PERTURBATION ANALYSIS FOR THE STATIONARY DISTRIBUTION OF A MARKOV CHAIN

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Abstract: We consider a stationary distribution of a finite, irreducible, homogeneous Markov chain. Our aim is to perturb the transition probabilities matrix using approximations to find regions of feasibility and optimality for a given basis when the chain is optimized using linear programming. We also explore the application of perturbations bonds and analyze the effects of these on the construction of optimal policies.

Key–Words: Markov chains, deviation matrix, linear pogramming, perturbation matrix analysis

1. Introduction

A perturbation in a Markov chain can be referred as a slight change in the entries of the corresponding transition stochastic matrix, resulting in structural changes in the underlying process, for example, sets of states which in the original case do not communicate, do so after a perturbation is imposed. Also, passages times that originally were not well defined random variables, may become so after the perturbation. In this sense, a square matrix is stochastic if its entries are real and non-negative and the sum of the entries in each row is equal 1.

Their importance is related with the dynamics that these represent, particularly, the singularly perturbed Markov chains have a few time scales. One time scale may correspond to the more frequent transitions occurring among states which communicate also in the unperturbed case. In this document we are interested in the matrix perturbation procedure from a probabilistic point of view, where the perturbation quantity of the original stochastic matrix ϕ, can be approximated by a given matrix A such that $\phi(\epsilon) = \phi + A(\epsilon) = \phi + \epsilon A$.

Given the perturbed $\phi(\epsilon)$ matrix we approach the problem of analyzing the effects of the perturbation on the optimal policies of a Markovian decision process, sustained in the Frobenius norm of $\phi(\epsilon)$. The markovian process describes the productive and reproductive lifespan of herd sows, where, under an infinite planning horizon, the linear programming (LP) is used as an optimization technique.

This investigation constitutes an alternating focus to the problem of replacement management of animals in a herd, sows in this case. This consists in to consider at regular time intervals whether it should be kept to a sow in the herd for an additional period or it should be replace by a new animal (gilt) and to optimize the expected return associated to the decisions made during the process (Tijms, 1994). Several authors have approached this problem with Markovian models or some of their variants, see for instance, Howard (1960), van der Wal and Wessels (1985), White and White (1989), Kristensen (1996) and Plá (2002). In this document we are devoted to study the properties of the transition probabilities matrix of the process when this is perturbed in random form, and, to analyze the effects of such perturbations on the optimal policies of the process. To illustrate our proposal we consider the sow replacement problem developed in Plá, Pomar and Pomar (2003). The system consist in a sow farm where sows are allowed to reach nine reproductive cycles as a maximum and at the end of each cycle, two actions can be taken: keep or replace. The problem is represented as a regular Markov decision process and solved using a linear programming model. Transition probabilities and reward values are arbitrary but near to what are observed in actual systems; the corresponding transition probabilities matrix is perturbed using the mentioned techniques and the optimal policies are characterized in terms of these. We report the theoretical and practical results.

2. Preliminary

A stochastic process $\{M(n)\}_{n=0,1,\ldots}$ with finite state space $\mathcal{Z} = \{z_1, \ldots, z_S\}$ is a Markov chain with discrete time, if for all $n \in \mathbb{N}$ and all $w_0, \ldots, w_n \in \mathcal{Z}$

$$\mathbf{P}\left(M(0) = w_0, M(1) = w_1, \ldots, M(n) = w_n\right) =$$

$$\mathbf{P}(M(0) = w_0)\ \gamma(i, i-1),$$

where $\gamma(i, i-1) = \Pi_{i=1}^{n} \mathbf{P}(M(i) = w_i \mid M(i-1) = w_{i-1})$

Consider a Markov chain with S states $z_1, \ldots, z_S$ where, in each stage $k = 1, 2, \ldots$, the analyst should made a decision d, among ξ possible. Denote by $z(n) = z_i$ and $d(n) = d_k$ the state and the decision made in stage n respectively, then the systems moves at the next stage, $n+1$, into the state z_j with perhaps, an unknown probability given by

$$\phi_{ij}^{k} = \mathbf{P}\left[z(n+1) = z_j \;\mid\; z(n) = z_i,\, d(n) = d_k\right].$$

When the transition occurs, it is followed by the reward r_{ij}^{k}, and the payoff at state z_i after the decision d_k is made is given by $\psi_i^k = \sum_{j=1}^{S} \phi_{ij}^{k}\, r_{ij}^{k}$. Since we assume that for every policy $\vartheta(k_1, \ldots k_S)$, the corresponding Markov chain is ergodic, then, the steady state probabilities of this chain are given by $\phi_i^{\vartheta} = \lim_{n\to\infty} \mathbf{P}\ [Z(n) = z_i],\ \ i = 1, \ldots, S$, and the problem is to find a policy ϑ for which the expected payoff

$$\Omega^{\vartheta} = \sum_{i=1}^{S} \phi_i^{\vartheta}\, \psi_i^{k}, \tag{1}$$

is maximum.

When the model involves an infinite horizon, the LP can be used to optimize (1), i.e., if the termination stage is unknown, usually the problem is described by an infinite planning horizon where the number N of stages is considered infinite. In this case the optimal policy is constant over stages and the objective function is given by

$$g^{\vartheta} = \sum_{i=1}^{S} \phi_i^{\vartheta} r_i^{\vartheta}, \tag{2}$$

where ϕ_i^{ϑ} is the limiting state probability under the policy ϑ (i.e., when the policy is kept constant over an infinit number of stages). This criterion maximizes the average net revenues per stage. Thus, the LP problem associated to the chain is (Kristensen 1996).

$$\begin{aligned}
&\max \textstyle\sum_{i=1}^{S} \sum_{d=1}^{\xi} r_i^d x_i^d \\
&\text{Subject to} \\
&\textstyle\sum_{d=1}^{\xi} x_i^d - \sum_{j=1}^{S} \sum_{d=1}^{\xi} \phi_i^d x_j^d = 0, \\
&\textstyle\sum_{i=1}^{S} \sum_{d=1}^{\xi} x_i^d = 1,\ x_i^d \geq 0,
\end{aligned} \tag{3}$$

where d is optimal in state i if and only if x_i^d from the optimal solution is strictly positive, and the x_i^d are the unconditional steady-state probabilities that the system is in the state i and decision d is made.

A replacement policy is a specification of a sequence of "keep" or "replace" actions, one for each period. An optimal policy is a policy that achieves the greatest reward (or the smallest total net cost) of ownership over the entire planning horizon. In Pérez et al. (2006) is demonstrated that the problem (3) has a degenerate solution.

3. The approximations method

In this section we discuss the following question: given the Markov chain of the problem (2), which is optimized using LP, ¿how affects to the optimal policy of the chain a perturbation on the optimal solution of the LP problem?.

To begin this discussion, consider the general LP problem:

$$\begin{aligned}
&\text{minimize } f(x) = c^t x \\
&\text{subject to } \quad A\,x = \delta,\ x \geq 0,\ A_{m\times n}, \\
&c,\ x\ \in \mathbb{R}^n,\ \delta\ \in \mathbb{R}^m
\end{aligned} \tag{4}$$

The number ρ of basic feasible solutions that the problem has, is less than or equal to $\binom{n}{m}$, and $\mathcal{B}_{m\times m}$ (submatrix of A) is a feasible basis of the LP model if $\mathcal{B} \in \mathcal{S}$, where $\mathcal{S} = \{\mathcal{B}_i \in A : \mathcal{B}_i^{-1}\,\delta\ \geq 0\,\}$.

Suppose $\mathcal{B}$ is perturbed to a matrix $\tilde{\mathcal{B}}$, that is the transition probability matrix of an n finite state, irreducible, homogeneous Markov chain as well. Denoting the stationary distribution vector of $\mathcal{B}$ by $x^{\star}$, and of $\tilde{\mathcal{B}}$ by $\tilde{x}$, the goal is to describe the change $-dx = (x^{\star} - \tilde{x})$ in the stationary distribution in terms of the changes $d\,\mathcal{B}$ using an approximations method. In this sense, $x^{\star}$ and $\tilde{x}$ satisfy the systems

$$x^{\star}\,\mathcal{B} = x^{\star}, \quad x^{\star} > 0, \quad x^{\star}\,\mathbf{e} = 1$$

and

$$\tilde{x}\,\tilde{\mathcal{B}} = \tilde{x}, \quad \tilde{x} > 0, \quad \tilde{x}\,\mathbf{e} = 1$$

where $\mathbf{e}$ is the column vector of all ones.

The approximations method used can be described as follows. Given a basis $\mathcal{B} \in \mathcal{S}$, we difference the matrix equation $\mathcal{B}x = b$, and obtain, $d\mathcal{B}x + \mathcal{B}dx = 0$, i.e., $dx = -\mathcal{B}^{-1}d\mathcal{B}\,x$.

Let $d_{ij} \in d\mathcal{B}$ be the perturbation on $b_{ij} \in \mathcal{B}$, and $x^\star$ an optimal solution of the problem (4). Defining $f^\star = f(x^\star) = c^t x^\star \leftarrow$ min, the resulting perturbation $\tilde{b}_{ij} \in \tilde{\mathcal{B}}$ can be written as

$$\tilde{b}_{ij} = b_{ij} + d_{ij}, \tag{5}$$

and therefore,

$$\tilde{x} = x^\star + dx, \tag{6}$$

constitutes a perturbated solution around of $x^\star$. Thus,

$$\tilde{f} = f(\tilde{x}) = f^\star + c^t dx, \tag{7}$$

is a new solution, not necessarily feasible (since $A\tilde{x} = \delta + Adx$) of the problem (4) evaluated in the perturbed point $\tilde{x}$. This is also an approximate solution to the modified problem

$$\begin{aligned} &\text{minimize } f(x) = c^t\,x \\ &\text{subject to } \tilde{A}\,x = \delta,\ x \geq 0,\ \tilde{A}_{m\times n}, \\ &c,\ x \in \mathbb{R}^n,\ \delta \in \mathbb{R}^m \end{aligned} \tag{8}$$

where $\tilde{A}$ is the resulting matrix after incorporating the perturbations d_{ij} in $\mathcal{B}$. Let $\hat{x}$ be an optimal solution of the problem (8), then we can write

$$\hat{x} = \tilde{x} + \varepsilon, \quad \varepsilon \in \mathbb{R}^n, \tag{9}$$

and there holds

$$\hat{f} = f(\hat{x}) = \tilde{f} + c^t\,\varepsilon\,, \tag{10}$$

The quantities, $\tilde{x}+\varepsilon$ and, $\tilde{f}+c^t\,\varepsilon$ can be viewed as approximations to $\hat{x}$ and $\hat{f}$ respectively, and ε is an error measure of the approximation. Naturally, we would want an error zero.

To evaluate the existent relationships among the ε quantity and the matrix $d\mathcal{B}$ we use the Frobenius norm $\|\cdot\|_F$ of $d\mathcal{B}$, and the Euclidian norm of ε defined as

$$\|\,d\mathcal{B}\,\|_F^2 = \text{ Trace } (d\mathcal{B}^t\,d\mathcal{B}),$$

and (11)

$$\|\,\varepsilon\,\|^2 = (\hat{x}-\tilde{x})^t(\hat{x}-\tilde{x})$$

3.1. Perturbation bounds

The norm perturbation bound used in this section is of the following form (Schweitzer 1968)

$$\|\,x^\star - \tilde{x}\,\|_1 \leq \|\,Z\,\|_\infty\,\|\,d\mathcal{B}\,\|_\infty, \tag{12}$$

where $\|\,x^\star - \tilde{x}\,\|_1$ is the 1-norm of the vector $x^\star - \tilde{x}$ defined as the absolute entry sum, $\|\,\varphi\,\|_\infty$ is the ∞-norm of the matrix φ defined as the maximum absolute row sum, and Z is the fundamental matrix associated to the matrix $\mathcal{B}$. Z has the form

$$Z \equiv \left[I - \mathcal{B} + \mathbf{e}\,(x^\star)^t\right]^{-1}, \tag{13}$$

Likewise, the stationary distribution vector $\tilde{x}$, of the perturbed matrix $d\mathcal{B}$ can be expressed in terms of $x^\star$ and the fundamental matrix Z as (Kemeny and Snell 1960)

$$(x^\star - \tilde{x})^t = \tilde{x}^t\,d\mathcal{B}\,Z \tag{14}$$

Using (14) we can now formalize an important result that relates to $\hat{f}$ and $\tilde{x}$ with $f^\star$.

Pre multiplying both sides of (14) by c we have

$$c^t\,x^\star - c^t\,\tilde{x} = c^t\,Z^t d\mathcal{B}^t\tilde{x}$$

or

$$-c^t\,dx = c^t\,Z^t d\mathcal{B}^t\tilde{x}$$

i.e.,

$$f^\star - \tilde{f} = c^t\,Z^t d\mathcal{B}^t\tilde{x}$$

equivalently

$$\tilde{f} = f^\star - c^t\,Z^t d\mathcal{B}^t\tilde{x}, \tag{15}$$

using (10) we have finally

$$\hat{f} = f^\star - c^t\,[Z^t\,d\mathcal{B}^t\,\tilde{x} + \varepsilon], \tag{16}$$

3.2. The LP model of d_{ij}

To evaluate the permissible maximum value for each perturbation, we propose the alternative LP problem

$$\text{Maximize } \varphi(d) = \{d\mathbf{e} : -\mathcal{B}\, d\mathcal{B}\, dx \leq x^\star\}\,, \quad (17)$$

where $\mathbf{e} \in \mathbb{R}^\zeta$, ζ is the number of elements of the matrix $\mathcal{B}$ that will be perturbed, and $d = d_{ij}$ is the perturbations vector. If the problem (4) has an optimal solution, then, the problem (12) also has an optimal solution because the inequality allows to slack the constrains.

In this sense, an important problem for this kind of perturbations consists on finding a feasible region φ for the perturbed basis $\tilde{\mathcal{B}}$. To solve this, we define the functions $g(dx_i) = -C^t\mathcal{B}_i^{-1}d\mathcal{B}\,x^\star$, $i = 1\ldots,\rho$. Then, a feasible region for $\tilde{\mathcal{B}}$ is given by

$$\begin{aligned} \varphi = \{d_{ij} \in g(dx_k) \,:\, \\ g(dx_k) \leq g(dx_i),\ i = 1, 2\ldots,\rho\}, \end{aligned} \quad (18)$$

where the basis $\mathcal{B}_k$ used to evaluate $g(dx_k)$ is that on which the perturbation will be made.

4. Numerical example

Consider the following transition probabilities matrices reported in Pla et al. (2003), which represent a markovian decision process with $D = 2$:

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.30 & 0 & 0.70 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0 & 0.75 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0.80 & 0 & 0 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0 & 0.80 & 0 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0 & 0 & 0.80 & 0 & 0 & 0 \\ 0.20 & 0 & 0 & 0 & 0 & 0 & 0 & 0.80 & 0 & 0 \\ 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 & 0 \\ 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.75 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{d = 1 } (m \equiv \text{keep})}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\text{d = 2 } (r \equiv \text{replace})}$$

The corresponding LP problem is to maximize the objective function $f(y)$ given by[1]:

$$190y_{1m} + 226y_{2m} + 232y_{3m} + 202y_{4m} + 202y_{5m} + 202y_{6m} + 202y_{7m} + 202y_{8m} + 202y_{9m} - 200B_r$$

subject to

$$\begin{aligned} & y_{1m} + y_{1r} - B_m = 0,\ y_{2m} + y_{2r} - 0.70y_{1m} = 0, \\ & y_{3m} + y_{3r} - 0.75y_{2m} = 0,\ y_{4m} + y_{4r} - 0.8y_{3m} = 0, \\ & y_{5m} + y_{5r} - 0.8y_{4m} = 0,\ y_{6m} + y_{6r} - 0.8y_{5m} = 0, \\ & y_{7m} + y_{7r} - 0.8y_{6m} = 0,\ y_{8m} + y_{8r} - 0.75y_{7m} = 0, \\ & y_{9m} + y_{9r} - 0.75y_{8m} = 0,\ B_r + B_m - y_{1r} - y_{2r} - \\ & y_{3r} - y_{4r} - y_{5r} - y_{6r} - y_{7r} - y_{8r} - y_{9r} - 0.3y_{1m} - \\ & 0.25y_{2m} - 0.2y_{3m} - 0.2y_{4m} - 0.2y_{5m} - 0.2y_{6m} - \\ & 0.25y_{7m} - 0.25y_{8m} - y_{9m} = 0,\ \ B_r + B_m + y_{1m} + \\ & y_{2m} + y_{3m} + y_{4m} + y_{5m} + y_{6m} + y_{7m} + y_{8m} + \\ & y_{9m} + y_{1r} + y_{2r} + y_{3r} + y_{4r} + y_{5r} + y_{6r} + y_{7r} + \\ & y_{8r} + y_{9r} = 1 \\ & y_{1m}, y_{2m}, y_{3m}, y_{4m}, y_{5m}, y_{6m}, y_{7m}, y_{8m}, y_{9m} \geq 0, \\ & y_{1r}, y_{2r}, y_{3r}, y_{4r}, y_{5r}, y_{6r}, y_{7r}, y_{8r}, y_{9r}, B \geq 0. \end{aligned}$$

The optimal solution and the basic variables of the inverse basis are (presented in order): $B_m = 0.2106, y_{1m} = 0.2106, y_{2m} = 0.1474, y_{3m} = 0.1105, y_{4m} = 0.08847, y_{5m} = 0.07078, y_{6m} = 0.05662, y_{7m} = 0.04529, y_{8m} = 0.03397, y_{9m} = 0.02548, S_{10} = 0$. The optimal objective function is $f^\star = 163.7765$. The basis $\mathcal{B}$ that will be perturbed is formed by the columns: $y_{1m}, y_{2m}, y_{3m}, y_{4m}, y_{5m}, y_{6m}, y_{7m}, y_{8m}, y_{9m}, B_{m,10}$

and

$$\begin{aligned} &\| \, d\mathcal{B} \, \|_F^2 = (d_{21} - 1)^2 + (d_{32} - 1)^2 + (d_{43} - 1)^2 + \\ &(d_{54} - 1)^2 + (d_{65} - 1)^2 + (d_{76} - 1)^2 + (d_{87} - 1)^2 + \\ &(d_{98} - 1)^2 + d_{21}^2 + d_{32}^2 + d_{43}^2 + d_{54}^2 + d_{65}^2 + d_{76}^2 + \\ &d_{87}^2 + d_{98}^2. \end{aligned}$$

Note that the convex function $\| \, d\mathcal{B} \, \|_F^2$ achieves its minimum in $d_{ij}^\star = 0.5,\ i = 2,\ldots,9,\ j = 1,\ldots,8$, and $\| \, d\mathcal{B}^\star \, \|_F$ = 2. In this point, $\| \, \varepsilon \, \| = 0.7280$.

By (12) we have $\| \, Z \, \|_\infty = 25.8248$, $\| \, d\mathcal{B} \, \|_\infty = 4$, and $\| \, x^\star - \tilde{x} \, \|_1 = 0.9993$

Using x as the optimal solution of the LP problem, the perturbed solution $\tilde{x} \approx x^\star - \mathcal{B}^{-1}d\mathcal{B}x$ is given by

$$\begin{aligned} & \tilde{B}_m \approx 0.2106 - 0.1741d_{21} - 0.1729d_{32} - 0.1124d_{43} - 0.0763d_{54} \\ & -0.0530d_{65} - 0.0344d_{76} - 0.0208d_{87} - 0.0095d_{98}; \\ & \tilde{y}_{1m} \approx 0.2106 - 0.1741d_{21} - 0.1729d_{32} - 0.1124d_{43} - 0.0763d_{54} \\ & \ -0.0530d_{65} - 0.0344d_{76} - 0.0208d_{87} - 0.0095d_{98}; \\ & \tilde{y}_{2m} \approx 0.1474 + 0.0886d_{21} - 0.1210d_{32} - 0.0787d_{43} - 0.0534d_{54} \\ & \ -0.0371d_{65} - 0.0241d_{76} - 0.0146d_{87} - 0.0066d_{98}; \\ & \tilde{y}_{3m} \approx 0.1105 + 0.0665d_{21} + 0.1198d_{32} - 0.0591d_{43} - 0.0401d_{54} \\ & \ -0.0278d_{65} - 0.0180d_{76} - 0.0109d_{87} - 0.0050d_{98} \\ & \tilde{y}_{4m} \approx 0.0884 + 0.0532d_{21} + 0.0958d_{32} + 0.1000d_{43} - 0.0320d_{54} \\ & \ -0.0222d_{65} - 0.0144d_{76} - 0.0087d_{87} - 0.0039d_{98} \\ & \tilde{y}_{5m} \approx 0.0707 + 0.0425d_{21} + 0.0766d_{32} + 0.0800d_{43} + 0.0848d_{54} \\ & \ -0.0178d_{65} - 0.0115d_{76} - 0.0070d_{87} - 0.0032d_{98} \end{aligned}$$

[1]The cost coefficients are arbitrary.

$\tilde{y}_{6m} \approx 0.0566 + 0.0341d_{21} + 0.0612d_{32} + 0.0641d_{43} + 0.0678d_{54}$
$+0.0741d_{65} - 0.0092d_{76} - 0.0056d_{87} - 0.0025d_{98}$
$\tilde{y}_{7m} \approx 0.0452 + 0.0271d_{21} + 0.0490d_{32} + 0.0512d_{43} + 0.0542d_{54}$
$+0.0593d_{65} + 0.0632d_{76} - 0.0044d_{87} - 0.0020d_{98}$
$\tilde{y}_{8m} \approx 0.0339 + 0.0204d_{21} + 0.0368d_{32} + 0.0384d_{43} + 0.0407d_{54}$
$+0.0444d_{65} + 0.0474d_{76} + 0.0532d_{87} - 0.0015d_{98}$
$\tilde{y}_{9m} \approx 0.0254 + 0.0153d_{21} + 0.0275d_{32} + 0.0288d_{43} + 0.0304d_{54}$
$+0.0333d_{65} + 0.0356d_{76} + 0.0399d_{87} + 0.0440d_{98}$
$\tilde{S}_{10} \approx 0.9400$

For the previously developed system we use the perturbations: $d_{21} = 0.20, d_{32} = 0.20, d_{43} = 0.12, d_{54} = 0.14, d_{65} = 0.18, d_{76} = 0.10, d_{87} = 0.15, d_{98} = 0.20$; and from these, we obtain $\tilde{f} = 184.9326, c^t dx = 21.2314$.

Similarly, the optimal solution $\hat{x}$ of the perturbed problem is: $(B_m = 0.3062, y_{1m} = 0.3062, y_{2m} = 0.1531, y_{3m} = 0.0842, y_{4m} = 0.0572, y_{5m} = 0.0377, y_{6m} = 0.0234, y_{7m} = 0.0164, y_{8m} = 0.0098, y_{9m} = 0.0054)$ and $\hat{f} = 142.6643$. Using (9) we get the ε value defined as: $(-0.0160, -0.0160, -0.0303, -0.0112, -0.0019, 0.0062, 0.0141, 0.0160, 0.0179, 0.0103)$, and the inner product $c^t\varepsilon = -42.2814$. Note that these values satisfy the equations (6), (7), (9) y (10).

The Frobenius norm, the $\tilde{x} - x^\star$ norm, the ε error and other parameters were evaluated for different values of d_{ij} (using $d_{ij} = d_{kl}, i = 2, \ldots, 9,\ j = 1, \ldots, 8$). In table 1 we summarize our findings and figure 1 sketch the numerical results. Table 2 shows the samples of $\hat{x}$, $x^\star$, dx and ε^2 for the proposed d_{ij}.

d_{ij}	$\parallel d\mathcal{B} \parallel_F$	$\parallel \tilde{x} - x^\star \parallel$	$\parallel \varepsilon \parallel$	$\tilde{f}$	$\hat{f}$
0	0	0	0.4704	163.7765	95.1337
0.1	2.5612	0.1148	0.5219	177.2500	102.0440
0.2	2.3323	0.2296	0.5842	190.7235	109.6893
0.3	2.1540	0.3443	0.6337	204.1971	240.1141
0.4	2.0396	0.4591	0.6841	217.6704	265.7273
0.5	2	0.5739	0.7280	231.1442	291.1452
0.6	2.0396	0.6887	0.7633	244.6177	315.0898
0.7	2.1540	0.8035	0.7904	258.0910	335.7125
0.8	2.3323	0.9183	0.8140	271.5647	351.2073
0.9	2.5612	1.0330	0.8423	285.0382	360.8219
1.0	2.8284	1.1478	0.8835	298.5116	365.4715

Table 1: Comparative aspects of the proposed d_{ij}

Let us consider the linear programming model defined in (12). In our example it become

maximize $= d_{21}+d_{32}+d_{43}+d_{54}+d_{65}+d_{76}+d_{87}+d_{98}$
Subject to

$0.1741d_{21} + 0.1729d_{32} + 0.1124d_{43} + 0.0763d_{54}$
$+0.0344d_{76} + 0.0208d_{87} + 0.0095d_{98} \leq 0.2106$
$-0.0886d_{21} + 0.1210d_{32} + 0.0787d_{43} + 0.0534d_{54}$
$+0.0371d_{65} + 0.0180d_{76} + 0.0109d_{87} + 0.0050d_{98}$
≤ 0.1474
$-0.0665d_{21} - 0.1198d_{32} + 0.0591d_{43} + 0.0401d_{54}$
$+0.0278d_{65} + 0.0180d_{76} + 0.0109d_{87} + 0.0050d_{98}$
≤ 0.1105

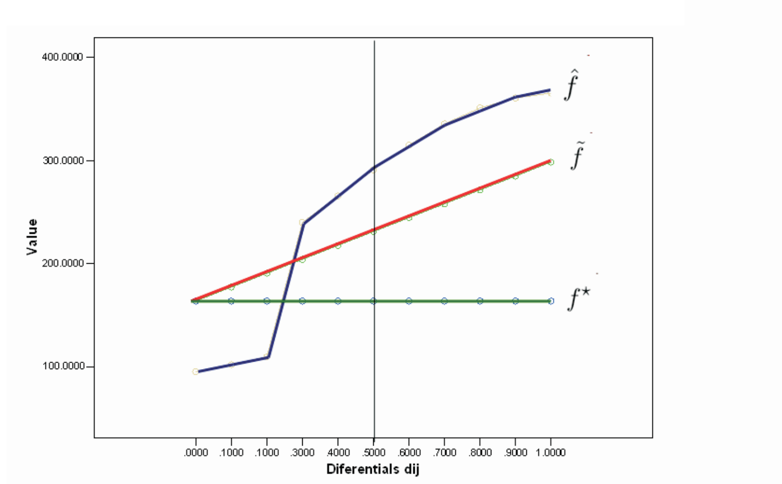

Fig 1: Functions $f^\star$, $\tilde{f}$ and $\hat{f}$ generated from the proposed d_{ij}

$-0.0532d_{21} - 0.0958d_{32} - 0.1d_{43} + 0.0320d_{54}$
$+0.0222d_{65} + 0.0144d_{76} + 0.0087d_{87} + 0.0039d_{98}$
≤ 0.0884
$-0.0425d_{21} - 0.0766d_{32} - 0.0800d_{43} - 0.0848d_{54}$
$+0.01784d_{65} + 0.0115d_{76} + 0.0070d_{87} + 0.0032d_{98}$
≤ 0.0707
$-0.0341d_{21} - 0.0612d_{32} - 0.0641d_{43} - 0.0678d_{54}$
$-0.0741d_{65} + 0.0092d_{76} + 0.0056d_{87} + 0.0025d_{98}$
≤ 0.0566
$-0.0271d_{21} - 0.0490d_{32} - 0.0512d_{43} - 0.0542d_{54}$
$-0.0593d_{65} - 0.0632d_{76} + 0.0044d_{87} + 0.0020d_{98}$
≤ 0.0452
$-0.0204d_{21} - 0.0368d_{32} - 0.0384d_{43} - 0.0407d_{54}$
$-0.0444d_{65} - 0.0474d_{76} - 0.0532d_{87} + 0.0015d_{98}$
≤ 0.0399
$-0.0153d_{21} - 0.0275d_{32} - 0.0288d_{43} - 0.0304d_{54}$
$-0.0333d_{65} - 0.0356d_{76} - 0.0399d_{87} - 0.04407d_{98}$
$\leq 0.0254,\ 0 \leq d_{ij} \leq 1,\ i = 2, \ldots, 9,\ j = 1, \ldots, 8$

which solution is $d_{21} = 0.1222, d_{32} = 0.0407, d_{43} = 0.3672, d_{54} = 1, d_{65} = 1, d_{76} = 1, d_{87} = 1, d_{98} = 1, \varphi(d^\star) = 5.5302$. The corresponding Frobenius norm is $\parallel d\mathcal{B} \parallel_F = 2.6912$, and $\parallel \epsilon \parallel = 0.8688$.

5. Conclusion

The approximations method is a good alternative to evaluate the sensitivity of the optimal solution in a markovian decision process. The norm perturbation bound associated to the fundamental matrix is a measure of the error made when changing the values of the transition probabilities matrix. This method is promising when evaluating the changes in the entrances of it, but considering now that these can be represented like probabilty density functions making the pertinent changes in the used norms.

d_{ij}		B_m	y_{1m}	y_{2m}	y_{3m}	y_{4m}	y_{5m}	y_{6m}	y_{7m}	y_{8m}	y_{9m}
0	$\hat{x}$	0.5	0.5	0	0	0	0	0	0	0	0
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	0	0	0	0	0	0	0	0	0	0
	ε^2	0.0837	0.0837	0.0217	0.0122	0.0078	0.0049	0.0032	0.0020	0.0011	0.0006
0.1	$\hat{x}$	0.4736	0.4736	0.0473	0.0047	0.0004	0	0	0	0	0
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.0653	−0.0653	−0.0247	0.0025	0.0167	0.0244	0.0284	0.0297	0.0280	0.0255
	ε^2	0.1078	0.1078	0.0056	0.0117	0.0109	0.0090	0.0072	0.0056	0.0038	0.0025
0.2	$\hat{x}$	0.4444	0.4444	0.0889	0.0178	0.0036	0.0007	0.0001	0.0000	0	0
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.1308	−0.1308	−0.0494	0.0050	0.0335	0.0489	0.0568	0.0596	0.0560	0.0511
	ε^2	0.1329	0.1329	0.0001	0.0096	0.0140	0.0141	0.0128	0.0110	0.0081	0.0058
0.3	$\hat{x}$	0.4118	0.4118	0.1235	0.0371	0.0111	0.0033	0.0010	0.0003	0.0001	0
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.1961	−0.1961	−0.0741	0.0076	0.0503	0.0733	0.0852	0.0894	0.0840	0.0766
	ε^2	0.1579	0.1579	0.0025	0.0066	0.0163	0.0198	0.0198	0.0080	0.0071	0.0059
0.4	$\hat{x}$	0.3750	0.3750	0.1500	0.0600	0.0240	0.0096	0.0038	0.0015	0.0006	0.0002
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.07070	0.0566	0.0452	0.0339	0.0254
	dx	−0.2615	−0.2615	−0.0988	0.0100	0.0670	0.0977	0.1136	0.1191	0.1120	0.1021
	ε^2	0.1814	0.1814	0.0102	0.0036	0.0172	0.0252	0.0276	0.0079	0.0070	0.0058
0.5	$\hat{x}$	0.3337	0.3337	0.1668	0.0834	0.0417	0.0208	0.0104	0.0052	0.0026	0.0013
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.3268	−0.3268	−0.1235	0.0126	0.0838	0.1222	0.1420	0.1489	0.1400	0.1276
	ε^2	0.2025	0.2025	0.0204	0.0015	0.0170	0.0296	0.0354	0.0079	0.0070	0.0058
0.6	$\hat{x}$	0.2877	0.2877	0.1726	0.1036	0.0621	0.0372	0.0223	0.0134	0.0080	0.0048
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.3922	−0.3922	−0.1482	0.0151	0.1005	0.1466	0.1704	0.1787	0.1680	0.1531
	ε^2	0.2203	0.2203	0.0301	0.0004	0.0160	0.0324	0.0418	0.0079	0.0070	0.0058
0.7	$\hat{x}$	0.2381	0.2381	0.1667	0.1166	0.0816	0.0571	0.0400	0.0280	0.0196	0.0137
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.4576	−0.4576	−0.1729	0.0176	0.1173	0.1710	0.1988	0.2085	0.1961	0.1786
	ε^2	0.2354	0.2354	0.0369	0.0001	0.0153	0.0340	0.0464	0.0079	0.0070	0.0058
0.8	$\hat{x}$	0.1876	0.1876	0.1501	0.1201	0.0960	0.0768	0.0614	0.0491	0.0393	0.0314
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.5230	−0.5230	−0.1976	0.0201	0.1340	0.1955	0.2272	0.2383	0.2241	0.2042
	ε^2	0.2500	0.2500	0.0401	0.0001	0.0159	0.0358	0.0494	0.0079	0.0070	0.0058
0.9	$\hat{x}$	0.1403	0.1403	0.1263	0.1136	0.1023	0.0920	0.0828	0.0745	0.0671	0.0604
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.5884	−0.5884	−0.2223	0.0226	0.1508	0.2199	0.2556	0.2680	0.2521	0.2297
	ε^2	0.2684	0.2684	0.0405	0.0003	0.0187	0.0394	0.0526	0.0079	0.0070	0.0058
1	$\hat{x}$	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
	$x^\star$	0.2106	0.2106	0.1474	0.1105	0.0884	0.0707	0.0566	0.0452	0.0339	0.0254
	dx	−0.6537	−0.6537	−0.2471	0.0252	0.1676	0.2444	0.2840	0.2978	0.2801	0.2552
	ε^2	0.2950	0.2950	0.0398	0.0012	0.0243	0.0462	0.0579	0.0079	0.0070	0.0058

Table 2: Samples of $\hat{x}$, $x^\star$, dx and ε^2 for the proposed d_{ij}.

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