Clifford analysis formulation of electromagnetism

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Abstract: We present a formulation of electromagnetism in vacuum and in the presence of an arbitrary gravitational field, based on Clifford analysis over a pseudo-Riemannian space with signature (1,3). We show that it is possible to solve the direct electromagnetic radiation problem in four dimensional form, without invoking electromagnetic potentials and solely by analytical means. Our method reveals that the full electromagnetic field can be completely expressed in terms of a particular solution of the underlying scalar wave equation, so that calculating the customary Green's dyadic is superfluous.

Key–Words: Electromagnetism, Gravity, Clifford analysis, Differential forms.

1 Introduction

The aim of this paper is to elucidate part of the, widely unrecognized, intrinsic simplicity of the physical manifestation that we call electromagnetism. We do this by modelling electromagnetism in terms of modern mathematical language and concepts. We identify and review the mathematical structures that are required to elegantly formulate electromagnetism in vacuum and in the presence of an arbitrary gravity field. The benefit of such an approach is that it severely simplifies the solution of direct radiation problems.

We show that it is possible to solve the direct electromagnetic radiation problem in four dimensional form, i.e., without assuming the customary time harmonic regime, without making the detour of invoking electromagnetic potentials and without the need for a Green's dyadic. We obtain the particular solution for the considered radiation problem purely by analytical means and in terms of a fundamental solution of the underlying scalar wave equation.

Our result is a direct consequence of the model that we use to represent electromagnetism. The model for electromagnetism, still widely in use today in applied sciences, is a virtually unchanged version that goes back to O. Heaviside, who simplified two earlier models by J. Maxwell. Its mathematical formulation is not only very outdated, but also obscured all this time the intrinsic simplicity and beauty of this physical phenomenon. Responsible for this state of affairs is a vector algebra, invented by J. Gibbs and adopted by Heaviside to build his model, but which is very inappropriate for describing electromagnetism. Heaviside's equations are widely considered to be the correct model for (classical) electromagnetism, because they predict numerical values for the magnitudes of the field components that are in agreement with experimental values. However, Heaviside's equations do not correctly model the geometrical content of the electromagnetic field, nor all its physical invariances. Modern physical insight requires that a good model not only predicts correct magnitudes, but also correctly models the geometric content and physical invariances of the physical phenomenon. Once one is willing to give attention to these requirements, by changing to a mathematical model that is also correct in this broader sense, fascinating new progress becomes possible.

We use here the Clifford algebra and the thereupon based Clifford analysis over a pseudo-Riemannian space with signature $(1,3)$, [2], [3], [6], [7], [10]. This allows us to model the above mentioned electromagnetic radiation problem by a single and simple equation. This equation is equivalent to Heaviside's equations in the narrow sense that both models produce the same field component magnitudes (in the absence of gravity of course). We derive a new expression for the particular solution of the full electromagnetic field in terms of the fundamental solution of the scalar wave equation. The general solution of the electromagnetic boundary value problem can also be expressed in a similar analytical form, but this development requires a somewhat more advanced and longer derivation and will be presented elsewhere.

We use natural units in our model for electromagnetism. This has the advantage that all superfluous unit conversion factors disappear from the equation, which so acquires its most simple form. Natural units are not unique, but all such unit systems are equivalent in the sense that they all result in the same equation. The use of a natural unit system reveals that there are no fundamental physical constants associated with electromagnetism. A physical constant is regarded as being fundamental iff it is a dimensionless quantity and different from 0 and 1. A convenient natural unit system for electromagnetism (and gravity) is obtained by defining the dimensionless constants $c \triangleq 1$ (c: "speed of light"), $8\pi G \triangleq 1$ (G: "gravitational constant") and $4\pi\varepsilon_0 \triangleq 1$ (ε_0: "permittivity of the vacuum").

2 Mathematical preliminaries

A finite subset of consecutive integers will be denoted by $\mathbf{Z}_{[i_1,i_2]} \triangleq \{i \in \mathbf{Z} : i_1 \leq i \leq i_2\}$. Let M designate a real, connected, oriented, smooth (i.e., C^∞) differential manifold, [4].

2.1 Contravariant tensor fields on M

At any point $x \in M$, we consider the tangent space T_xM, which is a linear space over $\mathbf{R}$ of some dimension $n \in \mathbf{N}$ and whose elements are called contravariant (tangent) vectors at x.

Denote further by $\wedge^k T_xM$, with $0 \leq k \leq n$, the linear space over $\mathbf{R}$, of totally antisymmetric contravariant tensors of order k at x, having dimension $\binom{n}{k}$. Elements of $\wedge^k T_xM$ are usually called in the Clifford algebra literature (contravariant) k-vectors and the order k of a k-vector is there called its grade. In particular, contravariant 0-vectors are by definition identified with the base field $\mathbf{R}$ and contravariant 1-vectors are identified with elements of the tangent space, i.e., $\wedge^1 T_xM \cong T_xM$.

We now assume that our manifold M admits a bilinear (generalized) inner product $\cdot : T_xM \times T_xM \to \mathbf{R}$, defined by a symmetric, 2-covariant, non-degenerate (i.e., of maximal rank), indefinite, inner product (so called "metric") smooth tensor field g such that $(u, v) \mapsto u \cdot v = g_x(u, v) = g_{\mu\nu}(x)\, u^\mu v^\nu$, with g_x the tensor obtained by evaluating g at x. This makes the structure (M, g) a smooth pseudo-Riemannian manifold.

The inner product of any pair of contravariant k-vectors, with respect to a basis for $\wedge^k T_xM$, is defined by

$$(a, b) \quad \mapsto \quad a \cdot b = \frac{1}{k!} a^{\mu_1 \ldots \mu_k} b^{\nu_1 \ldots \nu_k} g_{\mu_1\nu_1}(x) \ldots g_{\mu_k\nu_k}(x), \tag{1}$$

using the implicit Einstein summation convention over pairs of corresponding covariant and contravariant indices.

The manifold M together with the set of linear spaces $\wedge^k T_xM$, $\forall x \in M$, can be given the structure of a linear bundle, denoted $\wedge^k TM$ and called the k-th exterior power of the tangent bundle of M. Any section of $\wedge^k TM$ is called a totally antisymmetric contravariant tensor field of order (or grade) k on M, or in short a contravariant k-vector field. We will denote the set of contravariant k-vector fields by $\Gamma\left(\wedge^k TM\right)$. The manifold M together with the set of bases for $\wedge^k T_xM$, $\forall x \in M$, can also be given the structure of a linear bundle, called the frame bundle for $\wedge^k TM$. Any section of this frame bundle is called a contravariant frame field of order (or grade) k on M, or in short a (moving) contravariant k-frame. Any contravariant k-vector field has a representative with respect to any contravariant k-frame.

At any $x \in M$, we can always choose local coordinates on M such that the tensor field g at that point, g_x, becomes $g_x \triangleq [g_{\mu\nu}(x)] = [\eta_{\mu\nu}]$ with η the following diagonal tensor with components in matrix form given by

$$[\eta_{\mu\nu}] \triangleq diag\left[\underbrace{+1, +1, ..., +1}_{p \text{ times}}, \underbrace{-1, -1, ..., -1}_{q \text{ times}}\right], \tag{2}$$

and where $n = p + q$. The couple (p, q) is called the signature of the pseudo-Riemannian manifold M (and is independent of x). When $p > 0$ and $q > 0$, g_x is indefinite and M is said to be a pseudo-Riemannian manifold with p time dimensions and q space dimensions. If $q = 0$, g_x is positive definite and M is said to be a Riemannian manifold. If a pseudo-Riemannian manifold has zero curvature (i.e., is flat), a global coordinate system can be found on M such that the tensor field g takes the constant diagonal form (2) everywhere. Flat pseudo-Riemannian manifolds for which $p = 1$ are called Lorentzian manifolds and the particular case $p = 1$ and $q = 3$ is called Minkowski space.

In practice, g will represent a gravitational field present on M or/and will be induced by a particular choice of local coordinates, used to chart M in the vicinity of x.

2.2 Covariant tensor fields on M

The contravariant inner product structure on M defines a canonical isomorphism between the tangent space T_xM at x, and its dual T_x^*M, $\forall x \in M$. This dual T_x^*M is also a linear space over $\mathbf{R}$, of the same dimension n, called the cotangent space at x, and its elements are called covariant (cotangent) vectors at x.

Denote further by $\wedge^k T_x^* M$, with $0 \le k \le n$, the space of totally antisymmetric covariant tensors of order k at x. Elements of $\wedge^k T_x^* M$ are sometimes called in the Clifford algebra literature covariant k-vectors. In particular, covariant 0-vectors are again identified with the base field $\mathbf{R}$ and covariant 1-vectors with elements of the cotangent space, i.e., $\wedge^1 T_x^* M \cong T_x^* M$.

The canonical isomorphism from $T_x M \to T_x^* M$, $\forall x \in M$, induced by g, together with the non-degeneracy of g, enables us to define also an inner product on $T_x^* M$, by $\cdot : T_x^* M \times T_x^* M \to \mathbf{R}$ such that $(u^*, v^*) \mapsto u^* \cdot v^* = g_x^{-1}(u^*, v^*) \triangleq g_x(u, v)$.

With respect to a basis for $T_x^* M$, $u^* \cdot v^* = g_x^{-1}(u^*, v^*) = g^{\mu\nu}(x)\, u_\mu^* v_\nu^*$, with $[g^{\mu\nu}(x)] \triangleq [g_{\mu\nu}(x)]^{-1}$, $\forall x \in M$. The inner product of any pair of covariant k-vectors, with respect to a basis for $\wedge^k T_x^* M$, is defined by

$$(\alpha, \beta) \mapsto \alpha \cdot \beta = \frac{1}{k!} \alpha_{\mu_1 \ldots \mu_k} \beta_{\nu_1 \ldots \nu_k} g^{\mu_1 \nu_1}(x) \ldots g^{\mu_k \nu_k}(x) . \quad (3)$$

The manifold M together with the set of linear spaces $\wedge^k T_x^* M$, $\forall x \in M$, can be given the structure of a linear bundle, denoted $\wedge^k T^* M$ and called the k-th exterior power of the cotangent bundle of M. Any section of the linear bundle $\wedge^k T^* M$ is called a totally antisymmetric covariant tensor field of order (or grade) k on M, or in short a covariant k-vector field. In the mathematical literature, a covariant k-vector field is usually called a k-form. We will denote the set of k-forms by $\Gamma\left(\wedge^k T^* M\right)$. The manifold M together with the set of bases for $\wedge^k T_x^* M, \forall x \in M$, can also be given the structure of a linear bundle, called the frame bundle for $\wedge^k T^* M$. Any section of this frame bundle is called a covariant frame field of order (or grade) k on M, or in short a (moving) covariant k-frame. Any covariant k-vector field has a representative with respect to any covariant k-frame.

2.3 Differential forms on M

Let $\mathcal{F}_M \triangleq (C^\infty(M, R), +,)$ denote the unital ring of smooth real functions defined on M, $C^\infty(M, R)$, together with function pointwise addition $+$ and function pointwise multiplication (denoted by juxtaposition). The set of k-forms $\Gamma\left(\wedge^k T^* M\right)$, $\forall k \in \mathbf{Z}_{[0,n]}$, together with $\mathcal{F}_M$ and an external operation (also denoted by juxtaposition) from $\mathcal{F}_M \times \Gamma\left(\wedge^k T^* M\right) \to \Gamma\left(\wedge^k T^* M\right)$, becomes a left module. The elements of this structure are called left k-forms over $\mathcal{F}_M$.

This module of k-forms over $\mathcal{F}_M$, together with an exterior (wedge) product $\wedge : \Gamma\left(\wedge^k T^* M\right) \times \Gamma\left(\wedge^l T^* M\right) \to \Gamma\left(\wedge^{k+l} T^* M\right)$, becomes a left exterior (or graded) module over $\mathcal{F}_M$.

This exterior module over $\mathcal{F}_M$ is further supplemented with three parity preserving operations: (i) a left interior (contraction) product i_a by a contravariant vector field $a \in \Gamma(TM)$, (ii) a left exterior derivative operator d and (iii) Hodge's left star operator $*$. The elements of this structure are called (exterior) differential k-forms over $\mathcal{F}_M$.

Let $\mathcal{D}'_M \triangleq \left(\mathcal{D}'_+(M), +, \star\right)$ denote the integral domain consisting of distributions based on M with support in a closed forward light cone, together with distributional addition $+$ and distributional convolution $\star$. The set of smooth functions with support in a closed forward light cone is embedded in $\mathcal{D}'_+(M)$ by the regular distributions that they generate. In a similar way as above we define (exterior) differential k-forms over $\mathcal{D}'_M$ (the exterior derivative operator will now involve the generalized partial derivative).

2.3.1 The exterior product of forms

The exterior product is a bilinear map $\wedge : \Gamma\left(\wedge^k T^* M\right) \times \Gamma\left(\wedge^l T^* M\right) \to \Gamma\left(\wedge^{k+l} T^* M\right)$, which is just the antisymmetric tensor product $\alpha \wedge \beta \triangleq \alpha \otimes \beta - \beta \otimes \alpha$. With respect to natural covariant frame fields, the wedge product of any k-form α and any l-form β is given by

$$\alpha \wedge \beta = \frac{1}{(k+l)!} \frac{1}{k!} \frac{1}{l!} \delta^{\kappa_1 \ldots \kappa_k \lambda_1 \ldots \lambda_l}_{\mu_1 \ldots \mu_{k+l}} \alpha_{\kappa_1 \ldots \kappa_k} \beta_{\lambda_1 \ldots \lambda_l} \left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k+l}}\right), \quad (4)$$

wherein δ stands for the generalized Kronecker tensor, [4, p. 142]. If $k = 0$ of $l = 0$, the exterior product is defined to equal the external product of the module of k-forms. If $k + l > n$, the wedge product is defined to be zero (since there are no totally antisymmetric tensor fields of order greater than the dimension of the manifold).

Resulting from the usual product compatibility property for algebras, the product between the (non strict) components $\alpha_{\kappa_1 \ldots \kappa_k}$ and $\beta_{\lambda_1 \ldots \lambda_l}$ is the pointwise product between functions of the ring $\mathcal{F}_M$, in case α and β are k-forms over $\mathcal{F}_M$. If α and β are k-forms over $\mathcal{D}'_M$, the product between components is the convolution product $\star$ between distributions.

2.3.2 The interior product

The left interior product of a contravariant vector field and a k-form is a bilinear map $i_a : \Gamma(TM) \times \Gamma\left(\wedge^k T^* M\right) \to \Gamma\left(\wedge^{k-1} T^* M\right)$ such that the action of any contravariant vector field a on any k-form β is given, with respect to natural frame fields, by

$$\beta \mapsto i_a\beta = \frac{1}{(k-1)!} a^{\nu_1} \beta_{\nu_1\nu_2\ldots\nu_k} \left(dx^{\nu_2} \wedge \ldots \wedge dx^{\nu_k}\right). \tag{5}$$

In our case, we have assumed the existence of an inner product structure g on M. This structure allows us to naturally define i_a as a left inner product between the 1-form α, obtained from a under the canonical isomorphism from $T_xM \to T_x^*M$, and any k-form β. We have $\alpha = \alpha_\mu dx^\mu$, with $\alpha_\mu = g_{\mu\nu}a^\nu$. We can thus define this left inner product $\cdot : \Gamma\left(T^*M\right) \times \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k-1} T^*M\right)$ such that $\beta \mapsto \alpha \cdot \beta \triangleq i_a\beta$, with

$$\alpha \cdot \beta = \frac{1}{(k-1)!} g^{\mu\nu_1} \alpha_\mu \beta_{\nu_1\nu_2\ldots\nu_k} \left(dx^{\nu_2} \wedge \ldots \wedge dx^{\nu_k}\right). \tag{6}$$

If $k = 0$, the interior product (6) is defined to be zero (since there are no forms of order -1).

The remark related to the product between components as stated in the previous subsection, equally applies here.

2.3.3 The exterior derivative

The left exterior derivative operator is a linear map $d : \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k+1} T^*M\right)$ such that its action on any k-form is given, with respect to natural covariant frame fields, by

$$\begin{aligned} \alpha \quad \mapsto \quad d\alpha = {} & \frac{1}{(k+1)!}\frac{1}{k!}\delta^{\nu\nu_1\ldots\nu_k}_{\mu_1\mu_2\cdots\mu_{k+1}} \\ & \frac{\partial \alpha_{\nu_1\ldots\nu_k}}{\partial x^\nu}\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k+1}}\right). \end{aligned} \tag{7}$$

We will rewrite (7) in terms of partial covariant derivatives defined by the unique Riemannian connection generated by g. Due to the antisymmetry of $d\alpha$ (for $k > 0$) and since the Riemannian connection is torsion free, the extra connection terms (involving the Christoffel symbols) in (8) cancel out, so we get

$$\begin{aligned} d\alpha \quad = \quad & \frac{1}{(k+1)!}\frac{1}{k!}\delta^{\nu\nu_1\ldots\nu_k}_{\mu_1\mu_2\cdots\mu_{k+1}} \\ & \left(\nabla_\nu \alpha_{\nu_1\ldots\nu_k}\right)\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_{k+1}}\right), \end{aligned} \tag{8}$$

wherein ∇_ν is the directional covariant derivative in the direction of the basis vector field ∂_ν. Although $d\alpha$ does not depend on the inner product structure g and despite the fact that (7) is simpler than (8), we will find it convenient to add the extra connection terms in order to cast our results in Clifford algebra form later.

Define a 1-form ∂ by

$$\partial \triangleq \nabla_\nu dx^\nu. \tag{9}$$

The operator ∂, defined in (9), generalizes the ∂ operator encountered in Clifford analysis over Euclidean spaces (and which is there called Dirac operator, [3]) to the setting of contravariant and covariant k-vector fields on pseudo-Riemannian manifolds. We can now write (8) in terms of the exterior product (4) as

$$d\alpha = \partial \wedge \alpha, \tag{10}$$

Thus, the exterior derivative operator d is at the same time an analytical directional covariant derivative on the components of α and an algebraic wedge operator on the natural covariant frame field of α.

When the exterior derivative operator d is applied to differential k-forms over $\mathcal{D}'_M$, the generalized partial derivative D_ν replaces the ordinary partial derivative ∂_ν in ∇_ν.

The operation $d = \partial\wedge$ generalizes the curl operation, defined in classical vector calculus, to totally antisymmetric covariant tensor fields on pseudo-Riemannian manifolds. When acting on any 0-form f, we have $df = \partial \wedge f = \left(\nabla_\nu dx^\nu\right) \wedge f = \left(\nabla_\nu f\right)\left(dx^\nu \wedge 1\right) = \left(\nabla_\nu f\right) dx^\nu = \left(\partial_\nu f\right) dx^\nu$, so df coincides with the ordinary differential of the scalar function f.

2.3.4 Hodge's star operator

Hodge's left covariant star operator is a linear map $* : \Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{n-k} T^*M\right)$ such that, with respect to natural covariant frame fields,

$$\begin{aligned} \alpha \quad \mapsto \quad *\alpha = {} & \frac{1}{(n-k)!}\frac{1}{k!}\epsilon_{\mu_1\ldots\mu_k\mu_{k+1}\cdots\mu_n} \\ & g^{\mu_1\nu_1}\ldots g^{\mu_k\nu_k} \\ & \alpha_{\nu_1\ldots\nu_k}\left(dx^{\mu_{k+1}} \wedge \ldots \wedge dx^{\mu_n}\right), \end{aligned} \tag{11}$$

In (11), ϵ is the covariant Levi-Civita tensor field, which is equal to the oriented volume form ω, [4, p. 294],

$$\epsilon = \omega \triangleq \sqrt{\left|\det\left[g_x\right]\right|}\left(dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n\right). \tag{12}$$

The inverse star operator, acting on any k-form, is given by

$$*^{-1} = (-1)^{k(n-k)+q} *, \tag{13}$$

with q the number of space dimensions (see (2), the signature (p, q) of M).

2.3.5 The interior derivative

The main use of Hodge's left star operator and its inverse is to define the left interior derivative operator (also called codifferential) $\delta : \Gamma\left(\wedge^k T^*M\right) \to$

$\Gamma\left(\wedge^{k-1}T^*M\right)$, from the exterior differential d, such that

$$\alpha \mapsto \delta\alpha = -\left(-1\right)^k *^{-1} d * \alpha. \qquad (14)$$

Our definition (14) differs by an extra minus sign from the standard definition in the theory of exterior differential forms, in order to let our results agree with standard conventions in Clifford analysis. With respect to natural covariant frame fields, we get

$$\begin{aligned}\alpha \quad \mapsto \quad \delta\alpha = & \frac{1}{(k-1)!}\left(g^{\tau\sigma}\nabla_\tau \alpha_{\sigma\nu_2\ldots\nu_k}\right) \\ & \left(dx^{\nu_2}\wedge\ldots\wedge dx^{\nu_k}\right). \qquad (15)\end{aligned}$$

Similarly as for the exterior derivative, we can write the action of δ on any k-form α in terms of the operator defined in (9) and the inner product (6),

$$\delta\alpha = \partial\cdot\alpha, \qquad (16)$$

The interior derivative operator δ is at the same time an analytical directional covariant derivative on the components of α and an algebraic contraction operator on the natural covariant frame field of α.

The operation $\delta = \partial\cdot$ generalizes the divergence operation, defined in classical vector calculus, to totally antisymmetric covariant tensor fields on pseudo-Riemannian manifolds.

When acting on any 0-form f, $\delta f = \partial\cdot f = \left(\nabla_\nu dx^\nu\right)\cdot f = \left(\nabla_\nu f\right)\left(dx^\nu\cdot 1\right) = 0$. Further, we will need $\delta df = \left(\nabla_\mu dx^\mu\right)\cdot\left(\left(\partial_\nu f\right)dx^\nu\right) = g^{\mu\nu}\nabla_\mu\left(\partial_\nu f\right)$ or, see e.g., [4, p. 319],

$$\delta df = \frac{1}{\sqrt{\left|\det\left[g_x\right]\right|}}\partial_\mu\left(\sqrt{\left|\det\left[g_x\right]\right|}g^{\mu\nu}\partial_\nu f_x\right). \qquad (17)$$

2.3.6 Clifford product of a 1-form and a k-form

Finally, we can define a left Clifford product (denoted by juxtaposition) of any 1-form with any k-form as a bilinear map from $\Gamma\left(T^*M\right)\times\Gamma\left(\wedge^k T^*M\right) \to \Gamma\left(\wedge^{k-1}T^*M\right)\oplus\Gamma\left(\wedge^{k+1}T^*M\right)$ such that $(\alpha,\beta)\mapsto\alpha\beta$ with

$$\alpha\beta \triangleq \alpha\cdot\beta + \alpha\wedge\beta. \qquad (18)$$

Herein is $\alpha\cdot\beta$ the in (6) defined left inner product defined between any 1-form and any k-form and $\wedge$ the exterior product defined by (4).

The Clifford product can be defined more generally between forms of arbitrary grade, but we do not need this generalization here.

2.4 Covariant multivector fields on M

The direct sum of the modules of differential k-forms over $\mathcal{F}_M$, $\oplus_{k=0}^n\Gamma\left(\wedge^k T^*M\right)$, will be called the set of multiforms over $\mathcal{F}_M$. The exterior product for multiforms is naturally defined by linearity and the resulting structure becomes a left module. This module of multiforms inherits, from its grade components by linearity, (i) an interior (contraction) product i_a by a contravariant vector field $a\in\Gamma(TM)$, (ii) an exterior derivative operator d and (iii) a covariant Hodge's star operator $*$. The elements of this final structure are then called (exterior) differential multiforms over $\mathcal{F}_M$.

Since k-forms are covariant k-vector fields, multiforms are just covariant multivector fields. The concept of a multiform is the covariant analogue of the more common concept of a contravariant multivector field in Clifford analysis over a manifold, and there called a (contravariant) Clifford-valued function on M. We can thus similarly call a multiform a covariant Clifford-valued function on M.

When k-forms are considered over $\mathcal{F}_M$, this leads to the set of multiforms over $\mathcal{F}_M$, denoted $Cl^*\left(\mathcal{F}_M\right)$. When k-forms are considered over $\mathcal{D}'_M$, this leads to the set of multiforms over $\mathcal{D}'_M$, denoted $Cl^*\left(\mathcal{D}'_M\right)$.

The Clifford product, defined in (18), for a given 1-form α, is readily extended to multiforms β by linearity.

Contravariant multivector fields over $\mathcal{F}_M$ and $\mathcal{D}'_M$ on M can also be defined, but will not be needed here.

3 Electromagnetism in vacuum

3.1 Equation

Electromagnetism in space-time can be correctly and elegantly described, i.e., with respect for its geometrical content and physical invariances, in terms of the algebra of differential forms and is an idea that goes back to E. Cartan. We get the following two equations, see e.g., [5],

$$\begin{aligned} dF &= -K, \qquad (19)\\ \delta F &= -J. \qquad (20)\end{aligned}$$

Herein stands $J\in\Gamma\left(T^*M\right)$ for the electric monopole charge-current density source field, $K\in\Gamma\left(\wedge^3T^*M\right)$ for the magnetic monopole charge-current density source field and $F\in\Gamma\left(\wedge^2T^*M\right)$ for the resulting electromagnetic field. We will make the reasonable physical assumption that both J and K are of compact support in M.

In expectation that any magnetic monopoles are discovered in our universe, we can always put $K=0$. However, for the mathematical structure that we wish to expose here it is instructive to keep K in our model.

Eqs. (19)–(20) hold in the presence of any gravitational field, being represented by the inner product structure g on M. The first equation (19) is independent of g, but the second equation (20) depends on g since the interior derivative δ depends on it, through Hodge's star operator.

Being both tensor equations, (19)–(20) are form invariant under any change of bases, so they are in particular invariant under any change of coordinates. Hence, (19)–(20) hold for any coordinate system.

Eqs. (16), (10) and (9) allow us to add eqs. (19)–(20) and combine them into the single equation,

$$\partial F = -\left(J + K\right). \tag{21}$$

In the process of adding we have extended our set of mathematical quantities, k-forms, to the set of multiforms. For instance, $J + K$ is a multiform consisting of the 1-form J and the 3-form K. Clearly, the left-hand side of (21) also contains a multiform consisting of the 1-form $\partial \cdot F$ and the 3-form $\partial \wedge F$.

Eq. (21) is a very compact formulation for electromagnetism on a pseudo-Riemannian (vacuum) manifold. In addition to being compact, eq. (21) is also a fertile starting point to derive an analytical expression for the solution of electromagnetic radiation problems in vacuum, in the presence of any gravitational field, in terms of any coordinates, and for any compact sources J and K, as explained in the next section.

For more information about other uses of Clifford algebra in electromagnetism, see e.g., [1], [8], [9].

3.2 Particular solution

By definition, [4, p. 303], the directional covariant derivative ∇_u, along a contravariant vector field u, commutes with contracted multiplication (in particular, with the inner product (6)) and is a derivation with respect to the tensor product $\otimes$ (and hence by linearity also with respect to the wedge product $\wedge$). Consequently, ∇_u is also a derivation with respect to the Clifford product (18) for multiforms. Therefore, for any 1-form α over $\mathcal{D}'_M$ and any multiform β over $\mathcal{F}_M$, holds

$$\nabla_u \left(\alpha\beta\right) = \left(\nabla_u \alpha\right) \beta + \alpha \left(\nabla_u \beta\right). \tag{22}$$

The product between the components of α and β in the Clifford products $\alpha\beta$, $\left(\nabla_u \alpha\right) \beta$ and $\alpha \left(\nabla_u \beta\right)$ in (22) is the multiplication product between distributions and smooth functions and is always defined.

Letting $u = \partial_\mu$, $\alpha = C_x$ and $\beta = dx^\mu F$ and contracting, we get

$$\nabla_\mu \left(C_x dx^\mu F\right) = \left(\left(\nabla_\mu C_x\right) dx^\mu\right) F + C_x \left(\left(\nabla_\mu dx^\mu\right) F\right),$$

or

$$\nabla_\mu \left(C_x dx^\mu F\right) = \left(C_x \underset{\leftarrow}{\partial}\right) F + C_x \left(\underset{\rightarrow}{\partial} F\right), \tag{23}$$

wherein the under arrows indicate the direction of operation of the 1-form operator ∂. Substituting (21) in (23) gives

$$\left(C_x \underset{\leftarrow}{\partial}\right) F = C_x \left(J + K\right) + \nabla_\mu \left(C_x dx^\mu F\right). \tag{24}$$

We now choose C_x such that

$$C_x \underset{\leftarrow}{\partial} = \underset{\rightarrow}{\partial} C_x = \delta_x, \tag{25}$$

with δ_x the delta distribution concentrated at a parameter point $x \in M$. Let further $\Omega \subset M$ denote an open bounded region, with boundary $\partial\Omega$ and closure $\overline{\Omega} = \Omega \cup \partial\Omega$, such that $\Omega \supset \operatorname{supp}\left(J + K\right)$ and $\operatorname{supp}\left(\delta_x\right) \subset \Omega$ for δ_x in (25). Let $\varphi \in C_c^\infty \left(M, R\right)$, the set of smooth function of compact support, denote a test function being 1 over $\overline{\Omega}$. Let further $\langle , \rangle : \mathcal{D}'_+ \left(M\right) \times C_c^\infty \left(M, R\right) \to \mathbf{R}$ be the scalar product over M between our set of distributions $\mathcal{D}'_+ \left(M\right)$ and the set of test functions $C_c^\infty \left(M, R\right)$.

Substituting (25) in (24) and calculating the scalar product of eq. (24) with φ gives

$$\langle \delta_x F, \varphi \rangle = \langle C_x \left(J + K\right), \varphi \rangle + \langle \nabla_\mu \left(C_x dx^\mu F\right), \varphi \rangle,$$

or, due to the choice of support of φ,

$$\langle \delta_x, F\varphi \rangle = \langle C_x, J + K \rangle + \langle \nabla_\mu \left(C_x dx^\mu F\right), \varphi \rangle,$$

or, by definition of the delta distribution,

$$F\left(x\right) = \langle C_x, \left(J + K\right) \rangle + \langle \nabla_\mu \left(C_x dx^\mu F\right), \varphi \rangle. \tag{26}$$

The second term in the right-hand side of (26) contains a closed multiform over $\overline{\Omega}$. By Stokes' theorem, this term can be converted to a scalar product over $\partial\Omega$ between a multiform concentrated on $\partial\Omega$ with the 1 function on $\partial\Omega$. Then, eq. (26) gives the general solution of the boundary value problem, consisting of eq. (21) together with prescribed boundary values for the electromagnetic field F on $\partial\Omega$. The first term in the right-hand side of (26) gives the particular solution, F^{src}, caused by the sources J and K,

$$F^{src}\left(x\right) = \langle C_x, \left(J + K\right) \rangle. \tag{27}$$

The construction of C_x is simplified by noting that $\partial \wedge C_x = 0$ (i.e., C_x is closed) implies that (locally) $C_x = \partial \wedge f_x$ (i.e., C_x is exact) for some 0-form f_x over $\mathcal{D}'_M$ (Poincaré's lemma). Since $\partial \cdot f_x = 0$ (by definition of i_a), we can write this also as $C_x = \partial f_x$.

Substituting this representation for C_x in its defining equation (25) gives the equation to be satisfied by f_x,

$$\partial^2 f_x = \delta_x. \tag{28}$$

The grade preserving operator $\partial^2 = (d+\delta)^2 = d\delta + \delta d$ is just the Laplace-de Rham operator. From (17) follows that, with respect to natural frame fields, (28) becomes

$$\frac{1}{\sqrt{|\det [g_x]|}} D_\mu \left(\sqrt{|\det [g_x]|} g^{\mu\nu} D_\nu f_x \right) = \delta_0, \tag{29}$$

which is the generalized (i.e., the distributional) scalar wave equation in curved space-time.

Collecting results, we see that any fundamental solution f_x of the generalized scalar wave equation in curved space-time generates a distribution C_x, which in turn generates the general solution by (26). It can be shown that the general solution F is independent of the particular choice of fundamental solution f_x of (28).

In flat Minkowski space, (28) reduces to the ordinary wave equation $\Box f_x = \delta_x$, involving the generalized d'Alembertian $\Box \triangleq \eta^{\mu\nu} D_\mu D_\nu$. In this case, f_x is obtained by a simple shift from f_0, satisfying $\Box f_0 = \delta_0$, with f_0 being available in analytical closed form.

It is remarkable that our mathematical model for electromagnetism has in general no particular solution (this also holds for Heaviside's model). Indeed, operating on the left with ∂ shows that any solution of $\partial F = J$ is necessarily also a solution of

$$\partial^2 F = -\partial (J + K). \tag{30}$$

The operator ∂^2 is grade preserving, hence the grade of the left-hand side of eq. (30) equals the grade of F, which is 2. The right-hand side of eq. (30) has a grade 0 part, $\partial \cdot J$, a grade 2 part $\partial \wedge J + \partial \cdot K$ and a grade 4 part $\partial \wedge K$. For eq. (30) to have a solution it is thus necessary that both the grade 0 part and the grade 4 part vanishes. This requires that J and K must satisfy $\partial \cdot J = 0$ and $\partial \wedge K = 0$, or with respect to natural bases,

$$\frac{1}{\sqrt{|\det [g_x]|}} \frac{\partial}{\partial x^\tau} \left(\sqrt{|\det [g_x]|} g^{\tau\sigma} J_\sigma \right) = 0, \tag{31}$$

$$\frac{1}{3!} \epsilon^{\tau\nu_1\nu_2\nu_3} \frac{\partial K_{\nu_1\nu_2\nu_3}}{\partial x^\tau} = 0. \tag{32}$$

Eqs. (31)–(32) are the integrability conditions of our model for electromagnetism on a pseudo-Riemannian (vacuum) manifold in the presence of gravity. Eqs. (31)–(32) amount physically to the local conservation of electric monopole charge and of magnetic monopole charge, respectively. It can be shown that these conditions are necessary and sufficient for the existence and uniqueness of a particular solution of our model for electromagnetism.

In (24), the Clifford product of C_x with $J + K$ produces the multiform

$$C_x \cdot (J + K) + C_x \wedge (J + K). \tag{33}$$

It can also be shown that conditions (31)–(32) in turn guarantee that in (27) only the grade 2 part remains.

A more detailed derivation of the general solution of eq. (21) and a discussion of the boundary conditions (e.g., a generalization of the Sommerfeld radiation conditions) requires as its starting point a generalization of Stokes theorem. This will be considered elsewhere.

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