A comparison of magnetostatic field calculations associated with thick solenoids in the presence of iron using an integral formula derived in terms of the quaternion variable and a Maclaurin series solution.

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Abstract- The effect of iron on the uniformity of the field produced by an axisymmetric thick solenoid is considered. Using an integral equation derived for brevity using the quaternion variable of Hamilton the components of the magnetic induction are computed. The solution to the vector potential and field components is also achieved using a Maclaurin series expansion in ρ, the radial coordinate with numerical results using both methods of solution calculated.

Key Words: Time independent field, the quaternion variable, Maclaurin series, magnetic induction.

1 Introduction

The complex form of Green's theorem is:

$$\oint f(z,\bar{z})dz = 2i\iint_R \frac{\partial f}{\partial \bar{z}}dxdy$$

where $f(z,\bar{z})$ is a complex valued function that depends on z and its conjugate $\bar{z}$, with z = x + iy. The region R is bounded by the curve C where first order derivatives of are assumed continuous. Introducing a simple pole in R at z_0 and imposing that $f(z,\bar{z})$, has unit residue, then by construction:

$$f(z,\bar{z}) = w(z,\bar{z})g(z,z_0)$$

where g has unit residue at $z = z_0$ and $w(z,\bar{z})$ is analytic in R. By enclosing the singularity in a circle $\sum$ centre z_0 and with the usual connecting contour, then for this punctured region as shown in figure 1,

$$\oint_{C'} g(z,z_0)w(z,\bar{z})dz = \oint_C g(z,z_0)w(z,\bar{z})dz - \oint_\Sigma g(z,z_0)w(z,\bar{z})dz$$

$$\to \oint_C g(z,z_0)w(z,\bar{z})dz - 2\pi i w_0 \quad as\, \Sigma \to 0$$

so that

$$2\pi i w_0 = \oint_C g(z,z_0)w(z,\bar{z})dz - 2i\iint_R g(z,z_0)\frac{\partial w}{\partial \bar{z}}dxdy \qquad (1)$$

If the pole lies on the curve C then it can be shown using the Plemelj formulae, or by indenting the contour, that

$$\pi i w_0 = \oint_C g(z,z_0)w(z,\bar{z})dz - 2i\iint_R g(z,z_0)\frac{\partial w}{\partial \bar{z}}dxdy$$

Using this equation one would be able to solve a variety of potential rewritten as Fredholm integral equations of the first and second kind.

2 The three dimensional counterpart

Here the four dimensional quaternion operator of Hamilton will be used to derive the three dimensional counterpart of equation (1). The quaternion variable:

$[t, \underline{r}] = (t, x, y, z) = (t, ix, jy, kz)$

where $\underline{r} = \hat{i}x + \hat{j}y + \hat{k}z$, the separators i, j, k satisfy the following multiplication table

*	i	j	k
i	-1	k	-j
j	-k	-1	i
k	J	-i	-1

Where the * denotes multiplication, the separators must not be confused with the unit vectors $\hat{i}, \hat{j}, \hat{k}$. Writing the quaternion v as $[V, \underline{v}]$ where V is a scalar and $\underline{v}$ is the vector $\underline{v} = \hat{i}v_1 + \hat{j}v_2 + \hat{k}v_3$, then it follows that

$$vw = [V, \underline{v}][W, \underline{w}]$$
$$= [VW - \underline{v}.\underline{w}, V\underline{w} + W\underline{v} + \underline{v} \wedge \underline{w}]$$

where the . and $\wedge$ are the usual scalar and cross product of vectors. Thus for the vector operator $\underline{\nabla}$, it follows that

$$\int_R [0,\underline{\nabla}][V,\underline{v}]dv = \int_S [0,\underline{s}][V,\underline{v}]ds$$

i.e.

$$\int_R [-\underline{\nabla}.\underline{v}, V\underline{\nabla} + \underline{\nabla}\wedge\underline{v}]dv = \int_S [-\underline{n}.\underline{v}, V\underline{n} + \underline{n}\wedge\underline{v}]ds \quad (2)$$

Equation (2) is remarkable in the sense that this simple looking expression contains Gauss' divergence theorem and Stoke's theorem of vector calculus, where $\underline{n}$ is the unit outward normal to the surface S, enclosing the region R and dv and ds are the usual volume and surface differentials respectively. Equation (2) forms the basis of the three dimensional counterpart of equation (1). The functions involved will be dependent on two position vectors $\underline{P}$ and $\underline{Q}$, with W chosen to represent the reciprocal of the distance from $\underline{P}$ to $\underline{Q}$, so that

$W = \underline{r}^{-1}$

and $\underline{\nabla}_P W = -\underline{r}r^{-3}$,

so that $\underline{\nabla}_P^2 W = \underline{\nabla}_Q^2 W = 0$, where differentiation is carried out with respect to the coordinates of $\underline{P}$ with $\underline{Q}$ fixed denoted by the subscript p. When differentiating with respect to the coordinates of $\underline{Q}$ a suffix Q will be used. Now it can be shown that:

$$\begin{aligned}\underline{[0,\nabla]}[0,W\underline{w}] &= [0,\underline{\nabla}][W,0][0,\underline{w}] \\ &= [W,0][0,\underline{\nabla}][0,\underline{w}] \\ &\quad +[0,\underline{\nabla}][W,0][0,\underline{w}]\end{aligned}$$

integrating this last identity over a region R, bounded externally by a closed surface S and internally by a small sphere s_0 of radius r_0 and centre Q then

$$\int_{S+s_0} [0,\underline{n}][0,W\underline{w}]ds = \int_R [W,0][0,\underline{\nabla}][0,\underline{w}]dv - \int_R [0,\underline{\nabla}_Q][W,0][0,\underline{w}]dv \quad (3)$$

Applying the operator $[0,\underline{\nabla}_Q]$ to both sides of equation (3) gives the left hand side as

$$\begin{aligned}&\int_{S+s_0} [0,\underline{\nabla}_Q][0,\underline{n}][W,\underline{0}][0,\underline{w}]ds \\ &= \int_{S+s_0} [0,\underline{\nabla}_Q][W,\underline{0}][0,\underline{n}][0,\underline{w}]ds \\ &= -\int_{S+s_0} [0,\underline{\nabla W}][0,\underline{n}][0,\underline{w}]ds \\ &= \int_S [0,\underline{r}r^{-3}][0,\underline{n}][0,\underline{w}]ds - \\ &\int_{s_0} [0,\underline{n}r_Q^{-2}][0,\underline{n}][0,\underline{w}]ds \\ &= \int_S [0,\underline{r}r^{-3}][0,\underline{n}][0,\underline{w}]ds + [0,4\pi\underline{w}_0]\end{aligned}$$

as $r_Q \to 0$

similarly for the right hand side of equation (3) i.e.,

$$\begin{aligned}&\int_R [0,\underline{\nabla}_Q][W,0][0,\underline{\nabla}][0,\underline{w}]dv \\ &\quad - \int_R [\underline{\nabla}_Q^2, 0][W,0][0,\underline{w}]dv \\ &= \int_R [0,\underline{\nabla}_Q W][0,\underline{\nabla}][0,\underline{w}]dv - \int_R [\underline{\nabla}_Q^2 W, 0][0,\underline{w}]dv \\ &= \int_R [0,\underline{r}r^{-3}][0,\underline{\nabla}][0,\underline{w}]dv\end{aligned}$$

hence by equating both sides the following is valid

$$[0,4\pi\underline{w}_0] + \int_S [0,\underline{r}r^{-3}][0,\underline{n}][0,\underline{w}]ds = \int_R [0,\underline{r}r^{-3}][0,\underline{\nabla}][0,\underline{w}]dv \quad (4)$$

the vector part of equation (4) is the three dimensional counterpart of (1).

3 Application to magnetostatic field problem

Equation (4) will now be applied to calculate the filed components associated with an axisymmetric conductor of rectangular cross section situated equidistant from two semi-infinite regions of iron of finite permeability are computed. The magnetostatic field associated with iron-free axisymmetric systems has been considered by Boom and Livingstone [1], Garrett [2] and many others. Caldwell [3], Caldwell and Zisserman [4] and [5] have carried out work which takes account of the effects of the presence of iron on such systems. The main advantages of introducing iron are:

i. Higher fields are provided for the same current, producing substantial power savings over conventional conductors.

ii. The field uniformity is improved even for superconducting solenoids by placing the iron in a suitable position. The geometry considered is shown in figure 2, a toroidal conductor V' of rectangular cross section having inner radius A, outer radius B and length L-2ε, is located equidistant between two semi-infinite regions of iron of finite permeability a distance L apart, the axis of the torus being perpendicular to the iron boundaries. The region V between the conductor and the iron is assumed insulating. Cylindrical polar coordinates (ρ,ϕ,z) are used where ρ and z are normalized in terms of L.

Prior to Caldwell [3] the presence of iron in axisymmetric systems had been largely ignored see Loney [6] and Garrett [2] et al.

Using cylindrical coordinates (ρ,φ,z), for the conductor of figure 2 in the presence of iron of finite permeability, the vector part of equation (4) is:

$$4\pi \underline{B}_0(\underline{P}) = \int_S \left\{ \frac{\underline{r}}{|\underline{r}|^3} \wedge (\underline{n}\wedge\underline{B}) - \frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r} \right\} ds + \int_V \left\{ \frac{\underline{r}}{|\underline{r}|^3} \wedge (\underline{\nabla}\wedge\underline{B}) - \frac{(\underline{\nabla}.\underline{B})}{|\underline{r}|^3}\underline{r} \right\} dv \qquad (5)$$

the governing equations are those of Maxwell thus:

$$\underline{\nabla}\wedge\underline{B} = \begin{cases} 0 \ \ in\ V \\ -Ce_\phi \ \ in\ V' \end{cases}$$

where e_ϕ is a unit vector in the direction of increasing ϕ and C is a constant with

$\underline{\nabla}.\underline{B} = 0$ in V and V'

with boundary conditions

$\underline{n}\wedge\underline{B} = \underline{0}\ \ on\ \ z = 0,1$

$B_\rho(\rho,z) \to 0\ \ as\ \ \rho\to\infty$

$B_z(\rho,z) \to M\ \ as\ \ \rho\to\infty, (M\in\mathbb{R})$

The position vector of a point $\underline{r}$ in cylindrical coordinates is:

$$\underline{r} = (z-z')\hat{i} - x\sin\vartheta\hat{j} + (\rho - x\cos\vartheta)\hat{k}$$

and

$$|r|^3 = \left((z-z')^2 + x^2 + \rho^2 - 2x\rho\cos\vartheta\right)^{3/2}$$

considering the volume integral over V' in equation (5) and calculating the triple cross product gives

$$\underline{r}\wedge\underline{\nabla}\wedge\underline{B} = \mu_0 j\left((x-\rho\cos\vartheta)\hat{i} + (z-z')\sin\vartheta\hat{j} + (z-z')\cos\vartheta\hat{k}\right)$$

and hence the volume integral becomes

$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi} \frac{x(x-\rho\cos\vartheta)dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{i}$$
$$+$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi} \frac{x(z-z')\sin\vartheta dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{j}$$
$$+$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi} \frac{x(z-z')\cos\vartheta dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{k}$$

the $\hat{j}$ component vanishes due to the integrand being an odd function, so that

$$\int_V \left\{ \frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{\nabla}\wedge\underline{B}) \right\} dv =$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi} \frac{x(x-\rho\cos\vartheta)dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{i}$$
$$+$$
$$\mu_0 j\int_{\rho_a}^{\rho_b}\int_{z_a}^{z_b}\int_0^{2\pi} \frac{x(z-z')\cos\vartheta dxdz'd\vartheta}{((z-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\hat{k} \qquad (6)$$

Expanding these integrals in a Maclaurin series in ρ it can be shown (see Pavlika [8]) that the i and j components of expression (6) are given by

$$\mu_0 j\pi\left[\left[\begin{array}{l} 2x(w^2+x^2)^{1/2} - \rho\dfrac{(x^2+2w^2)}{xw(w^2+x^2)^{1/2}} \\ +\dfrac{\rho^2}{2!}\dfrac{(x^4+6x^2w^2+4w^4)}{xw(w^2+x^2)^{3/2}} \end{array}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b} + O(\rho^3)$$

and

$$\frac{\rho\mu_0 j\pi}{2}\left[\left[-\log\left|\frac{(x^2+w^2)^{1/2}-x}{(w^2+x^2)^{1/2}+x}\right| - \frac{2x^2w}{xw(w^2+x^2)^{3/2}}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}$$

$+O(\rho^3)$

where w=z-z'. Now to consider the surface integral S where

$$S = \int_{S_1}\left\{ \frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B}) - \frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r} \right\} ds + \int_{S_2}\left\{ \frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B}) - \frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r} \right\} ds + \int_{S_3}\left\{ \frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B}) - \frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r} \right\} ds \equiv S_1 + S_2 + S_3.$$

Where the discs s_i (i=1,2,3) are shown in figure 3. On the discs s_1 and s_2, $\underline{n}\wedge\underline{B} = \underline{0}$, so that the integral S_1 becomes

$$S_1 = \int_{S_1}\left\{ \frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B}) - \frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r} \right\} ds = -\int_{S_1}\frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}ds,$$

with z=0 on s_1. Using the expression for $\underline{r}$ with outward drawn normal to s_1 equal to –i, then

$$S_1 = -\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi} B_z\left(\frac{z'\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{(z'^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$

similarly S_2 (with z=1) becomes

$$S_2 = -\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi} B_z\left(\frac{(1-z')\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{((1-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$

in the integrals S_1 and S_2 the j component vanishes. Similarly on s_3 considering S_3 such that

$$S_3 = \int_{S_3}\left\{\frac{\underline{r}}{|\underline{r}|^3}\wedge(\underline{n}\wedge\underline{B}) - \frac{(\underline{n}.\underline{B})}{|\underline{r}|^3}\underline{r}\right\}ds$$

and using the vector identity:

$$\underline{A}\wedge(\underline{B}\wedge\underline{C}) = (\underline{A}.\underline{C})\underline{B} - (\underline{A}.\underline{B})\underline{C}$$
$$\Rightarrow \underline{r}\wedge(\underline{n}\wedge\underline{B}) - (\underline{n}.\underline{B})\underline{r} = (\underline{r}.\underline{B})\underline{n} - (\underline{r}.\underline{n})\underline{B} - (\underline{n}.\underline{B})\underline{r} \tag{7}$$

with outward drawn normal to s_3 radial, so that $\underline{n} = \sin\vartheta\hat{j} + \cos\vartheta\hat{k}$, equation (7) gives

$$\begin{pmatrix} B_z(-(\rho - x\cos\vartheta)\cos\vartheta + x\sin^2\vartheta) \\ -B_\rho\cos\vartheta(z-z') \end{pmatrix}\hat{i}$$
$$+$$
$$\begin{pmatrix} B_z\sin\vartheta(z-z') \\ +B_\rho\sin\vartheta((\rho - x\cos\vartheta) + x\sin\vartheta\cos\vartheta) \end{pmatrix}\hat{j}$$
$$+$$
$$\begin{pmatrix} B_z\cos\vartheta(z-z') \\ +B_\rho(\cos\vartheta(\rho - x\cos\vartheta) + x\sin^2\vartheta) \end{pmatrix}\hat{k}$$

so that

$$S_3 = \lim_{\rho\to\infty}\left\{\begin{pmatrix} B_z(-(\rho-x\cos\vartheta)\cos\vartheta+x\sin^2\vartheta) \\ -B_\rho\cos\vartheta(z-z') \end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{i}\right\}$$
$$+\lim_{\rho\to\infty}\left\{\begin{pmatrix} B_z\sin\vartheta(z-z') \\ +B_\rho\sin\vartheta((\rho-x\cos\vartheta)+x\sin\vartheta\cos\vartheta) \end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{j}\right\}$$
$$+\lim_{\rho\to\infty}\left\{\begin{pmatrix} B_z\cos\vartheta(z-z') \\ +B_\rho(\cos\vartheta(\rho-x\cos\vartheta)+x\sin^2\vartheta) \end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{k}\right\}$$

therefore

$$4\pi\underline{B}_0(\underline{P}) =$$
$$\hat{i}\mu_0 j\pi\left[\left[2x(w^2+x^2)^{1/2} - \rho\frac{(x^2+2w^2)}{xw(w^2+x^2)^{1/2}} + \frac{\rho^2}{2!}\frac{(x^4+6x^2w^2+4w^4)}{xw(w^2+x^2)^{3/2}}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}$$
$$+$$
$$\frac{\hat{j}\rho\mu_0 j\pi}{2}\left[\left[-\log\left|\frac{(x^2+w^2)^{1/2}-x}{(w^2+x^2)^{1/2}+x}\right| - \frac{2x^2w}{xw(w^2+x^2)^{3/2}}\right]_{\rho_a}^{\rho_b}\right]_{z_a}^{z_b}$$
$$-\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi} B_z\left(\frac{z'\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{(z'^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$
$$-\lim_{\rho\to\infty}\left\{\int_{z_a}^{z_b}\int_0^{2\pi} B_z\left(\frac{(1-z')\hat{i}+x\sin\vartheta\hat{j}-(\rho-x\cos\vartheta)\hat{k}}{((1-z')^2+x^2+\rho^2-2x\rho\cos\vartheta)^{3/2}}\right)\rho d\vartheta dz'\right\}$$
$$+\lim_{\rho\to\infty}\left\{\begin{pmatrix} B_z(-(\rho-x\cos\vartheta)\cos\vartheta+x\sin^2\vartheta) \\ -B_\rho\cos\vartheta(z-z') \end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{i}\right\}$$
$$+\lim_{\rho\to\infty}\left\{\begin{pmatrix} B_z\sin\vartheta(z-z') \\ +B_\rho\sin\vartheta((\rho-x\cos\vartheta)+x\sin\vartheta\cos\vartheta) \end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{j}\right\}$$
$$+\lim_{\rho\to\infty}\left\{\begin{pmatrix} B_z\cos\vartheta(z-z') \\ +B_\rho(\cos\vartheta(\rho-x\cos\vartheta)+x\sin^2\vartheta) \end{pmatrix}.\left(\frac{\rho d\vartheta dz'}{R^{3/2}}\right)\hat{k}\right\}$$
$$+O(\rho^3) \tag{8}$$

In this last equation the point $\underline{P}$ is allowed to occupy the boundary points $\underline{Q}$ giving rise to a diagonally dominant system of algebraic equations for the unknown field values B_z and B_ρ on the iron boundaries. Once these have determined, to calculate the field components off the iron boundaries at (ρ, z) the coordinates are input into expression (8) used as a formula. The respective field components $B_z(\rho, z)$ and $B_\rho(\rho, z)$ can then be determined near the axis.

4 Solution to the vector potential using a Maclaurin series in ρ

Using the integral representation of the vector potential this gives

$\underline{A}(\underline{r}) = \int_{V'}\frac{j(\underline{r})}{|\underline{r}-\underline{r}'|}dv'$, hence for finite μ,

$$A_\phi(\rho,z)=\frac{\mu_0 j}{4\pi}\sum_{n=-\infty}^{\infty}K^{|n|}\int_a^b\int_0^{2\pi}\int_\varepsilon^{1-\varepsilon}\frac{x\cos\vartheta\, dx\, d\vartheta\, dz'}{\{(z-z'-n)^2+\rho^2+x^2-2x\rho\cos\vartheta\}^{1/2}}$$

where $K=\dfrac{\mu-1}{\mu+1}$, known as the image factor. By expanding $A_\phi(\rho,z)$ in a Maclaurin series in ρ it follows that

$$A_\phi(\rho,z)=A_\phi(0,z)+\rho\frac{\partial A_\phi(0,z)}{\partial\rho}+\frac{\rho^2}{2!}\frac{\partial^2 A_\phi(0,z)}{\partial\rho^2}+O(\rho^3)$$

letting $I_n=\int_0^{2\pi}\frac{\cos\vartheta}{R}d\vartheta$,

where

$$R=\{(z-z'-n)^2+x^2+\rho^2-2x\rho\cos\vartheta\}^{1/2}$$

$$\Rightarrow A_\phi(\rho,z)=\frac{\mu_0 j\rho}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[w\log_e(x+(w^2+x^2)^{1/2})]_{x=b}^{a}]_{w=\varepsilon+n-z}^{1-\varepsilon+n-z}+O(\rho^2) \quad (9)$$

Similarly, it can be shown (see for example Pavlika [7]) that:

$$B_\rho(\rho,z)=\frac{\mu_0 j\rho}{4}\sum_{n=-\infty}^{\infty}K^{|n|}[[w\log_e(x+\alpha^{1/2})-x\alpha^{-3/2}]_{x=b}^{a}]_{w=\varepsilon+n-z}^{1-\varepsilon+n-z}+O(\rho^2) \quad (10)$$

where $\alpha=w^2+x^2$ and $w=z-z'-n$, and

$$B_z(\rho,z)=\frac{\mu_0 j}{2}\sum_{n=-\infty}^{\infty}K^{|n|}[[w\log_e(x+\alpha^{1/2})]_{x=b}^{a}]_{w=\varepsilon+n-z}^{1-\varepsilon+n-z}+O(\rho^2) \quad (11)$$

Results for $A_\phi(\rho,z)$, $B_\rho(\rho,z)$ and $B_z(\rho,z)$ using expressions (9), (10) and (11) with a=0.9, b=1.1, ϵ = 0.05 and μ_0j = 100 were found to be in good agreement with the solution using the quaternion method of solution as shown in tables 1, 2, 3 and 4.

5 Conclusions

The two methods of solution were found to be in good agreement. The summations were performed from -200 to 200 with a change only in the fourth decimal place occurring when the number of terms in the summation was doubled. The effect of the permeability of the iron is shown in figures 4, 5, 6 and 7.

6 References

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7 Tables

Table 1. Values of $B_\rho(\rho,z)$ using the Maclaurin Series Expansion.

ρ	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0.1	0.1	5.584-3	0.0127	0.0718	0.2815
0.2	0.1	1.131-2	0.0272	0.1472	0.5776
0.3	0.1	2.350E-2	0.0451	0.2297	0.9026
0.4	0.1	3.826-2	0.0680	0.3226	1.2710
0.5	0.1	5.896-2	0.0976	0.4297	1.6972
0.1	0.2	8.727-3	0.0141	0.0607	0.2316
0.1	0.3	8.493-3	0.0122	0.0443	0.1647
0.1	0.4	5.153-3	0.0070	0.0234	0.0855
0.1	0.5	0	0	0	0

Table 2. Values of $B_z(\rho,z)$ using the Maclaurin Series Expansion.

ρ	z	$\mu=10^3$	$\mu=10^2$	$\mu=1$
0	0.1	17.9169	17.6163	6.9821
0.1	0.1	17.0149	17.6150	7.0022
0.2	0.1	17.9090	17.6111	7.0627
0.3	0.1	17.8990	17.6046	7.1634
0.4	0.1	17.8851	17.5964	7.3045
0.5	0.1	17.8672	17.5838	7.4860
0.1	0.2	17.9731	17.6545	7.5232
0.1	0.3	17.9722	17.6770	7.9258
0.1	0.4	17.9860	17.6995	8.1802
0.1	0.5	17.9866	17.7014	8.2672

Table 3. Values of B_ρ (ρ,z) using the quaternion method of solution.

ρ	z	$\mu=10^3$	$\mu=10^2$	$\mu=10$	$\mu=1$
0.1	0.1	5.831E-3	0.0162	0.1041	0.0361
0.2	0.1	1.314E-2	0.0342	0.2119	0.0775
0.3	0.1	2.343E-2	0.0555	0.3673	0.1425
0.4	0.1	3.818E-2	0.0819	0.4520	0.1598
0.5	0.1	5.886E-2	0.1150	0.5913	2.0971
0.1	0.2	8.425E-3	0.0165	0.0851	0.2936
0.1	0.3	8.082E-3	0.0135	0.0606	0.2071
0.1	0.4	4.897E-3	0.0070	0.0315	0.0106
0.1	0.5	0	0	0	0

Table 4. Values of $B_z(\rho,z)$ using the quaternion method of solution

ρ	z	$\mu=10^3$	$\mu=10^2$	$\mu=1$
0	0.1	17.9167	17.6165	6.9822
0.1	0.1	17.0147	17.6154	7.0032
0.2	0.1	17.9092	17.6121	7.0637
0.3	0.1	17.8991	17.6048	7.1644
0.4	0.1	17.8853	17.5967	7.3055
0.5	0.1	17.8675	17.5838	7.487
0.1	0.2	17.9732	17.6549	7.5242
0.1	0.3	17.9723	17.6780	7.9268
0.1	0.4	17.9861	17.6998	8.1812
0.1	0.5	17.9867	17.7018	8.2682

9 Figures

Figure 1.The region R bounded by the curve C showing the singularity z_0 inside R

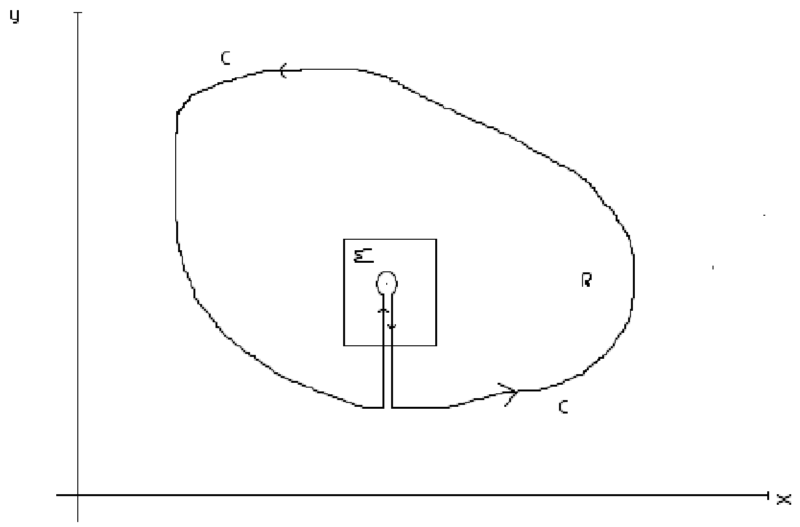

Figure 2. A toroidal conductor V' of rectangular cross section located midway between two semi infinite regions of iron of finite permeability. The region V is assumed to be insulating.

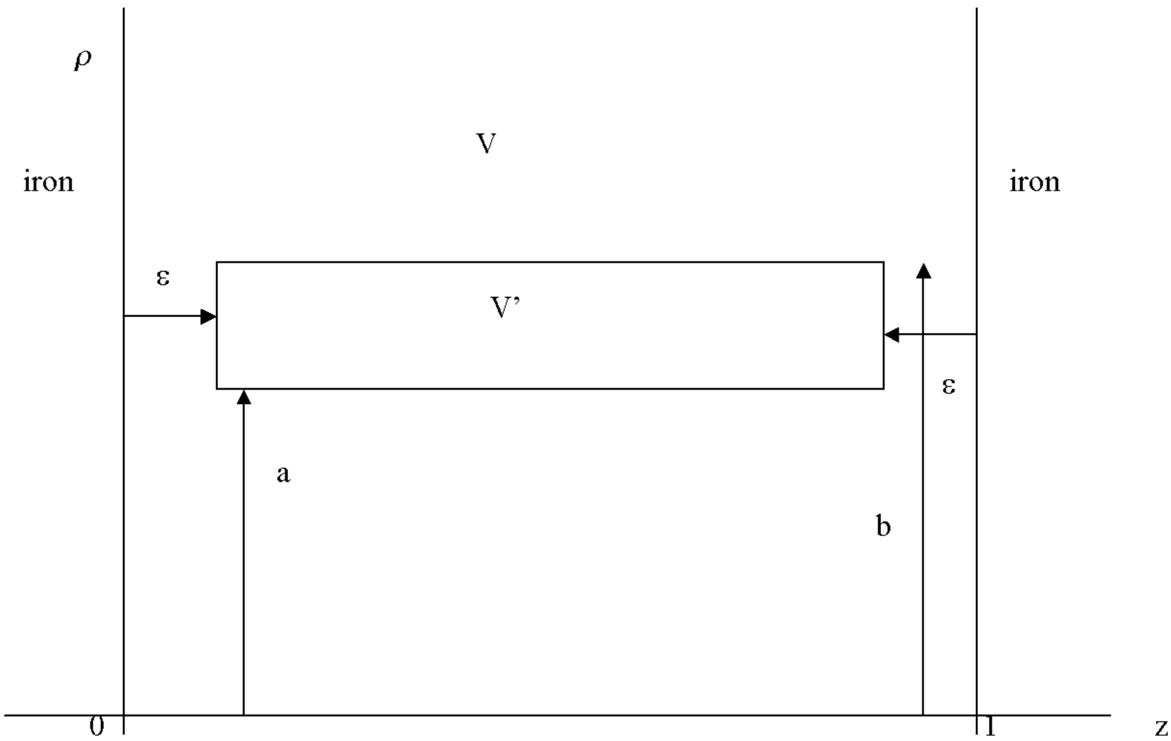

Figure 3. The volume of interest over which the integrations are performed.

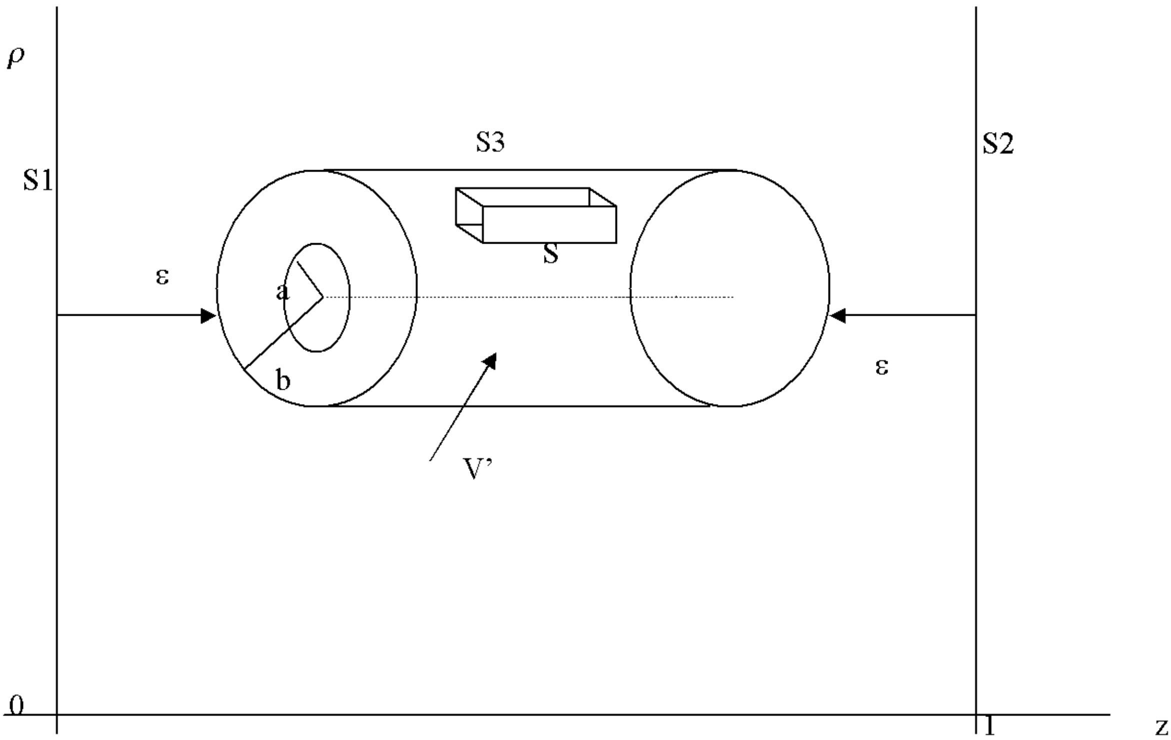

Figure 4. The variation of $B_z(\rho,z)$ with ρ and z for two semi-infinite regions of iron of unit permeability. ■:$\rho=0.3$, ✠:$\rho=0.2$, ●:$\rho=0.1$

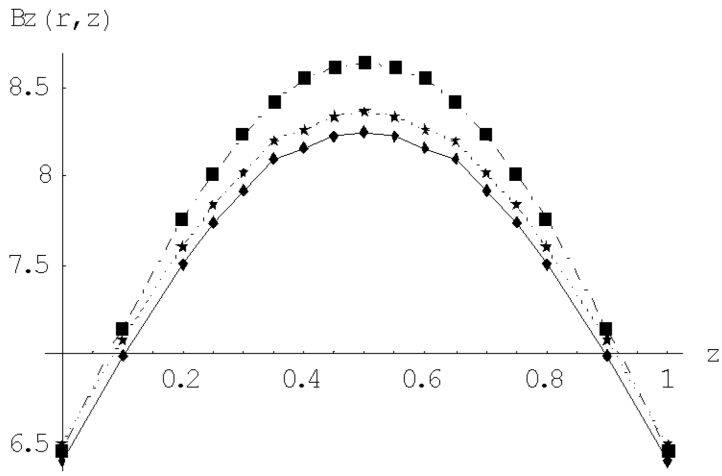

Figure 5. The variation of $B_z(\rho,z)$ with ρ and z for two semi-infinite regions of iron of infinite permeability. ■:$\rho=0.1$, ✠:$\rho=0.2$, ●:$\rho=0.3$

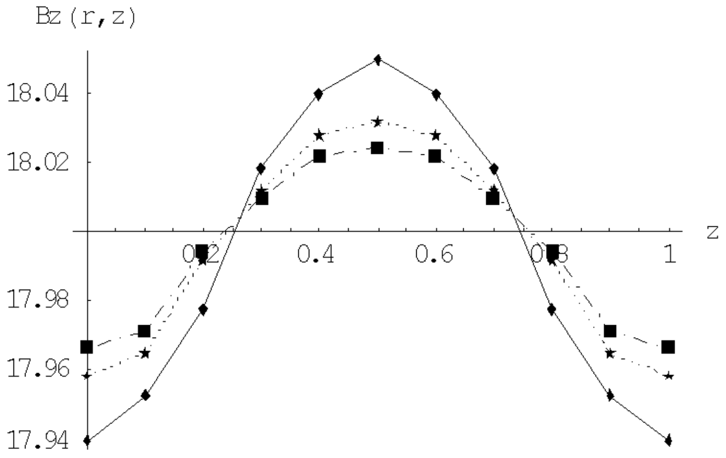

Figure 6. The variation of $B_r(\rho,z)$ with ρ and z for two semi-infinite regions of iron of unit permeability. ■:$\rho=0.1$, ✠:$\rho=0.2$, ●:$\rho=0.3$

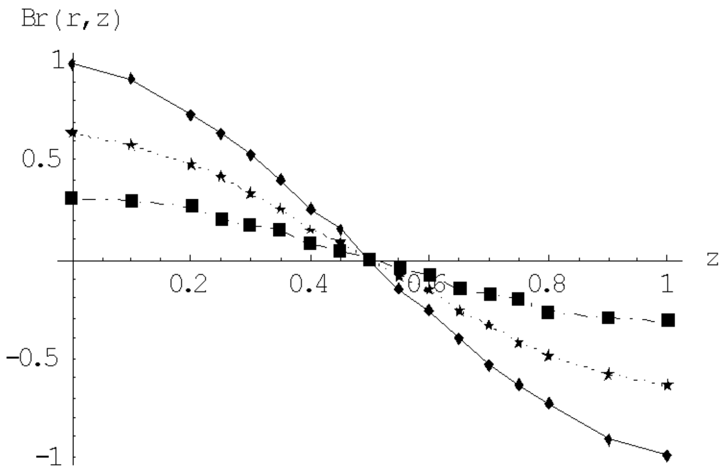

Figure 7. The variation of $B_\rho(\rho,z)$ with ρ and z for two semi-infinite regions of iron of infinite permeability. ■:$\rho=0.1$, ✠:$\rho=0.2$, ●:$\rho=0.3$