A New Hybrid Method for Finding an Eigenpairs of a Symmetric Quadratic Eigenvalue Problem in an Interval

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Abstract: The symmetric quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C + K)u = 0,$$

where M, C, and K are given $n \times n$ matrices and (λ, u) is an eigenpair, arises in a wide variety of practical applications, including **vibration, acoustic, and noise control analysis**[5]. In the most practical application, the problem is often of a very large dimension. Unfortunately because of the nonlinearity, the problem is extremely hard to solve numerically, and the state-of-art computational techniques, such as the Jacobi-Davidson method, are capable of computing only a few extremal eigenvalues and eigenvectors [2,3]. Fortunately, there are engineering applications that require only some of the eigenvalues lying within an interval. In this paper, a new hybrid method combining a Parametrized Newton-type method described in [1] with the Jacobi-Davidson method [2] is proposed to compute an eigenpair of quadratic pencil within an interval. The experimental results show that this method is much faster than the Jacobi-Davidson method. The results of this paper generalize those of an earlier work on parameterized Newton's Algorithm for finding an eigenpair of a symmetric matrix [1].

Key–Words: Quadratic eigenvalue problem, Parametrized Newton's method, Jacobi-Davidson method, Hybrid method, Symmetric positive definitive matrix.

1 Introduction

A standard way to solve the quadratic eigenvalue problem

$$(\lambda^2 M + \lambda C + K)u = 0, \tag{1}$$

where M, C, and K are given matrices and (λ, u) is an eigenpair, is to transform the quadratic eigenvalue problem to a linear generalized eigenvalue problem of the form $Ay = \lambda By$ and then apply the well-known QZ algorithm [4]. Unfortunately, the QZ algorithm is not effective for large and sparse problems. Moreover it can not take advantage of the exploitable structures of the matrices M, C, and K, such as the symmetry, positive definiteness, sparsity, etc., which are generally offered by practical problems.

The state-of-the-art technique, such as the Jacobi-Davidson method [3], is capable of only computing a few extremal eigenvalues. In [1], a Newton type of method was proposed for approximating an eigenpair of a large symmetric matrix. In the present paper, we first show how to extend Newton type of method in [1] to iteratively approximate an eigenpair of the **symmetric quadratic pencil**

$$Q(\lambda) = \lambda^2 M + \lambda C + K \tag{2}$$

and then propose a new Hybrid method, combining our quadratic eigenvalue method, with the Jacobi-Davidson method, to approximate an eigenpair in a given interval of the real-line. We will assume through out the paper the symmetric pencil $Q(\lambda)$ has all its eigenvalues real. Our results of numerical experiments show that the Hybrid method converges faster than the Jacobi-Davidson method. In fact, for some examples, the Jacobi-Davidson did not converge at all.

2 Parametrized Newton's Method for the Symmetric QEP:

In this section, we state our Parametrized Newton's Method [1] for the symmetric quadratic eigenvalue Problem.

We define the function $f : R^{n+1} \to R^{n+1}$

$$f(\lambda, u) = \begin{pmatrix} Q(\lambda)u \\ u^T u - 1 \end{pmatrix},$$

where $Q(\lambda) = \lambda^2 M + \lambda C + K$,
M, C, and $K \in R^{n\times n}$ are symmetric positive definite matrices and $u \in R^n$ and $\lambda \in R$.

It follows that $f(\lambda, u) = 0$ if and only if $Q(\lambda)u = 0$ and $u^T u = 1$, where (λ, u) is an eigenpair of the QEP($Q(\lambda)$).

Given an initial approximation $\begin{pmatrix} x_i \\ \alpha_i \end{pmatrix}$ of an eigenpair $\begin{pmatrix} u \\ \lambda \end{pmatrix}$ lying in the given interval $[a, b]$. In order to use Newton's Method, we need the Jacobian matrix J_f of f which can be calculated as

$$J_f(\lambda, u) = \begin{pmatrix} Q(\lambda) & Q'(\lambda)u \\ 2u^T & 0 \end{pmatrix},$$

where $Q'(\lambda)$= $2\lambda M + C$ is the derivative of the matrix polynomial $Q(\lambda)$.
The Parametrized Newton's method for $Q(\lambda)$ to refine the approximation of $\begin{pmatrix} u \\ \lambda \end{pmatrix}$ iteratively can be stated as follows:

$$\begin{pmatrix} x_{i+1} \\ \alpha_{i+1} \end{pmatrix}$$

$$= \begin{pmatrix} x_i \\ \alpha_i \end{pmatrix} - \begin{pmatrix} Q(\alpha_i) & Q'(\alpha_i) \\ 2x_i^T & 0 \end{pmatrix}^{-1} \begin{pmatrix} Q(\alpha_i)x_i \\ x_i^T x_i - 1 \end{pmatrix}.$$

Assuming $\alpha_i \neq 0$ then we can write $\begin{pmatrix} Q(\alpha_i) & Q'(\alpha_i)x_i \\ 2x_i^T & 0 \end{pmatrix} \begin{pmatrix} x_{i+1} \\ \alpha_{i+1} \end{pmatrix}$

$$. \qquad = \begin{pmatrix} 0 & Q'(\alpha_i)x_i \\ x_i^T & \frac{1}{\alpha_i} \end{pmatrix} \begin{pmatrix} x_i \\ \alpha_i \end{pmatrix}.$$

Now choose a parameter $t > 0$ so that the method takes the form:

$$\begin{pmatrix} Q(\alpha_i) & Q'(\alpha_i)x_i \\ 2x_i^T & 0 \end{pmatrix} \begin{pmatrix} x_{i+1} \\ \alpha_{i+1} \end{pmatrix}$$

$$. \qquad = \begin{pmatrix} I & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & Q'(\alpha_i)x_i \\ x_i^T & \frac{1}{\alpha_i} \end{pmatrix} \begin{pmatrix} x_i \\ \alpha_i \end{pmatrix}.$$

The above can be written as:

$$Q(\alpha_i)x_{i+1} + \alpha_{i+1}Q'(\alpha_i)x_i = \alpha_i Q'(\alpha_i)x_i$$

or

$$x_{i+1} = (\alpha_i - \alpha_{i+1})Q^{-1}(\alpha_i)Q'(\alpha_i)x_i \qquad (3)$$

and

$$2x_i^T x_{i+1} = t(x_i^T x_i + 1). \qquad (4)$$

From (3) and (4), we obtain

$$x_i^T x_{i+1} = \tfrac{t}{2}(x_i^T x_i + 1)$$

$$= (\alpha_i - \alpha_{i+1})x_i^T Q^{-1}(\alpha_i)Q'(\alpha_i)x_i.$$

Set $\beta_i = x_i^T Q^{-1}(\alpha_i)Q'(\alpha_i)x_i$. Then we have

$$\frac{t}{2}(x_i^T x_i + 1) = \beta_i(\alpha_i - \alpha_{i+1})$$

or

$$(\alpha_i - \alpha_{i+1}) = \frac{t}{2}(x_i^T x_i + 1)\frac{1}{\beta_i}.$$

Since x_i is normalized vector,

$$\alpha_{i+1} = \alpha_i - \frac{t}{\beta_i}, \qquad (5)$$

from (3) and (5) we obtain,

$$x_{i+1} = \frac{t}{\beta_i}Q^{-1}(\alpha_i)Q'(\alpha_i)x_i. \qquad (6)$$

Set $\hat{\beta}_i = ||(Q^{-1}(\alpha_i)Q'(\alpha_i)x_i)||$

$$= (x_i^T Q'(\alpha_i)Q^{-2}(\alpha_i)Q'(\alpha_i)x_i)^{\frac{1}{2}}$$

and $y_i = Q'(\alpha_i)x_i$ then, $\hat{\beta}_i = (y_i^T Q^{-2}(\alpha_i)y_i)^{\frac{1}{2}}$.

Now compute

$$\frac{x_{i+1}}{||x_{i+1}||} = \frac{\frac{t}{\beta_i}Q^{-1}(\alpha_i)Q'(\alpha_i)x_i}{||\frac{t}{\beta_i}Q^{-1}(\alpha_i)Q'(\alpha_i)x_i||}$$

$$= \frac{|\beta_i|}{\beta_i}\frac{1}{\hat{\beta}_i}Q^{-1}(\alpha_i)y.$$

Since $\frac{|\beta_i|}{\beta_i} = \pm 1$, we ignore the sign and $||x_{i+1}|| = 1$, Then we can write

$$x_{i+1} = \frac{1}{\hat{\beta}_i}Q^{-1}(\alpha_i)y_i$$

where $y_i = Q'(\alpha_i)x_i$ and $\alpha_{i+1} = \alpha_i - \frac{t}{\beta_i}$.
Set $r_i = \frac{\beta_i}{\hat{\beta}_i}$. Let $t = r_i^2 s$, where s is an arbitary positive number.

Parametrized Newton's iteration for $Q(\lambda)$ now takes the form:

$$x_{i+1} = \frac{1}{\hat{\beta}_i}Q^{-1}(\alpha_i)y_i \qquad (7)$$

and

$$\alpha_{i+1} = \alpha_i - \frac{r_i}{\hat{\beta}_i}s, \quad (8)$$

where $y_i = Q'(\alpha_i)x_i$, $\hat{\beta}_i = ||(Q^{-1}(\alpha_i)y_i)||$ and $r_i = \frac{\beta_i}{\hat{\beta}_i}$.

The parameter s needs to be choosen so that the iteration defined by (7) and (8) converges to a desired eigenpair.

For this purpose, we define the the residual at the $(i+1)^{th}$ step of Parametrized Newton's Method by

$$Res_{i+1} = Q(\alpha_{i+1})x_{i+1}. \quad (9)$$

Using (7), (8) and (9), we can then show that:

$$||Res_{i+1}||^2 = \frac{1}{\hat{\beta}_i^{\,2}}[y_i^T y_i - 2\frac{r_i s}{\hat{\beta}_i}y_i^T p_i + \frac{r_i^2 s^2}{\hat{\beta}_i^{\,2}}(2y_i^T z_i + p_i^T p_i) - 2\frac{r_i^3 s^3}{\hat{\beta}_i^{\,3}}p_i^T z_i + \frac{r_i^4 s^4}{\hat{\beta}_i^{\,4}}z_i^T z_i]. \quad (10)$$

That is,

$$||Res_{i+1}||^2 \le \frac{1}{\hat{\beta}_i^{\,2}}[||y_i||^2 - 2\frac{r_i s}{\hat{\beta}_i}y_i^T p_i + 2\frac{r_i^2 s^2}{\hat{\beta}_i^{\,2}}||y_i||||z_i|| . + \frac{r_i^2 s^2}{\hat{\beta}_i^{\,2}}||p_i||^2 - 2\frac{r_i^3 s^3}{\hat{\beta}_i^{\,3}}p_i^T z_i + \frac{r_i^4 s^4}{\hat{\beta}_i^{\,4}}||z_i||^2]$$

where $\hat{\beta}_i^{\,2} = ||z_i||^2, \quad y_i = Q'(\alpha_i)x_i,$

$$z_i = Q^{-1}(\alpha_i)Q'(\alpha_i)x_i, \quad p_i = Q'(\alpha_i)z_i,$$

and

$$r_i = \frac{(Q^{-1}(\alpha_i)x_i)^T(Q'(\alpha_i)x_i)}{||Q^{-1}(\alpha_i)Q'(\alpha_i)x_i||}. \quad (11)$$

The right hand side of (10) is a polynomial in s of degree 4. Our objective then will be to choose value of s so that the residual is minimum at every iteration.

In the next section, we describe the Hybrid method for finding an eigenpair if a symmetric quadratic pencil $Q(\lambda)$ in a given interval $[a,\ b]$.

3 The New Hybrid Method:

The Hybrid method has two parts. In the first part, we choose 3 sets of random eigenpairs $\begin{pmatrix} \alpha_i \\ v_i \end{pmatrix}$ inside the interval $[a,\ b]$ where $\alpha_i \in [a,\ b]$, and the vectors v_i, $i = 1,\ 2,\ 3$ are orthogonal to each other. We use 3-iterations of Parametrized Newton's method as described in section 2 for each pair and then use these eigenvectors to run the Jacobi-Davidson method. Preliminary results show that this Hybrid method always converges to several eigenpairs in an interval and is faster than the Jacobi-Davidson method.

Algorithm 1 Hybrid Method:

INPUT:

- The matrix M = $I \in R^{n\times n}$, C, and $K \in R^{n\times n}$ symmetric positive matrices .
- Three real numbers α_i, i=$1, 2, 3$ as initial approximations of an eigenvalues inside the interval $[a\ b]$.
- An orthonormal matrix V = (v_1, v_2, v_3).

OUTPUT:

An approximate eigenpair of the QEP in the interval $[a\ b]$.

Step 1: Find three approximate eigenparis $(\alpha_i,\ v_i)$ using the Parametrized Newton's method described in Section 2 (run only maximum of three iterations).

Setp 2: Apply the Jacobi-Davidson method with the eigenvectors obtained in the Step 1 and a shift α choosing it as one of the eigenvalues α_i, $i = 1, 2, 3$ appropriately (depending upon the location of the eigenvalue sought).

Step 3: Check if norm of the residual defined in equation (9) is less than a given tolerance. If so Stop. Otherwise, expand the search space V and return to Step 2.

4 Results of Numerical Experiments:

In the following, we present our results of numerical experiments on comparison between the Jacobi-Davidson Method and the Hybrid method. In our experiments, M is chosen as an identity matrix, and C, K are choosen as arbitary symmetric positive definite matrices of order 500 and 800. Maximum number of iterations = 20 and tolerance = 10^{-4}. Each graph shows the log of the residual with every iteration.

Example 1

Matrix size(n) = 500, Interval $[41.5\ \ 43.5]$

The approximate initial eigenvalue determined by the Parametrized Newton method = 42.0307903.

Exact eigenvalues in interval $[41.5\ \ 43.5]$ are:

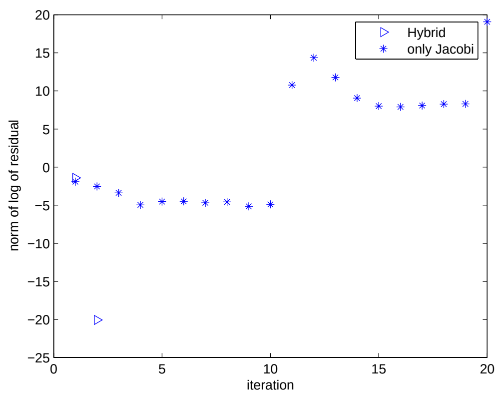
41.59431775969728, 41.62015325607620, 41.93211289355256, 43.00123828531239.

TABLE 1: Convergence comparison between

the Hybrid method and the Jacobi-Davidson method for Example 1

Methods	Residual	Iteration	Eigenvalue
Hybrid	$1.9e^{-9}$	2	41.93211
JD	No convergence	20	

Figure 1: Norm of log of Residual *verses* Iteration



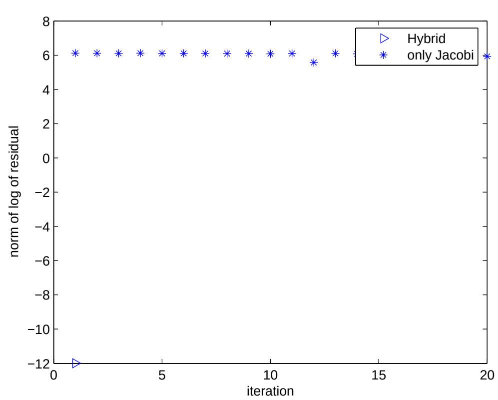
Example 2

Matrix size(n) = 800, Interval $[70 \;\; 73]$
The approximate initial eigenvalue determined by the Parametrized Newton method = 71.063630762458.
Exact eigenvalues in interval $[70 \;\; 73]$ are:
$71.48881531878268, \quad 72.89726841255406$.

TABLE 2: Convergence comparison between the Hybrid method and the Jacobi-Davidson method for Example 2

Methods	Residual	Iteration	Eigenvalue
Hybrid	$6.61e^{-6}$	1	71.4888153978
JD	No convergence	20	

Figure 2: Norm of log of Residual *verses* Iteration



5 Conclusion

The quadratic eigenvalue problem arises in a wide variety of engineering applications, including vibration analysis of structures, model updating in dynamics structures, acoustic studies, etc. In most of these applications, however, what is needed is the knowledge of a few eigenvalues and eigenvectors and in many cases just an estimate of an eigenvalue in a given interval. A Newton type method is quite suitable for this purpose. In this paper, a new Newton type method is first proposed to find an approximation of an eigenpair of a symmetric positive definite quadratic matrix pencil, and then a hybrid method blending this new method with the well-known Jacobi-Davidson method, is developed for finding an estimate of an eigenvalue in a given interval. Numerical experimental results show that the hybrid method converges faster than the Jacobi-Davidson method alone; indeed, in some cases when the Jacobi-Davidson method did not converge at all, the new method worked quite well.

The method is parametric in nature and the convergence and rate of convergence depends upon the appropriate choice of the parameter. Studies on how to choose it properly to guarantee or accelerate the convergence is currently underway and will be reported in a future paper.

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