Numerical approximations of Riemann solutions to multiphase flows used in petroleum engineering

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Abstract: We study two one-dimensional systems of hyperbolic conservation laws that are used to model the multiphase flow in the process of secondary oil recovery. We compute approximate solutions for several examples using a spacetime discontinuous Galerkin (SDG) method based on causal spacetime triangulations and a piecewise constant Galerkin basis. Even though convergence of the SDG method was shown only in the case of strictly hyperbolic genuinely nonlinear Temple class systems, numerical solutions presented in this paper show that the SDG method could be effectively used for approximating solutions to more general problems.

Key–Words: Secondary oil recovery, spacetime discontinuous Galerkin, Riemann problems, conservation laws

1 Introduction

After an underground source of oil is tapped, initially oil flows out due to high pressure in the reservoir. When this flow stops, usually there is still a large amount of oil left in the reservoir and the method of "secondary oil recovery" is used. This process assumes that water and gas are injected into some wells of the oil field to displace oil to the producing well.

We consider two models in one spatial dimension in which oil, in a long tube of porous medium of rock or sand, is displaced by water and gas pumped at one end. We study systems of conservation laws, describing these two models, and we compute approximate solutions to several initial value problems using a spacetime discontinuous Galerkin method with causal spacetime triangulations and the piecewise constant Galerkin basis. For more details on multiphase flows in porous media and computational techniques used in petroleum engineering, see [1] by Chen, Huan & Ma and references therein.

2 Definition of a spacetime discontinuous Galerkin method

In this paper we consider systems of conservation laws in one spatial dimension

$$\partial_t u + \partial_x f(u) = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \qquad (1)$$

with initial data

$$u(0,x) = u_0(x), \quad x \in \mathbb{R}. \qquad (2)$$

Here, $u : [0,\infty) \times \mathbb{R} \to \mathcal{D} \subseteq \mathbb{R}^n$ denotes densities of conserved variables, $f : \mathcal{D} \to \mathbb{R}^n$ is the spatial flux, and $u_0 : \mathbb{R} \to \mathcal{D}$ is a function of bounded total variation. Let Df denote the gradient matrix of f and let λ^i denote the i-th eigenvalue of Df. If $\lambda^1(u), \ldots, \lambda^n(u)$ are real for all $u \in \mathcal{D}$, the system is *hyperbolic* in the domain of conservation states $\mathcal{D}$. Moreover, if $\lambda^1(u), \ldots, \lambda^n(u)$ are real and distinct for all $u \in \mathcal{D}$, the system is *strictly hyperbolic* in $\mathcal{D}$. If there are states $u \in \mathcal{D}$, such that the matrix $Df(u)$ possesses complex eigenvalues, the system is either *elliptic* of *mixed hyperbolic-elliptic* there.

Discontinuous Galerkin (DG) methods for conservation laws have been studied extensively during the past two decades (for more details, see [3] by Cockburn, Karniadakis & Shu). They are based on the elementwise representation of the solution and enforce the conservation law locally. The Runge-Kutta DG methods use finite element discretizations of the spatial domain and a Runge-Kutta technique to march the solution in time (see [2] by Cockburn).

The spacetime discontinuous Galerkin (SDG) method that we consider here is based on spacetime triangulations of the spacetime domain $[0,\infty) \times \mathbb{R}$. The Galerkin basis consists of functions which are polynomials of degree $p \geq 0$ within each element, while the values on adjacent elements are coupled through the Godunov flux. The SDG method is closely related to the method proposed by Lowrie in [10], which is based on uniform layered triangulations and which uses an approximation of the Go-

dunov flux on certain element faces. The SDG method was introduced by Palaniappan, Haber & Jerrard in [12] with several numerical examples for scalar one-dimensional hyperbolic conservation laws, including the Buckley-Leverett equation which models a two-phase flow in oil recovery. It was also shown numerically that the SDG method is of the order $p+1$, and when $p \geq 1$, certain slope and compact limiters are used in order to avoid oscillations in the regions where the solutions are discontinuous. Analysis of the SDG method in the case $p \geq 1$ is rather challenging due to its spacetime format. As in [4, 5] by Jegdić and Jegdić & Jerrard, where convergence of the method was shown for certain strictly hyperbolic systems, we assume $p = 0$, i.e., we assume that the Galerkin basis consists of piecewise constant functions. Given a spacetime triangulation $\mathcal{T}_h$ (h stands for the maximal diameter of a triangle in the considered triangulation), we denote the corresponding Galerkin basis by $\mathcal{P}_h$. We also assume that the triangulations are *causal*, meaning that for each edge Γ of an element T with the outward unit spacetime normal $\nu = (\nu_t, \nu_x)$, we require

$$\text{either} \quad (1, \lambda^i(u)) \cdot \nu < 0 \quad \text{or} \quad (1, \lambda^i(u)) \cdot \nu > 0,$$

for every $i \in \{1, \ldots, n\}$ and all $u \in \mathcal{D}$. Given an edge Γ of an element T with the outward spacetime normal ν, if $(1, \lambda^i(u)) \cdot \nu$ is negative (positive) for all $i \in \{1, \ldots, n\}$ and all $u \in \mathcal{D}$, than the edge Γ is said to be *inflow (outflow)* for the element T. Therefore, the formulation of the causal spacetime discontinuous Galerkin method is:

Given a causal spacetime triangulation $\mathcal{T}_h$ of the domain $[0, \infty) \times \mathbb{R}$, find $u_h \in \mathcal{P}_h$ such that

$$\int_{\partial T^-} (u_h^-, f(u_h^-)) \cdot \nu \, d\mathcal{H}^1 + \int_{\partial T^+} (u_h, f(u_h)) \cdot \nu \, d\mathcal{H}^1 = 0,$$

holds on each element $T \in \mathcal{T}_h$, where ∂T^- (∂T^+) stands for the inflow (outflow) part of the boundary ∂T, u_h^- denotes the value of the approximate solution along ∂T^- which is computed on an adjacent mesh element, and $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure.

In the case of strictly hyperbolic systems of conservation laws, we show in [4, 5] that given a causal spacetime triangulation $\mathcal{T}_h$, if an approximation $u_h \in \mathcal{P}_h$ exists, then u_h satisfies certain discrete entropy inequalities. Also, in the case of strictly hyperbolic genuinely nonlinear Temple class systems, we show that given a sequence of causal spacetime triangulations $\{\mathcal{T}_h\}$, the corresponding sequence $\{u_h\}$ of approximations exists and is precompact in $L^1_{loc}([0, \infty) \times \mathbb{R}; \mathbb{R}^n)$. Moreover, we show that any limit of any convergent subsequence of $\{u_h\}$ is a weak solution to the initial value problem (1)-(2). SDG approximations for Riemann problems with solutions containing singular and transitional shocks were computed in [6] by Jegdić, showing that the SDG method could be successfully used for approximating solutions to more general problems. In this paper, we further confirm this claim by studying systems of conservation laws used in petroleum engineering.

3 Numerical simulations for an immiscible three-phase flow

The first model we study was considered by Marchesin & Plohr in [11]. They consider the idealized flow of water, gas and oil in a homogeneous linear core, neglecting gravity, compressibility, heterogeneity and the mass transfer among the phases, they assume that the displacement is immiscible and ignore fingering effects. Governing equations are given by

$$\begin{aligned} \partial_t s_w + \partial_x f_w(s_w, s_g) &= D_w, \\ \partial_t s_g + \partial_x f_g(s_w, s_g) &= D_g, \end{aligned} \tag{3}$$

where $(t, x) \in [0, \infty) \times \mathbb{R}$, $s_w, s_g : [0, \infty) \times \mathbb{R} \to [0, 1]$ stand for water and gas saturations, respectively, oil saturation is defined by $s_o = 1 - s_w - s_g$, and f_w and f_g are fractional flow functions given, in terms of relative permeability functions k_w, k_g and k_o and the fluid viscosities μ_w, μ_g and μ_o, by

$$f_w(s_w, s_g) = \frac{k_w/\mu_w}{k_w/\mu_w + k_g/\mu_g + k_o/\mu_o}$$

and

$$f_g(s_w, s_g) = \frac{k_g/\mu_g}{k_w/\mu_w + k_g/\mu_g + k_o/\mu_o}.$$

The diffusion terms, D_w and D_g, represent the effect of capillary pressure differences among fluids. As in [11] we assume that each permeability function depends only on its fluid saturation. More precisely,

$$k_w = s_w^2, \; k_g = s_g^2 \quad \text{and} \quad k_o = s_o^2 = (1 - s_w - s_g)^2,$$

and $\mu_w = 0.5$, $\mu_g = 0.3$ and $\mu_o = 1$. In this paper, we also assume $D_w = D_g = 0$. For each state $S := (s_w, s_g)$ within the saturation triangle, defined by $\mathcal{D} := \{(s_w, s_g) : 0 \leq s_w, s_g \leq 1, s_w + s_g \leq 1\}$, the Jacobian matrix

$$\begin{bmatrix} \partial f_w/\partial s_w & \partial f_w/\partial s_g \\ \partial f_g/\partial s_w & \partial f_g/\partial s_g \end{bmatrix}$$

has two real nonnegative eigenvalues, denoted by λ^1 and λ^2, implying that the system (3) is hyperbolic. We graph $(\lambda^1-\lambda^2)(s_w, s_g)$ for $0 \le s_w, s_g \le 1$ in Figure 1 and we note that the two eigenvalues coincide for exactly one point inside the saturation triangle $\mathcal{D}$. This point is called an "umbilic point". On each edge of the saturation triangle, the saturation of one of the fluids is zero, and the flow is described by the Buckley-Leverett equation for the two fluids.

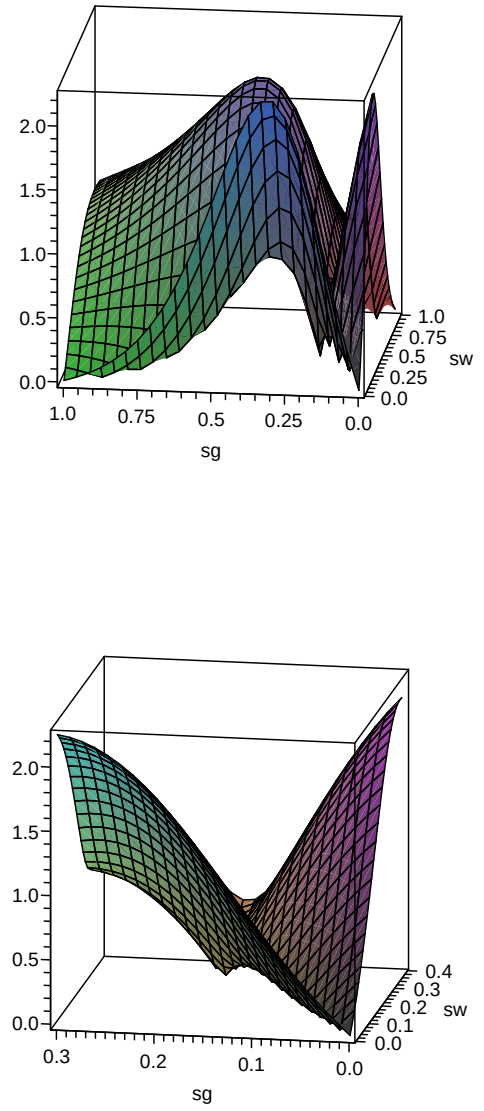


Figure 1: Graphs of $\lambda^1 - \lambda^2$.

In the two Riemann problems we consider next, the right state corresponds to almost pure oil and the left state corresponds to the mixture of water and gas at the injection well. We compute the approximate solutions for system (3) using the causal SDG method. We consider a layered spacetime triangulation with the time step $\Delta t = 0.002$ and the diameter of the elements in spatial direction $\Delta x \in (0.01, 0.02)$.

Example 1. *We consider Riemann data given by*

$$S(0,x) := \begin{cases} S_L = (0.613, 0.387) & \text{if} \quad x < 0, \\ S_R = (0.05, 0.15) & \text{if} \quad x \ge 0, \end{cases}$$

which models injection of water (61.3%) and gas (38.7%). The saturation of oil changes from 0% at the left side of the core to 80% at the right side of core. The approximate saturations of water, gas and oil at time $t = 2$ are depicted in Figure 2. The solution has a classical wave structure, consisting of a slow and a fast wave groups. The slow wave group consists of a "compound wave" (a rarefaction wave and an adjacent shock wave) connecting the left state to an intermediate state, and the fast wave group is represented by a Buckley-Leverett shock (i.e., an oil bank) from this intermediate state to the right state.

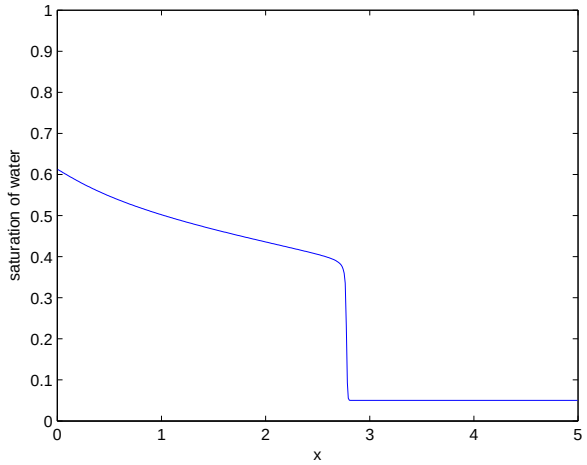
Example 2. *The Riemann initial data is given by*

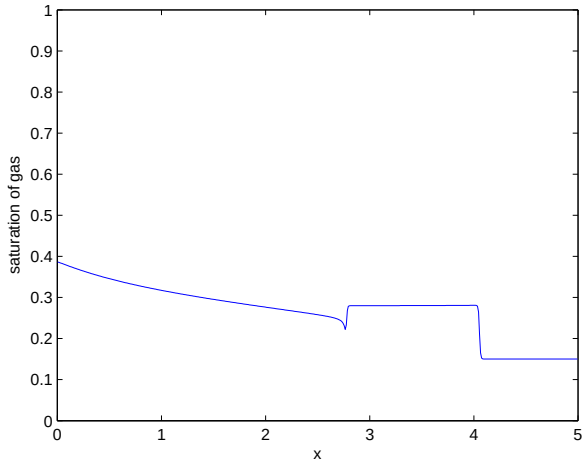
$$S(0,x) := \begin{cases} S_L = (0.721, 0.279) & \text{if} \quad x < 0, \\ S_R = (0.05, 0.15) & \text{if} \quad x \ge 0, \end{cases}$$

which models injection of water (72.1%) and gas (27.9%). Again, the saturation of oil is 0% at the left side and 80% at the right side of core. The approximate saturations of water, gas and oil at time $t = 2$ are plotted in Figure 3. In this case the solution consists of three wave groups. Again, the slow wave group consists of a rarefaction and an adjacent shock and the fast wave group is represented by a Buckley-Leverett shock. However, between the slow and the fast wave groups, there is a new kind of shock, called a "transitional shock".

4 Numerical simulations for a two-phase first-contact miscible flow

The second model we consider was studied by Juanes & Lie in [7, 8] using a front-tracking algorithm. The main assumptions are: (a) it is a three component model consisting of water, gas and oil, (2) the injection gas, called "solvent", and the resident oil mix and form a hydrocarbon phase h which is immiscible with water, (3) the fluids are incompressible, and (4) gravity, capillary forces and viscous fingering are neglected. Hence, this is a two-phase model consisting of water and a hydrocarbon phase. For front-tracking





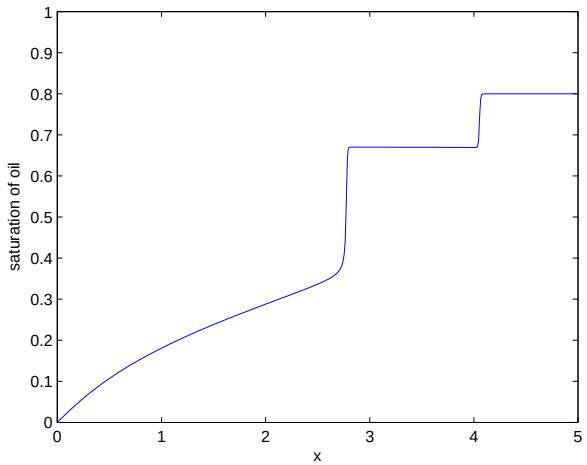
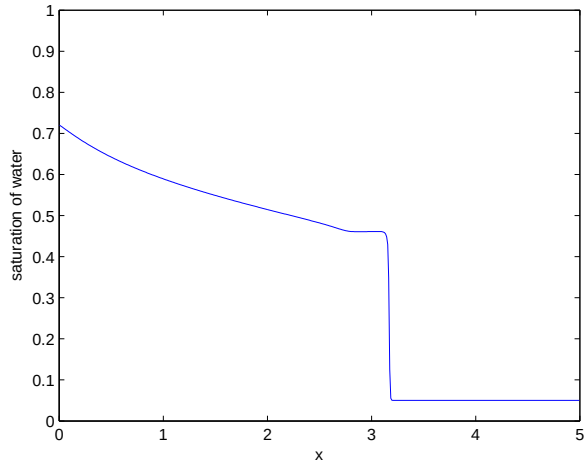
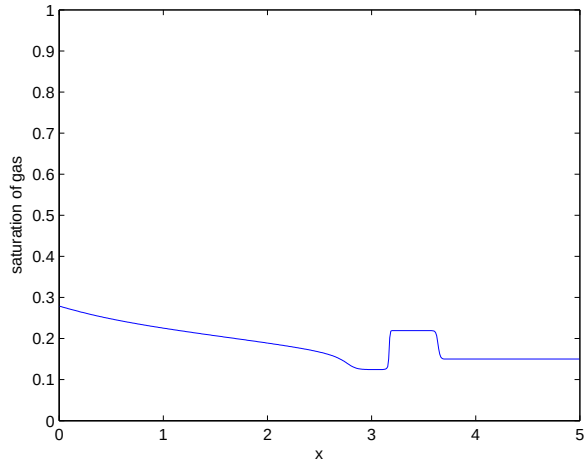


Figure 2: Approximate saturations at $t = 2$.





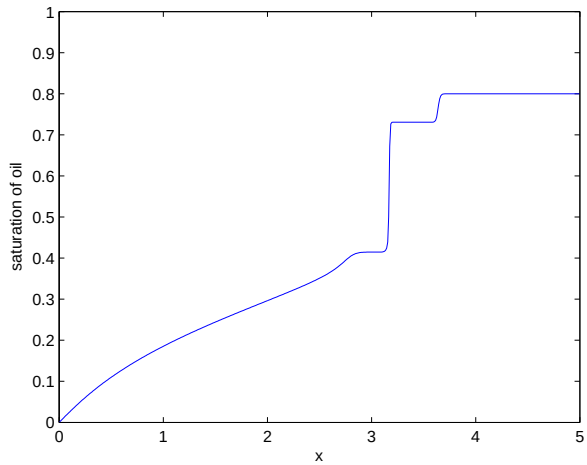


Figure 3: Approximate saturations at $t = 2$.

simulations of an immiscible three-phase flow, we refer to [9] by Lie & Juanes. Let s_w and s_h denote saturations of water and the hydrocarbon phase ($s_w+s_h = 1$), and let χ_o and χ_g denote the mass fractions of oil and gas in the hydrocarbon phase ($\chi_o+\chi_g = 1$). Then the governing equations are given by

$$\partial_t s_w + v_T \partial_x f_w = 0,$$
$$\partial_t((1-s_w)\chi_g) + v_T \partial_x((1-f_w)\chi_g) = 0,$$

where f_w is a fractional flow function given by

$$f_w = \frac{k_w/\mu_w}{k_w/\mu_w + k_h/\mu_h},$$

k_w and k_h are relative permeability functions, μ_w and μ_h denote viscosities, and $v_T = v_w+v_h$ is the total velocity. After rescaling of the time variable to eliminate v_T, and using the notation $s := s_w$, $c := (1-s_w)\chi_g$, $\chi := \chi_g$, $f := f_w$ and $g := (1-f_w)\chi_g$, we obtain

$$\begin{aligned} \partial_t s + \partial_x f(s,c) &= 0, \\ \partial_t c + \partial_x g(s,c) &= 0, \end{aligned} \tag{4}$$

where s and c stand for water saturation and solvent concentration, respectively, relative permeabilities are specified by

$$k_w = \left(\frac{s-s_{wc}}{1-s_{wc}}\right)^2 \quad \text{if} \quad s < s_{wc},$$

and $k_w = 0$, otherwise, and

$$k_h = 0.1\frac{1-s-s_{hc}}{1-s_{hc}} + 0.9\left(\frac{1-s-s_{hc}}{1-s_{hc}}\right)^2$$

if $1-s > s_{hc}$, and $k_h(s) = 0$, otherwise. Further, $s_{wc} = s_{hc} = 0.2$, and viscosities are given by $\mu_w = 1$, $\mu_g = 0.4$, $\mu_o = 4$ and

$$\mu_h(\chi) = \left(\frac{1-\chi}{\mu_o^{1/4}} + \frac{\chi}{\mu_g^{1/4}}\right)^{-4}.$$

For each state $U := (s,c)$ within the saturation triangle $\mathcal{D} := \{(s,c) : 0 \le s,c \le 1, s+c \le 1\}$, we compute the eigenvalues λ^1 and λ^2 of the Jacobian matrix

$$\begin{bmatrix} \partial f/\partial s & \partial f/\partial c \\ \partial g/\partial s & \partial g/\partial c \end{bmatrix}.$$

We depict the graphs of $|\lambda^1 - \lambda^2|$ in Figure 4. We note that the system is hyperbolic, but not everywhere strictly hyperbolic. The two eigenvalues λ^1 and λ^2 coincide along a curve (called a "transition curve", since $\lambda^1 < \lambda^2$ on the left, and $\lambda^1 > \lambda^2$ on the right of this curve). We notice this curve for $0.2 \le s \le 0.8$ and $0 \le c \le 1$, and we also depict a closer look for $0.2 \le s \le 0.8$ and $0 \le c \le 0.5$.

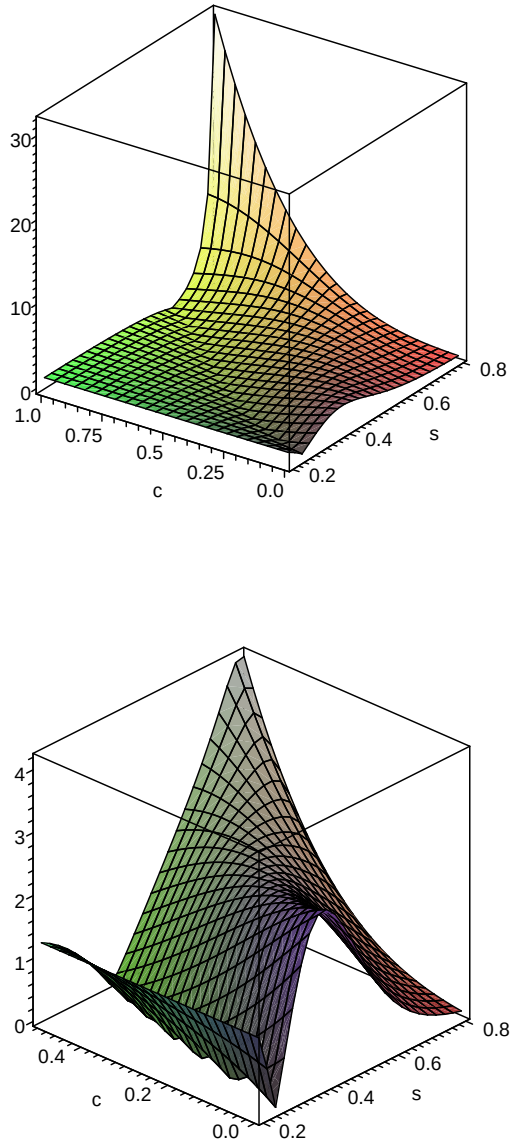


Figure 4: Graphs of $|\lambda^1 - \lambda^2|$.

Example 3. *The Riemann initial data we consider for system* (4) *is given by*

$$U(0,x) := \begin{cases} (0.8,0) & \text{if} \quad x < 0.5, \\ (0.2,0.7) & \text{if} \quad x \ge 0.5. \end{cases}$$

We compute the approximate solution at time $t = 2$ using the SDG method and depict the plot for the saturation of water in Figure 5.

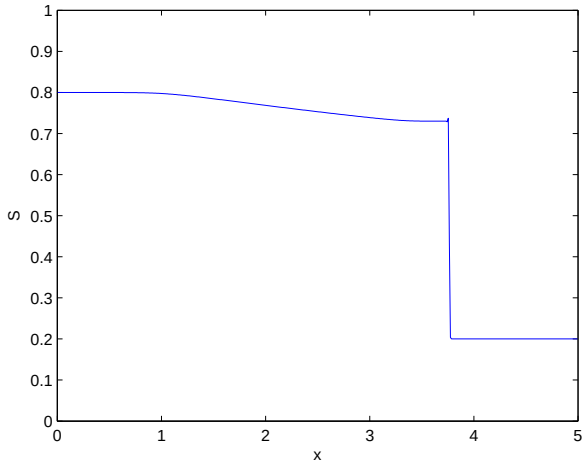


Figure 5: Saturation of water at $t = 2$.

5 Conclusion

We note again that convergence of the SDG method presented in §2 was so far proved only for a special class of strictly hyperbolic systems (Temple systems) using their special geometric properties. In this paper we consider two hyperbolic, but not everywhere strictly hyperbolic, systems of conservation laws that are used to model a process of secondary oil recovery. The SDG solutions for the above examples are consistent with analytical solutions in [11] (Examples 1 and 2) and front-tracking approximations in [8] (Example 3), indicating that the SDG method could be successfully used for approximating solutions to more general systems of hyperbolic conservation laws.

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