

# Mathematical programming elements: its application to the optimum power flow problem

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*Abstract:* This paper seeks to investigate the application of areas from the mathematical programming, more precisely, in the linear programming applied to the linearized optimum power flow problem. This investigation involved the study of linear programming and the analysis of the optimum power flow problem in its several formulations. The uses of linear programming were examined, particularly in Electrical Engineering, on the optimum power flow problem. Some problems and examples about energy networks were formulated and presented.

*Key-words:* Mathematical Programming; Linear Programming; Energy flow in electrical energy networks; Optimum Power Flow DC; Congestion.

## 1. Introduction

The Brazilian and world-wide electrical sector have been suffering several transformations [1]. The change from the monopoly model to the competitive model demands new operational and planning philosophies of electrical systems, which involve generation, transmission and distribution. Furthermore, in the biggest part of the system, the rapid demand of energy has forced the systems to operate in the limit of their capacity and, on the other hand, the tentative to expand has faced problems with environmental, social characteristics and also financial crisis that has reduced investments in this sector.

As an alternative to expansion, for example, it is possible to actuate in the systems' operation, resettling generators and/or working on the equipments' adjustments, having as objectives to diminish losses, to minimize the generation costs, to increase the system's transmission capacity. In other words, optimize on one or more of its performance indexes.

The main computerized tool used to determine the electrical systems' optimum operational point is denominated optimum power flow (OPF). The optimum power flow can be used together with the estimated state to, periodically, adjust the optimum exit control in order to maintain a feasible voltage and the sources of reactive power [9].

In 1984, Kamarkar, quoted by [10], published an article in which the optimizing method that was presented, rarely visits the extreme points before the optimum point is found; in other words, the

algorithm finds viable solutions in the interior of the polygon, avoiding, in this manner, the combinative complexity which is derivative from the vertex's solutions. Due to the procedure proposed by Karmarkar, the method is called "interior points method" (IPM) and has dispersed characteristics and has been widely used in the specialized literature.

The interior points method belongs to a class of optimizing algorithms originally designated for linear programming problems. However, due to its high performance level this method was extended to quadratic, convex programming problems and differentiable optimizing problems in general.

When applying the interior point method in optimum power flow problems, two distinct strategies are adopted. The first applies the method to a linear programming problem which is obtained by the linearization of the active and reactive power balance equations of the power flow algorithm. The second consists on applying the interior points method directly on the original non-linear programming problem of the optimum power flow.

## 2. Operational Research

### 2.1. Initial Considerations

According to [6], during the Second World War, a group of scientists were united in England to study strategy problems and the tactics associated to the country's defence. The objective was to decide about the most efficient use of the limited military resources. The call of this group's meeting is

identified as the first operational research formal activity.

The positive results that were obtained by the British operational research team motivated the Americans to initiate similar activities. Even though the origin of the Operational Research is accounted to England, its propagation is due mainly to a group of scientists led by George B. Dantzig from the United States of America, drafted during the Second World War. The result of the effort involved in this research, which was concluded in 1947, was named *Simplex Method*.

A very important characteristic of operational research and which made the process of analysis and of decision easier was the usage of models. They allow the experimentation of the proposed solution. This means that, before a decision is implemented, it can be better evaluated and tested. The obtained economy and the experience that is acquired by this experimentation, justifies its usage.

In the beginning of the 50s, several areas began to appear, which are today collectively known as *mathematical programming*. With the linear programming the mathematical programming sub-areas grew rapidly, having a fundamental performance in this development. Among these sub-areas are the non-linear programming, which started around 1951 with the famous Karush-Kuhn-Tucker condition, commercial utilization, network flows, linear programming, integer programming, dynamic programming and stocking programming.

The *linear programming* is used to analyze models where the restrictions and the objective function are linear; the *integer programming* is applied in models that have integer variables (or discrete); the *dynamic programming* is used in models where the entire problem can be decomposed into smaller sub-problems; the *stocking programming* is applied in a special class of models where the parameters are described by probability functions; finally, the *non-linear programming* is used in models containing non-linear functions.

A characteristic that is present in almost of all the mathematical programming is that the optimum solution of the problem can not be obtained with only one step, having to be obtained iteratively. An initial solution is chosen (which usually is not an optimum solution). One algorithm is specified to determine, starting from it, a new solution that normally is superior to the preceding one. This step is repeated until the optimum solution is achieved (supposing it does exist).

## 2.2. Linear Programming

The general problem of the linear programming, according to [3], is used to optimize (maximize or minimize) a linear function of variables, called "objective function", which is subject to a succession of linear equations or inequalities, called restrictions. The formulation of the problem to be solved by linear programming follows some basic steps, as described below:

1. The basic objective of the problem should be defined, in other words, the optimization to be reached. For example, the profit's maximization, or performance, or social welfare, cost, loss, time minimization. This objective will be represented by an objective function, to be maximized or minimized.
2. For this function to be mathematically specified, the variables of decision involved should be defined. For example, number of machines, the area to be explored, and the classes of investment that are available, etc. Normally, it is expected that all these variables can assume only positive values.
3. These variables normally are subjected to a series of restrictions, usually represented by equations. For example, quantity of equipment that is available, size of the area to be explored, the capacity of a reservoir, nutritional requirements of a determined diet, etc.

All these expressions, however, should be according to the main hypotheses of the linear programming, in other words, all the relations between the variables should be linear. This implies in the proportionality of the quantities involved. This linearity characteristic can be interesting as for simplifying the mathematical structure involved, but prejudicial when representing non-linear phenomenon (for example, cost functions that are typically quadratic).

The canonical form of a linear programming problem is presented as followed:

$$\begin{aligned} \text{Max.} \quad & c_1x_1 + c_2x_2 + \dots + c_nx_n \\ \text{subject to} \quad & \begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \leq b_2 \\ \dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \end{cases} \\ & x_1, x_2, \dots, x_n \geq 0 \end{aligned} \quad (1)$$

where  $c_1x_1 + c_2x_2 + \dots + c_nx_n$  represents a linear objective function to be maximized and can be expressed or represented by  $z$ . The coefficients  $c_1, c_2, \dots, c_n$  represent costs (known values) and  $x_1, x_2, \dots, x_n$  represent the decision variables; their values should be obtained by the solution, if the

solution of the problem exists. The inequality  $\sum_{j=1}^n a_{ij}x_j \leq b_i$  represents the set of linear restrictions with  $i \in \{1, 2, \dots, m\}$  and  $j \in \{1, 2, \dots, n\}$ . The set of all the coefficients  $a_{ij}$  make up the technological coefficients' matrix. And  $x_1, x_2, \dots, x_n \geq 0$  guarantees the non-negativity of the decision variables.

### 2.3. The Simplex Method

According to [6], a procedure is a finite sequence of instructions and algorithm is a procedure that ends in a finite number of operations.

The simplex method through its iterative algorithm searches for the solution, if it exists, by the vertices of a viable region until it finds a solution which does not have better neighbors than itself. This is an optimum solution. The optimum solution may not exist in two cases: when a viable solution does not exist, due to incompatible restrictions; or when the maximum does not exist (or minimum), in other words, one or more variables can incline to the infinitive and the restrictions continue to be satisfied which gives a value without limits to an objective function.

The simplex algorithm stands out as one of the greatest contributions to the mathematical programming of the twentieth century. It is an extremely efficient general algorithm, as mentioned by [6], for the solution of linear systems and adaptable for computational calculus. Its functional comprehension will give a base for several other methods. Refuting this statement, Latoree quoting [1], declares that even though the simplex method is in practice very efficient, it presents exponential complexity, in other words, the number of iterations grows exponentially with the number of the problem's variables.

### 2.4. The Interior Points Methods (IPM)

The interior points methods had their recognition in 1984, when Karmarkar proposed an polynomial algorithm that requires  $(n^{3.5}L)$  arithmetic operations and  $(nL)$  iterations in the worse cases, assuring that the iterative process is of an order of 50 times more rapid than the simplex method [10]. Initially the performance of this method was very criticized by the scientific community, but the results present by (ADLER, 1989) quoted in [8], gave a new impulse to the development of this class of methods. The revolution of the interior points' methods, as many other revolutions, includes earlier ideas which are

rediscovered or seen in a different way, together with genuine new ideas [10].

Given a solution's feasible region of a linear (or non-linear) programming problem, a interior point is that in which all the variables (coordinates) meet inside this region, named region of viable solutions.

The Karmarkar algorithm is significantly different from George Danzig's simplex method, which solves a Linear Programming Problem (LPP) starting from an extreme point along its limit to, finally, reach an optimum extreme point. The method that was projected by Karmarkar rarely visits the extreme points before the optimum point is reached, in other words, the algorithm finds viable solutions in the interior of the solution.

The IPM tries to find a solution in the center of the polygon, finding a better direction for the next move, in the direction to find an optimum solution for the problem. Choosing the correct steps, an optimum solution is reached after a few iterations.

Even though to find a direction of movement, the IPM approach requires a longer computing time than the traditional simplex method and a smaller number of iterations will be required by the IPM to reach an optimum solution. In this manner, the IPM approach has become a competitive tool with the simplex method and, for this reason, has attracted the attention of the optimization community.

Fig.1 illustrates how the two methods approach the optimum solution. In this example, the IPM algorithm requires approximately the same quantity of iterations than the simplex method. However, for a bigger problem, this method requires only a fraction of the numbers of repetitions demanded by the simplex method and, the IPM also works perfectly with non-linear problems.

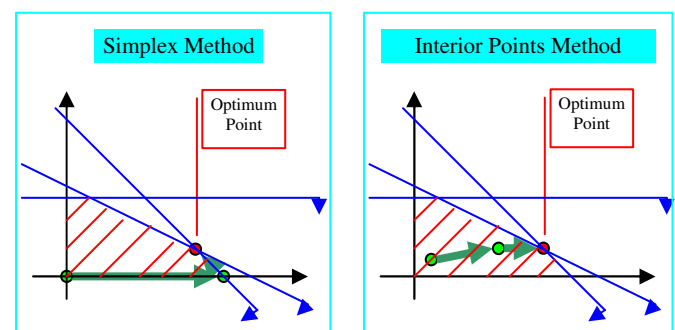


Fig.1 – Simplex Method X Interior Points Method

## 3. Optimum Power Flow

Electric energy has an important role in the society, from domestic, commercial usage to industrial usage. Knowing this, it is impossible conceive the

lack of this important input in any kind of activity. That is the reason of the importance of these studies related to the improvement of the generation and transmission of this energy.

According to [7], up to the year of 1970, the final energy consumed from electrical energy in Brazil had less than 20% of participation of the final consumption. After the first petroleum crises in 1975, the percentage of the final consumption of electrical energy reached 22% and in 1999 it reached a percentage of 40%. It is important to remember that the hydroelectric plants are responsible for about 80% of the generation of electrical energy, as declared by Oliveira (1999), quoted in [7].

The Brazilian electrical energy generation system has characteristics that make it unique in the world:

1. Hydroelectric predominance;
2. Great geographic extensions and great distances between the generation sources and the main consumer centers.
3. Several potentials to be utilized in the same river;
4. Diversity of hydrologic and pluviometric regimes;
5. Relative high degree of interconnection between the systems (south/southeast/centre-west regions), in comparison with other countries;
6. Great unexplored hydroelectric potential.

With these characteristics, it is possible to notice the importance of integrated expansion planning e usage of the generation and transmission system, so that it can work in an optimized manner.

### 3.1 The Optimum Power Flow Problem

There are, according to [1], several feasible points for the correct performance of the electric power system (EPS), but some of the operational points are more advantageous than others, depending in the aspects in which they are evaluated. For example, to diminish the system's losses, it is possible to distribute uniformly the generation by the generation systems; on the other hand, to minimize the generation costs, it is advantageous that this distribution stops being uniform and starts to being concentrated in generators with lower costs.

To solve this problem, it is common to use the optimum power flow (OPF) where, by means of an objective function, it is possible to find an optimum performance point to satisfy one or more

objectives, being the system subjected to physical, performance, reliability restrictions, among others.

[2] Maintains that OPF problem was proposed by Carpentier in the beginning of the 60s, starting from the economic dispatch (ED) problem. Historically, the ED problem, solved by equal incremental costs, was the predecessor of the optimum power flow problem, which marked the end of the ED classical period, which had been studied and developed during 30 years. Thus, the ED problem started to be approached as an OPF private case.

According to [9], the methods for the solution of the OPF can be united in four big families: Linear Programming (LP), Kuhn-Tucker (KT), Gradiente (GR) and metric variables (MV). During the last three decades, the problem's solutions used these different mathematical programming techniques.

[8] Affirm that the OPF can be applied in several analysis problems and power operational systems, such as generation and transmission reliability, security analysis, generation and transmission expansion planning and short term generation programming.

In most of these applications, the linearized representation (DC) of the power flow has been adopted, due to its bigger simplicity and the degree of satisfactory precision of its results. In the Fig.2 the functional structure of the EPSs are presented.

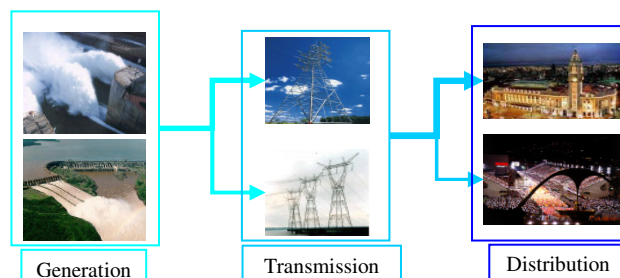


Fig. 2 – The EPSs' Functional Structure

According to [5], the functional structural components which are presented in the Fig.2 are:

- Generation: formed by generating plants or powerhouses. These powerhouses can be hydroelectric, thermal (coal, oil or natural gas) or nuclear. The hydroelectric powerhouses, generally, are located far from the consuming centers, making it necessary to have complex transmission systems and high tension.
- Transmission: constituted by the transmission auxiliary equipments which are needed to transmit the power produced in the generating powerhouses to the consumers' centers. The transmission systems can be of alternate current (AC) or continuous current (CC).

- Distribution: constituted by the substations and feeders which are responsible for the electric power distribution to the industrial, commercial and residence consumers.

The mathematical model of the OPF problem is represented by an optimizing problem formulated in the next section.

### 3.2 Formulation of the Optimum Power Flow Problem

The optimum power flow problem, as seen before, consists in determining the state of an electric network. It maximizes or minimizes an objective function while it satisfies a group of physical and operational restrictions.

The restrictions of equality correspond to the active and reactive power balance equations in each network's busbar. The inequalities are functional restrictions, such as flow monitoring in lines and physical and operational limits of the system.

The Optimum Power Flow problem can be formulated mathematically and, generically, by:

$$\begin{aligned} \min. & f(x, u, p) \\ \text{s.a.} & g_i(x, u, p) = 0, i = 1, \dots, m \\ & h_j(x, u, p) \leq 0, j = 1, \dots, r \\ & x_{\min} \leq x \leq x_{\max}, u_{\min} \leq u \leq u_{\max} \end{aligned} \tag{2}$$

where:  $(x, u, p) \in R^n$  represents the state, control and disturbance variables respectively;  $f(x, u, p)$  represents the performance index of the system;  $g(x, u, p) = 0$  represents power flow equations;  $h(x, u, p) \leq 0$  represents functional restrictions, in other words, active and reactive power limits in the transmission lines and transformers, reactive power injection limits in the controlling tension bars and injection of active power in the reference bar;  $x_{\min} \leq x \leq x_{\max}$  e  $u_{\min} \leq u \leq u_{\max}$  represent limits on the state and controlling variables, respectively.

## 4. Mathematical Programming Applied in the Optimum Power Flow Problem DC

### 4.1 Initial Considerations

An electric power system has a series of controlling devices which have a direct influence on the operational conditions and, therefore, should be included in the modeling of the system so that it can correctly simulate its performance.

The table 1 lists each of the cases that were investigated, the network that was used, the problem in question, the objective function, the problem's restrictions and the optimizing method that was used.

Table 1 – Synthesis of the cases to be investigated

Case	Network	Investigated Problem	Objective Function	Restrictions	Optimizing Method
I	6 bars 7 lines	Economical Dispatch	Minimum Cost of Active Generation	—	MatLab (LinProg)
II	6 bars 7 lines	Congestion Management	Minimum Load Cut	Line Flows	MatLab (LinProg)

Source: [4]

### 4.2. Economical Dispatch – Case I

Case 1 consists in solving an economical dispatch problem using an objective function of which goal it to minimize the cost of the active power generation. The MatLab's LINPROG routine will be used to solve this case.

The Fig.3 shows a unifilar diagram of the 6 bars system, being 2 generation bars, 3 of charge and one of 3 type, that is, one of reference and 7 lines, used to illustrate Case 1.

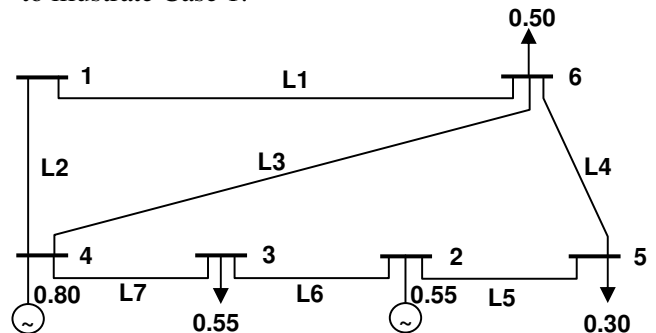


Fig.3 – System of 6 bars and 7 lines.

The data for the modeling, resolution and analysis of the problem, are found in the Tables 2 and 3. In the Table 2 the data about the lines are shown.

Table 2 – Data of the lines /Transformers for the 6 bars and 7 lines system

Lines	Initial Bar	Final Bar	X (pu)	Flows limits (pu)
L1	1	6	0.518	0.80
L2	1	4	0.370	0.80
L3	4	6	0.407	0.30
L4	5	6	0.300	0.18
L5	2	5	0.640	0.80
L6	2	3	1.050	0.90
L7	3	4	0.133	0.80

Source: [4]

In Table 3, the bar data are shown.

Table 3 – Bar Data for the 6 bars and 7 lines system

Bar	Bar Type	(+)P <sub>G</sub> (pu) (-)P <sub>D</sub> (pu)	P <sub>G min</sub> (pu)	P <sub>G max</sub> (pu)	Unitary Cost (\$ / pu)	
1	3	Ref.	-	-	-	
2	1	PV	0.55	0	1	0.4798
3	2	PQ	-0.55	-	-	-
4	1	PV	0.80	0	1	0.6535
5	2	PQ	-0.30	-	-	-
6	2	PQ	-0.50	-	-	-

Source: [4]

The problem can be formulated as:

$$\min \sum_{i \in I_G} CP_i \cdot P_{Gi}$$

s.a.

$$\sum_{i \in I_G} P_{Gi} = \sum_{i \in I_D} P_{Di}$$

$$P_{Gi}^{\min} \leq P_{Gi} \leq P_{Gi}^{\max}$$
(3)

Rewriting the formulated problem, you have:

$$\min C_T = 0.4P_{G2}^A + 0.9P_{G2}^B + 0.1P_{G4}^A + 0.8P_{G4}^B$$

s.a.

$$P_{G2}^A + P_{G2}^B + P_{G4}^A + P_{G4}^B = 1.35$$

$$\begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.0 \end{bmatrix} \leq \begin{bmatrix} P_{G2}^A \\ P_{G2}^B \\ P_{G4}^A \\ P_{G4}^B \end{bmatrix} \leq \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$
(4)

Fig.4 shows the dispatch, before and after the optimization.

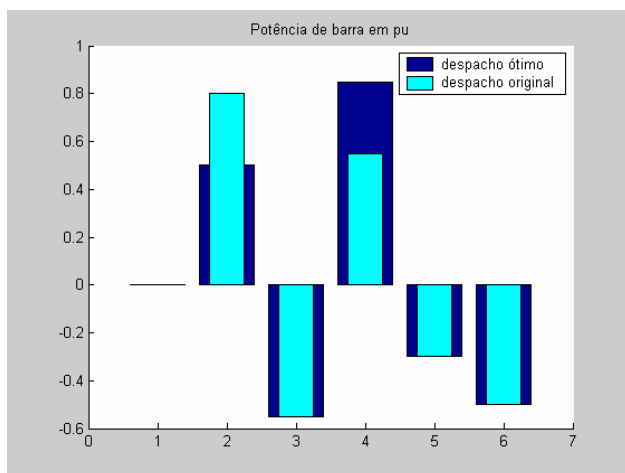


Fig.4 – Bar's Power  
Source: [4]

The powers' flows before and after the optimization are shown in Fig.5.

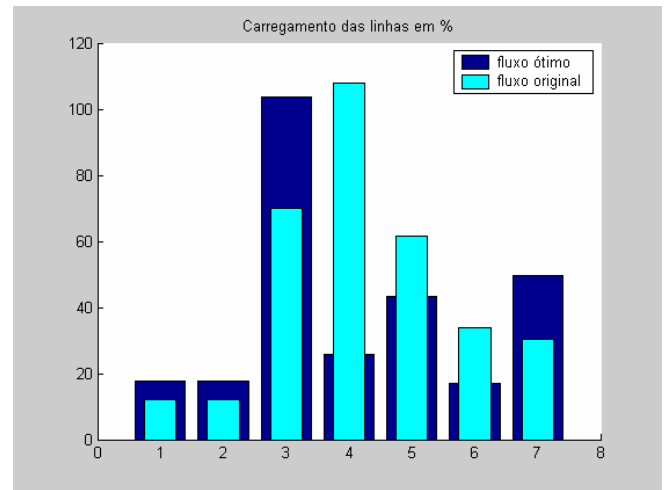


Fig.5 – Flows in the lines  
Source: [4]

### 4.3. Congestion Management – Case II

This case involves congestion management with the minimum of load cut. The network data are the same as in Case I. The limits of the controlling variables are described in Table 4. The limits of the power flows were reduced in 50%, aiming in creating situations of multiple congestions. To solve the problem, the MatLab toolbox optimization from LINPROG routine will be maintained.

Table 4 – Limits in the Controlling Variables

Controlling Variable	Inferior Limit (pu)	Superior Limit (pu)
P <sub>G2</sub>	0.40	0.80
P <sub>D3</sub>	-0.55	-0.35
P <sub>G4</sub>	0.10	0.55
P <sub>D5</sub>	-0.30	-0.25
P <sub>D6</sub>	-0.50	-0.40

The problem can be formulated in an incremental form:

$$\min f = [0.001 \ 0.5 \ 0.001 \ 0.5 \ 0.5] \cdot [\Delta P_{G2} \ -\Delta P_{D3} \ \Delta P_{G4} \ -\Delta P_{D5} \ -\Delta P_{D6}]^T$$

s.a.

$$[1 \ -1 \ 1 \ -1 \ -1] \cdot [0.8 + \Delta P_{G2} \ (0.55 + \Delta P_{D3}) \ 0.55 + \Delta P_{G4} \ (0.3 + \Delta P_{D5}) \ (0.5 + \Delta P_{D6})]^T = 0$$

$$\begin{bmatrix} -0.80 \\ -0.80 \\ -0.30 \\ -0.18 \\ -0.80 \\ -0.90 \\ -0.80 \end{bmatrix} \leq \begin{bmatrix} -0.4557 & -0.3183 & -0.3009 & -0.5394 & -0.5787 \\ -0.5443 & -0.6817 & -0.6991 & -0.4606 & -0.4213 \\ -0.0852 & 0.2146 & 0.2525 & -0.2679 & -0.3535 \\ 0.5409 & 0.1038 & 0.0484 & 0.8073 & -0.0678 \\ 0.5409 & 0.1038 & 0.0484 & -0.1927 & -0.0678 \\ 0.4591 & -0.1038 & -0.0484 & 0.1927 & 0.0678 \\ 0.4591 & 0.8962 & -0.0484 & 0.1927 & 0.0678 \end{bmatrix} \begin{bmatrix} P_{G2} \\ P_{D3} \\ P_{G4} \\ P_{D5} \\ P_{D6} \end{bmatrix} \leq \begin{bmatrix} 0.80 \\ 0.80 \\ 0.30 \\ 0.18 \\ 0.80 \\ 0.90 \\ 0.80 \end{bmatrix}$$

$$\begin{bmatrix} 0.40 \\ -0.55 \\ 0.10 \\ -0.30 \\ -0.50 \end{bmatrix} \leq \begin{bmatrix} 0.80 + \Delta P_{G2} \\ -(0.55 + \Delta P_{D3}) \\ 0.55 + \Delta P_{G4} \\ -(0.30 + \Delta P_{D5}) \\ -(0.50 + \Delta P_{D6}) \end{bmatrix} \leq \begin{bmatrix} 0.80 \\ -0.35 \\ 0.55 \\ -0.25 \\ -0.40 \end{bmatrix}$$

Fig.6 shows the controlling variables before and after the solution of the load shedding problem.

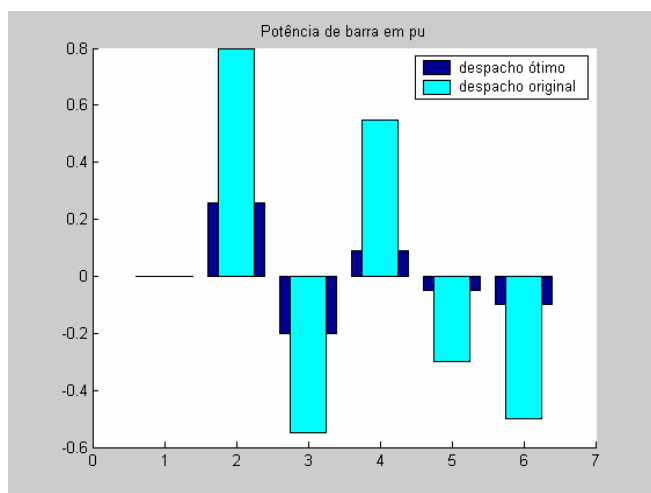


Fig.6 – Bar power

Source: [4]

The initial and final flows are represented in Fig.7.

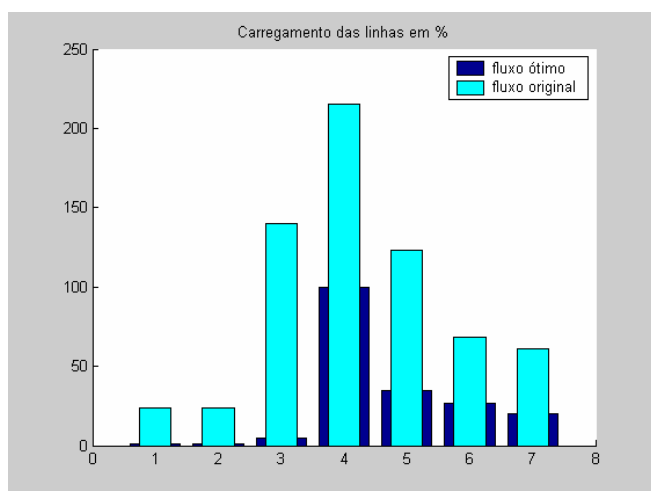


Fig.7 – Flows in the lines

Source: [4]

## 5. Final Considerations

This paper investigated the mathematic programming applications, more precisely, the linear programming to the linearized optimum power flow problem.

Different methods of problem solutions in linear programming were presented and discussed. The interior points method was reported as being indicated for larger problems. The Simplex algorithm showed itself to be adequate for smaller and medium size networks.

The optimum power flow problem, in its several formulations, was examined and its objective functions and typical problems' restrictions were described.

Formulations of Optimum Dispatch and Network Congestion problems, in its linear version, were presented and numerical demonstrative examples were also presented.

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