_{M_i}}{t_i}\right)^2 + \left(\frac{\sigma_{B_i}}{t_{bi}}\right)^2\right]^{\frac{1}{2}} \qquad (7)$$

From the mathematical background above, it is known that $\sigma_{M_i} = \sqrt{M_i}$ and $\sigma_{B_i} = \sqrt{B_i}$. Substituting into equation (7) yields:

$$\sigma_i = \left[\frac{M_i}{t_i^{\;2}} + \frac{B_i}{t_{bi}^{\;2}}\right]^{\frac{1}{2}} \qquad (8)$$

By definition, counting rates are $r_i + b_i = \dfrac{M_i}{t_i}$, and $b_i = \dfrac{B_i}{t_{bi}}$, then equation (8) becomes :

$$\sigma_i = \left[\frac{r_i + b_i}{t_i} + \frac{b_i}{t_{bi}}\right]^{\frac{1}{2}} \qquad (9)$$

Assuming that (n) materials are available for measurement and materials are independent of each others, the total associated standard deviation of counting rates for all materials is:

$$\sigma_T = \sigma_1 + \sigma_2 + \sigma_3 + \ldots\ldots\ldots + \sigma_n \qquad (10)$$

Substituting (9) into (10) one gets:

$$\sigma_T = \left[\frac{r_1 + b_1}{t_1} + \frac{b_1}{t_{b1}}\right]^{\frac{1}{2}} + \left[\frac{r_2 + b_2}{t_2} + \frac{b_2}{t_{b2}}\right]^{\frac{1}{2}}$$

$$+ \ldots\ldots + \left[\frac{r_n + b_n}{t_n} + \frac{b_n}{t_{bn}}\right]^{\frac{1}{2}} \qquad (11)$$

Equation (11) represents the objective function that needs to be minimized with the following constraint:

$$T = t_1 + t_2 + t_3 + \ldots\ldots\ldots + t_n \qquad (12)$$

5 Validation

The nonlinear objective function in (11) under the constraint in (12) is to be solved using any nonlinear programming package. Assuming that background radiation is negligible (i.e. $b_i = 0$), equation (11) may be simplified; and σ_T becomes:

$$\sigma_T = \left[\frac{r_1}{t_1}\right]^{\frac{1}{2}} + \left[\frac{r_2}{t_2}\right]^{\frac{1}{2}} + \ldots\ldots + \left[\frac{r_n}{t_n}\right]^{\frac{1}{2}} \qquad (13)$$

By differentiating σ_T with respect to $t_1, t_2, \ldots\ldots\ldots, t_n$, setting all the derivatives to zero and rearranging terms, the following set of equations is obtained:

$$\frac{t_2}{t_1} = \left[\frac{r_2}{r_1}\right]^{\frac{1}{3}}, \frac{t_3}{t_1} = \left[\frac{r_3}{r_1}\right]^{\frac{1}{3}}, \ldots, \frac{t_n}{t_1} = \left[\frac{r_n}{r_1}\right]^{\frac{1}{3}} \qquad (14)$$

This set of equations (14) ((n-1) equations & (n) unknowns), along with equation (12) can now be solved analytically. The results can then be compared with the results found using the proposed numerical nonlinear model.

Without loss of generality, equation (14) above can be rewritten as:

$$t_i = t_1\left[\frac{r_i}{r_1}\right]^{\frac{1}{3}} \quad (\mathit{for}\ i = 2, 3, \ldots, n) \qquad (15)$$

Substituting for t_i in the constraint equation (12) yields:

$$T = t_1\left\{1 + \left[\frac{r_2}{r_1}\right]^{\frac{1}{3}} + \left[\frac{r_3}{r_1}\right]^{\frac{1}{3}} + \cdots + \left[\frac{r_n}{r_1}\right]^{\frac{1}{3}}\right\} \qquad (16)$$

Since t_1 is the only unknown variable in equation (16) one can solve for t_1. The values for all t_i $(i = 2, 3, \cdots n)$ can then be found by substituting the value of t_1 in equations (15).

To illustrate the steps discussed above, two examples are presented. The first one will be solved using the proposed nonlinear model. The second example is solved using both, the proposed nonlinear model and the analytical method formulated specially for verification purposes.

6 Examples

The following examples are presented below along with their corresponding solutions:

6.1 Example 1:

The total time allotted for testing the radioactivity of 8 materials is T = 7224.9923 minutes. The counting rates in counts per minute due to both material and background and that due to background alone are given below:

r_1+b_1	r_2+b_2	r_3+b_3	r_4+b_4
611	1017	2022	1781

r_5+b_5	r_6+b_6	r_7+b_6	r_8+b_8
922	792	1415	921

And,

b_1	b_2	b_3	b_4
121	140	230	180

b_5	b_6	b_6	b_8
160	90	160	80

The above example was solved using the proposed nonlinear model presented in this paper. The minimum objective function is σ_T = 12.0944 and the optimal allocations for the 7224.9923 minutes allotted to the 8 materials are:

t_1	t_2	t_3	t_4
532.36	642.101	814.1235	784.1783

t_5	t_6	t_7	t_8
614.696	595.68	722.9819	633.1941

And,

t_{b1}	t_{b2}	t_{b3}	t_{b4}
236.909	238.2416	274.5851	249.3172

t_{b5}	t_{b6}	t_{b7}	t_{b8}
256.0668	200.8176	243.1223	186.6179

6.2 Example 2:(Background is assumed negligible (i.e. b = 0)).

This example was solved using the proposed nonlinear model and validated using the analytical method. The total time allotted for testing 6 materials is T = 2117 minutes. Counting rates per minute are given below:

r_1	r_2	r_3	r_4	r_5	r_6
1015	921	102	201	333	621

Using the proposed nonlinear model, the minimum value for the objective function is σ_T = 6.730412; the optimal allocation of the 2117 minutes allotted to 6 materials is:

t_1	t_2	t_3
464.8093	449.9932	216.0991

t_4	t_5	t_6
270.9265	320.5783	394.5935

The same example was solved using the analytical method, the minimum value for the objective function is σ_T = 6.7304; and the optimal allocations of the 2117 minutes allotted to 6 materials are:

t_1	t_2	t_3
464.8093	449.9933	216.0991

t_4	t_5	t_6
270.9265	320.5784	394.5936

7 Discussion

It should be noted that the objective function σ_T in equation (11) along with the constraint equation (12) constitute a nonlinear model; both are functions of $t_1, t_2, t_3, \ldots\ldots\ldots, t_n, t_b$. Fortunately, the objective function σ_T is a monotonically decreasing function of $t_1, t_2, t_3, \ldots\ldots\ldots, t_n, t_b$ and the constraint equation (12) is purely linear. Therefore, the solution found is a global minimum. Comparison of results found from the nonlinear model with those from the analytical one show a high degree of compatibility between them. This degree of consistency between results establishes a proof of effectiveness of the proposed nonlinear model presented in this paper.

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