Notes on the Superposition Scandal

IRWIN W. SANDBERG
Department of Electrical and Computer Engineering
The University of Texas at Austin
1 University Station C0803
Austin, Texas 78712 - 0240
USA

Abstract: -Attention is focused on continuous-space shift-invariant systems with continuous system maps and inputs and outputs that are elements of $L_\infty(I\!\!R^d)$. It is shown that infinite superposition can fail in this important setting. It is also shown that continuous shift-invariant linear mappings need not commute with the operation of integration (even when the two composite operations are well defined).

Key-Words: - linear systems, superposition, commutativity, shift-invariant systems, bounded measurable inputs

1 Introduction

One can give a very long list of books – many of them basically very good books – (see, for example, [1, pp. 268–269], [2, p. 53], [3, p. 67], [4, p. 176]) in which superposition is said to hold in the case of any linear system with an excitation that can be written as a sum of a countably infinite number of excitations. As is well known, this conclusion – which has been taught to decades of students in several fields – plays a central role in textbook material concerning the origins of both discrete-time and continuous-time convolution representations. In [5], and also in the brief note [6], attention is directed to the fact that the conclusion is not correct. One consequence of the oversight is that the usual representation for linear discrete-time systems has had to be corrected by adding an additional term [5].[1] In [6], in the setting of linear spaces with metrics and a standard definition of convergence, a simple criterion is given for (infinite) superposition to hold. A discrete-space example is given in [6] to illustrate that superposition can fail. However, the linear system map in the example is not continuous, and it is defined on only a certain unusual input space. In addition, the simple criterion given for superposition to hold involves a strong assunption on the convergence of the input sum representation of the input. With that assumption, superposition can fail only for system maps that lack continuity.

Here attention is focused on shift-invariant continuous-space systems with continuous system maps. Inputs and outputs are elements of the familiar space $L_\infty(I\!\!R^d)$ of Lebesgue measurable complex-valued functions defined on $I\!\!R^d$, where d is any positive integer. We show that superposition can fail in this important setting. Also given – and again in contrast with what is said in texts on linear systems – is a related result showing that continuous shift-invariant linear mappings need not commute with the operation of integration (even when the two composite operations are well defined).[2] Our main results are Theorems 1 and 2 of the following section. The section contains in addition results that provide sufficient conditions under which commutativity is valid.

[1] As mentioned in [5], this writer does not claim that cases in which the extra term is nonzero are necessarily of importance in applications, but he does feel that the existence of these cases illustrates that the analytical ideas in the books are flawed.

2 On Superposition and Commutativity

2.1 Preliminaries

In the signal-processing literature, $x(\alpha)$ typically denotes a function. In the following we distinguish between a function x and $x(\alpha)$, the latter meaning the value of x at the point (or time) α. Sometimes a function x is denoted by $x(\cdot)$, and also we use Gx to mean $G(x)$. This notation is often useful in studies of systems in which signals are transformed into other signals.

We use $\|\cdot\|_d$ to stand for a norm on $I\!\!R^d$. As in Section 1, $L_\infty(I\!\!R^d)$ denotes the linear space of Lebesgue measurable complex-valued functions defined on $I\!\!R^d$, where d is any positive integer. We view $L_\infty(I\!\!R^d)$ as a normed space with the norm given by

$$\|x\|_\infty = \sup_{\alpha \in I\!\!R^d} |x(\alpha)|. \qquad (1)$$

All integrals in this section and in the Appendix are Lebesgue integrals. For $1 \le p < \infty$, $L_p(I\!\!R^d)$ stands for the usual normed linear space space of pth power integrable functions defined on $I\!\!R^d$. $BL_1(I\!\!R^d)$ denotes the linear space of bounded $L_1(I\!\!R^d)$ functions.

[2] For related material motivated by the fact that there exist continuous shift-invariant linear maps whose input-output behavior is not determined by its impulse response, see [7] and [8].

2.2 Lack of Infinite Superposition

Our main result is the following.

Theorem 1 : There are elements x and $x_1, x_2, \ldots$ in $L_\infty(I\!R^d)$, and a continuous linear shift-invariant map G from $L_\infty(I\!R^d)$ into itself, such that

(i) $x = \sum_{n=1}^{\infty} x_n$ in the sense of pointwise convergence.

(ii) $\sum_{n=1}^{\infty} Gx_n$ converges in $L_\infty(I\!R^d)$, and we have

$$Gx \neq \sum_{n=1}^{\infty} Gx_n.$$

Proof:
Let x be any element of $L_\infty(I\!R^d)$ such that $\lim_{\|\alpha\|_d \to \infty} x(\alpha) = c$, in which c is a nonzero number, and define x_n by $x_n(\alpha) = x(\alpha)$ for $n-1 \leq \|\alpha\|_d < n$, and zero otherwise, for each positive integer n. It is clear that $x = \sum_{n=1}^{\infty} x_n$ in the sense of pointwise convergence. We will show that there is a continuous linear shift-invariant map G from $L_\infty(I\!R^d)$ into itself, such that (ii) holds, with in fact each Gx_n the zero function in $L_\infty(I\!R^d)$. We will use the following lemma.

Lemma 1 : There is a continuous linear shift-invariant map H from $L_\infty(I\!R^d)$ into itself such that

(a) H takes every element of $BL_1(I\!R^d)$ into the zero function in $L_\infty(I\!R^d)$.

(b) $(Hy)(\alpha) = \zeta$ for all $\alpha \in I\!R^d$ and each $y \in L_\infty(I\!R^d)$ with $\lim_{\|\alpha\|_d \to \infty} y(\alpha) = \zeta$, in which ζ is an arbitrary complex number.

Lemma 1 is all but stated in [9]. For the reader's convenience, a proof is given in the Appendix.

Select G to be any H of the kind described in the lemma, and observe that G satisfies (ii) because of the assumed limit property of x, and the fact that each x_n belongs to $BL_1(I\!R^d)$. This proves the theorem.

The fact that the series $\sum_{n=1}^{\infty} x_n$ of the theorem is not required to converge in $L_\infty(I\!R^d)$ is crucial, in that (see [6]) $\sum_{n=1}^{\infty} Gx_n$ converges in $L_\infty(I\!R^d)$, with

$$Gx = \sum_{n=1}^{\infty} Gx_n$$

when the series converges in $L_\infty(I\!R^d)$ (and G is as indicated). Also, a result entirely analogous to Theorem 1 holds in the corresponding discrete-space setting. Specifically, direct modifications of the proof show that Theorem 1 holds if $L_\infty(I\!R^d)$ is replaced with the usual normed linear space $\ell_\infty(\mathcal{Z}^d)$ of complex-valued functions defined on $\mathcal{Z}^d$, where $\mathcal{Z}$ denotes the integers. The theorem holds also if $L_\infty(I\!R^d)$ is replaced with its corresponding real-valued space, and similarly for $\ell_\infty(\mathcal{Z}^d)$.

2.3 Lack of Commutativity

The tools used to prove Theorem 1 also yield the following.

Theorem 2 : Let a be any element of $BL_1(I\!R^d)$ such that a has a nonzero integral. Then there is an element x in $L_\infty(I\!R^d)$ and a continuous linear shift-invariant map H from $L_\infty(I\!R^d)$ into itself such that H maps $BL_1(I\!R^d)$ into itself and

$$H \int_{I\!R^d} a(\cdot - \beta)x(\beta)\, d\beta \neq \int_{I\!R^d} Ha(\cdot - \beta)x(\beta)\, d\beta.$$

Proof:
Choose x to to be an element of $L_\infty(I\!R^d)$ with $\lim_{\|\alpha\|_d \to \infty} x(\alpha) = c$, in which c is a nonzero number, and notice that the theorem follows at once from Lemma 1 and Proposition 1 (in the Appendix). Here too, we arrive at a case in which the right side, but not the left side, is the zero function in $L_\infty(I\!R^d)$.

The theorem holds also if $L_\infty(I\!R^d)$ is replaced with its corresponding real-valued space. Also, a result entirely analogous to Theorem 2 holds in the corresponding discrete-space setting in which $L_\infty(I\!R^d)$ and $BL_1(I\!R^d)$ are replaced with the familiar spaces $\ell_\infty(\mathcal{Z}^d)$ and $\ell_1(\mathcal{Z}^d)$, respectively, and the integrals are replaced with the analogous infinite sums.[3]

2.4 Sufficient Conditions for Commutativity

We close Section 2 by stating two results that provide conditions under which a linear map does commute with integration:

Theorem 3 Suppose that $1 \leq p < \infty$, and that M is a linear shift-invariant map whose domain includes $L_p(I\!R^d) \cup L_1(I\!R^d)$. Assume that M is defined on $L_p(I\!R^d) \cup L_1(I\!R^d)$ such that the restriction of M to $L_p(I\!R^d)$ is a continuous map into $L_p(I\!R^d)$, and the restriction of M to $L_1(I\!R^d)$ is a continuous map into $L_1(I\!R^d)$. Let $x \in L_p(I\!R^d)$, and let $g \in L_1(I\!R^d)$. Then, with ℓ the element of $L_p(I\!R^d)$ given by

$$\ell = \int_{I\!R^d} g(\cdot - \beta)x(\beta)\, d\beta$$

we have

$$M\ell = \int_{I\!R^d} (Mg)(\cdot - \beta)x(\beta)\, d\beta.$$

Theorem 3 is an extension [10] of Lemma 21.2.2 of [14, pp. 568] where the $p = 1$ case is given. Although not stated in [14], the proof given of the lemma yields also the following.

Theorem 4: Suppose that $1 \leq p < \infty$, and that M is a continuous linear shift-invariant map of $L_p(I\!R^d)$ into itself.

[3] A classical observation related in a general sense to Theorem 2 is that differentation under the integral sign is not always valid.

Let $g \in L_p(I\!R^d)$ and let $x \in L_1(I\!R^d)$. Then, with ℓ the element of $L_p(I\!R^d)$ given by

$$\ell = \int_{I\!R^d} g(\cdot - \beta)x(\beta)\, d\beta$$

we have

$$M\ell = \int_{I\!R^d} (Mg)(\cdot - \beta)x(\beta)\, d\beta.$$

2.5 Conclusion

It is shown that infinite superposition can fail in a certain important setting. It is also shown that continuous shift-invariant linear mappings need not commute with the operation of integration (even when the two composite operations are well defined).

3 Appendix

3.1. Proof of Lemma 1

We use the following three propositions, in which F denotes the map defined on $L_\infty(I\!R^d)$ by

$$(Fy)(\alpha) = \int_{I\!R^d} f(\alpha - \beta)y(\beta)\, d\beta, \;\; \alpha \in I\!R^d$$

where $f \in BL_1(I\!R^d)$ with unit integral. Also, $\mathcal{C}$ stands for the normed linear space of bounded uniformly-continuous complex-valued functions defined on $I\!R^d$, with the norm described by (1).[4]

Proposition 1: If $y \in L_\infty(I\!R^d)$ satisfies

$$\lim_{\|\alpha\|_d \to \infty} y(\alpha) = \zeta \;\text{ for some }\; \zeta,$$

then

$$\lim_{\|\alpha\|_d \to \infty} (Fy)(\alpha) = \zeta.$$

Proof of Proposition 1:

Suppose that y is as indicated. We have

$$(Fy)(\alpha) = \int_{I\!R^d} f(\alpha - \beta)y(\beta)\, d\beta = \zeta \int_{I\!R^d} f(\beta)\, d\beta + \int_{I\!R^d} f(\alpha - \beta)[y(\beta) - \zeta]\, d\beta$$

for each α. Consider the last integral, and let any $\epsilon > 0$ be given. Choose a positive c_1 so that

$$\sup_{\|\alpha\|_d > c_1} |y(\alpha) - \zeta| \int_{I\!R^d} |f(\beta)|\, d\beta < \epsilon/2$$

and then, using the hypothesis that $f \in L_1(I\!R^d)$, select a $c_2 > 0$ for which

$$\sup_{\alpha \in I\!R^d} |y(\alpha) - \zeta| \int_{\|\beta\|_d \le c_1} |f(\alpha - \beta)|\, d\beta < \epsilon/2$$

[4] A complex-valued function x defined on $I\!R^d$ is uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ for which $|x(\alpha_1) - x(\alpha_2)| < \epsilon$ whenever $\|\alpha_1 - \alpha_2\|_d < \delta$.

for $\|\alpha\|_d > c_2$. Observe that for $\|\alpha\|_d > c_2$,

$$\left| \int_{I\!R^d} f(\alpha - \beta)[y(\beta) - \zeta]\, d\beta \right| \le \int_{\|\beta\|_d > c_1} |f(\alpha - \beta)[y(\beta) - \zeta]|\, d\beta + \int_{\|\beta\|_d \le c_1} |f(\alpha - \beta)[y(\beta) - \zeta]|\, d\beta < \epsilon/2 + \epsilon/2 = \epsilon,$$

which proves the proposition.

Proposition 2:

$$\lim_{\|\alpha\|_d \to \infty} \int_{I\!R^d} f(\alpha - \beta)y(\beta)\, d\beta = 0$$

for each $y \in BL_1(I\!R^d)$.

Proof of Proposition 2:

Both f and y belong to $L_2(I\!R^d)$. Using a version of the Parseval identity [11, p. 119],

$$(2\pi)^d \int_{I\!R^d} f(\alpha - \beta)y(\beta)\, d\beta = \int_{I\!R^d} \exp\{j(\omega \cdot \alpha)\}\hat{f}(\omega)\hat{y}(\omega)\, d\omega, \;\; \alpha \in I\!R^d \qquad (2)$$

in which $\omega \cdot \alpha$ stands for the dot product of ω and α, $j = \sqrt{-1}$, and $\hat{f}$ and $\hat{y}$ denote the Fourier transforms of f and x, respectively. These Fourier transforms belong to $L_2(I\!R^d)$. By the Schwarz inequality, z given by $z(\omega) = \hat{f}(\omega)\hat{y}(\omega)$ for all ω belongs to $L_1(I\!R^d)$. Therefore, by the Riemann-Lebesgue lemma for functions in $L_1(I\!R^d)$ [11, p. 57], the right side of (2) approaches zero as $\|\alpha\|_d \to \infty$, proving the proposition.

Proposition 3: There exists a continuous linear shift-invariant map $E : \mathcal{C} \to \mathcal{C}$ such that $(Ey)(\alpha) = \zeta$ for all $\alpha \in I\!R^d$ and each $y \in \mathcal{C}$ with $\lim_{\|\alpha\|_d \to \infty} y(\alpha) = \zeta$, in which ζ is an arbitrary complex number.

For a proof of this proposition, see the proof of the theorem in [13].

Continuing with the proof of the lemma, observe that

$$|(Fy)(\alpha_1) - (Fy)(\alpha_2)| \le \|y\|_\infty \int_{I\!R^d} |f(\alpha_1 - \beta) - f(\alpha_2 - \beta)|\, d\beta = \|y\|_\infty \int_{I\!R^d} |f(\beta) - f(\alpha_2 - \alpha_1 + \beta)|\, d\beta$$

in which the last integral approaches zero as $\|\alpha_2 - \alpha_1\|_d \to 0$. Thus, because $f \in L_1(I\!R^d)$, we see that F in fact maps into $\mathcal{C}$.

Set $H = EF$ with E as described in Proposition 3. We see that H is a linear shift-invariant continuous map of $L_\infty(I\!R^d)$ into $\mathcal{C}$, and therefore of $L_\infty(I\!R^d)$ into itself. By Propositions 2 and 3, the range of H restricted to $BL_1(I\!R^d)$ is the zero function, showing that part (a) of the lemma holds. Using Propositions 1 and 3, we see that part (b) also holds. This proves the Lemma.

It is of interest to note that we have proved a stronger result than is stated in the lemma, in that the range $H[L_\infty(\mathbb{R}^d)]$ of H can be taken to be contained in (the relatively simple space) $\mathcal{C}$.[5] There are several variations of Lemma 1. For example, for $d = 1$, one can show (using the result in [12]) that H can be taken to be causal.

References:

[1] W. M. Seibert, *Circuits, Signals, and Systems*, Cambridge, MA: MIT Press, 1997.

[2] A. V. Oppenheim, A. S. Willsky, and S. H. Nawab, *Signals and Systems*, Upper Saddle River, NJ: Prentice Hall, 1997.

[3] J. G. Proakis, and D. G. Manolakis, *Digital Signal Processing*, New York: Macmillan, 1992.

[4] M. J. Roberts, *Signals and Systems*, Boston: McGraw-Hill, 2004.

[5] I. W. Sandberg, "A Note on Representation Theorems for Linear Discrete-Space Systems," *Circuits, Systems, and Signal Processing*, vol. 17, no. 6, pp 703–708, 1998.

[6] __________, "The Superposition Scandal," *Circuits, Systems, and Signal Processing*, vol. 17, no. 6, pp. 733–735, 1998.

[7] __________, "Recent Representation Results for Linear System Maps: A Short Survey," *Journal of Circuit Theory and Applications*, September 2007 (to appear).

[8] M. Ciampa, M. Franciosi, and M. Poletti, "A Note on Impulse Response for Continuous, Linear, Time-Invariant, Continuous-Time Systems," *IEEE Transactions on Circuits and Systems I*, vol. 53, no. 1, pp. 106–113, 2006.

[9] I. W. Sandberg, "Bounded Inputs and the Representation of Linear System Maps," *Circuits, Systems, and Signal Processing*, vol. 24, no. 1, pp. 103–115, 2005.

[10] I. W. Sandberg, "On the Representation of Linear System Maps: Inputs that Need Not be Continuous," *Proceedings of the 6th WSEAS International Conference on Applied Mathematics*, Corfu Island, Greece, August 17–19, 2004 (five pages on CD).

[11] S. Bochner and K. Chandrasekharan, *Fourier Transforms*, Princeton: Princeton University Press, 1949.

[12] I. W. Sandberg, "Causality and the Impulse Response Scandal," *IEEE Transactions on Circuits and Systems I*, vol. 50, no. 6., pp. 810–811, 2003.

[13] I. W. Sandberg, "Continuous Multidimensional Systems and the Impulse Response Scandal," *Multidimensional Systems and Signal Processing*, 15, pp. 295–299, 2004.

[14] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Providence: American Mathematical Society, 1957.

[5] It is possible to prove Lemma 1 using an approach more along the lines of the proof in [13], but that route seems to require a delicate measurability argument involving the representation of linear functionals on $L_\infty(\mathbb{R}^d)$.