Aggregation Procedures in Intelligent Systems

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Abstract: The problem of aggregating information represented by fuzzy sets in a meaningful way has been of central interest since the late 1970s. In most cases, the aggregation operators are defined on a pure axiomatic basis and are interpreted either as logical connectives (such as t-norms and t-conorms) or as averaging operators allowing a compensation effect (such as the arithmetic mean). On the other hand, it can be observed by some empirical tests that the above-mentioned classes of operators differ from those ones that people use in practice. Therefore, it is important to find operators that are, in a sense, mixtures of the previous ones, and allow some degree of compensation. This paper summarizes the research results of the authors that have been carried out in recent years on generalization of conventional aggregation operators. This includes, but is not limited to, the class of uninorms and nullnorms, absorbing norms, distance- and entropy-based operators, quasi-conjunctions and nonstrict means.

Key–Words: t-norm, t-conorm, uninorm, entropy- and distance-based conjunctions and disjunctions, nonstrict means

1 Introduction

Aggregation of several inputs into a single output is an indispensable step in diverse procedures of mathematics, physics, engineering, economical, social and other sciences. Generally speaking, the problems of aggregation are very broad and heterogeneous.

The problem of *consistent aggregation* was posed by Klein [19] and, for ease of exposition, is formulated as follows, and is illustrated in Figure 1.

There are n inputs that contribute to the outputs of m producers. The jth producer's output depends upon the inputs $x_{j1}, \ldots, x_{jn}$ to that producer through possibly producer-specific (microeconomic) production functions F_j ($j = 1, \ldots, m$). The question is, do there exist aggregation functions for the outputs (G) and for each kind of inputs (G_k; $k = 1, \ldots, n$) so that the aggregated output depend only upon the n aggregated inputs through a macroeconomic function F.

We get the functional equation of $m \times n$ rectangular *generalized bisymmetry*:

$$G(F_1(x_{11}, \ldots, x_{1n}), \ldots, F_m(x_{m1}, \ldots x_{mn})) = \\ = F(G_1(x_{11}, \ldots, x_{m1}), \ldots, G_n(x_{1n}, \ldots x_{mn}))$$

Note that several district problems may lead to (particular forms of) this equation; see e.g. [13, 14].

Several questions can and should be asked. For example:

$$\begin{array}{ccccccc}
x_{11} & \ldots & x_{1k} & \ldots & x_{1n} & F_1 \longrightarrow & y_1 \\
\vdots & & \vdots & & \vdots & & \vdots \\
x_{j1} & \ldots & x_{jk} & \ldots & x_{jn} & F_j \longrightarrow & y_j \\
\vdots & & \vdots & & \vdots & & \vdots \\
x_{m1} & \ldots & x_{mk} & \ldots & x_{mn} & F_m \longrightarrow & y_m \\
 & & & & & & \\
G_1 & & G_k & & G_n & & G \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
z_1 & \ldots & z_k & \ldots & z_n & F \longrightarrow & u
\end{array}$$

Figure 1: Scheme of consistent aggregation.

1. Which of the functions $F_1, \ldots, F_m, G_1, \ldots, G_n, F, G$ should be considered as given and which as unknown?
2. What are the domains and ranges of these functions?
3. What kind of production functions $(F, F_1, \ldots, F_m)$ and aggregation functions $(G, G_1, \ldots, G_n)$ can appear during consistent aggregation?

Answers to these and similar questions have been given by many authors, especially by Maksa and his coauthors [1, 2, 23].

In this contribution, however, we restrict ourselves to information aggregation in intelligent systems.

The problem of aggregating information represented by membership functions (i.e., by fuzzy sets) in a meaningful way has been of central interest since the late 1970s. In most cases, the aggregation operators are defined on a pure axiomatic basis and are interpreted either as logical connectives (such as t-norms and t-conorms) or as averaging operators allowing a compensation effect (such as the arithmetic mean).

On the other hand, it can be recognized by some empirical tests that the above-mentioned classes of operators differ from those ones that people use in practice (see [28]). Therefore, it is important to find operators that are, in a sense, mixtures of the previous ones, and allow some degree of compensation.

One can also discern that people are inclined to use standard classes of aggregation operators also as a matter of routine. For example, when one works with binary conjunctions and there is no need to extend them for three or more arguments, as it happens e.g. in the inference pattern called generalized modus ponens, associativity of the conjunction is an unnecessarily restrictive condition. The same is valid for the commutativity property if the two arguments have different semantical backgrounds and it has no sense to interchange one with the other.

These observations advocate the study of enlarged classes of operations for information aggregation and have urged us to revise their definitions and study further properties.

2 Traditional Operations

The original fuzzy set theory was formulated in terms of Zadeh's standard operations of intersection, union and complement. The axiomatic skeleton used for characterizing fuzzy intersection and fuzzy union are known as *triangular norms (t-norms)* and *triangular conorms (t-conorms)*, respectively. For more details we refer to the books [11] and [21].

2.1 Triangular Norms and Conorms

Definition 1. *A* triangular norm *(shortly: a t-norm) is a function $T : [0,1]^2 \to [0,1]$ which is associative, increasing and commutative, and satisfies the boundary condition $T(1,x) = x$ for all $x \in [0,1]$.*

Definition 2. *A* triangular conorm *(shortly: a t-conorm) is an associative, commutative, increasing $S : [0,1]^2 \to [0,1]$ function, with boundary condition $S(0,x) = x$ for all $x \in [0,1]$.*

Notice that continuity of a t-norm and a t-conorm is not taken for granted.

The following are the four basic t-norms, namely, the minimum $T_{\mathbf{M}}$ the product $T_{\mathbf{P}}$, the Łukasiewicz t-norm $T_{\mathbf{L}}$, and the drastic product $T_{\mathbf{D}}$, which are given by, respectively:

$$\begin{aligned} T_{\mathbf{M}}(x,y) &= \min(x,y), \\ T_{\mathbf{P}}(x,y) &= x \cdot y, \\ T_{\mathbf{L}}(x,y) &= \max(x+y-1,0), \\ T_{\mathbf{D}}(x,y) &= \begin{cases} 0 & \text{if } (x,y) \in [\,0,1[\,^2, \\ \min(x,y) & \text{otherwise.} \end{cases} \end{aligned}$$

These four basic t-norms have some remarkable properties. The drastic product $T_{\mathbf{D}}$ and the minimum $T_{\mathbf{M}}$ are the smallest and the largest t-norm, respectively. The minimum $T_{\mathbf{M}}$ is the only t-norm where each $x \in [0,1]$ is an idempotent element. The product $T_{\mathbf{P}}$ and the Łukasiewicz t-norm $T_{\mathbf{L}}$ are prototypical examples of two important subclasses of t-norms (of strict and nilpotent t-norms, respectively).

Definition 3. *A non-increasing function $N : [0,1] \to [0,1]$ satisfying $N(0) = 1$, $N(1) = 0$ is called a* negation. *A negation N is called* strict *if N is strictly decreasing and continuous. A strict negation N is said to be a* strong negation *if N is also involutive: $N(N(x)) = x$ for all $x \in [0,1]$.*

The standard negation is simply $N_s(x) = 1 - x, \quad x \in [0,1]$. Clearly, this negation is strong. It plays a key role in the representation of strong negations.

We call a continuous, strictly increasing function $\varphi : [0,1] \to [0,1]$ with $\varphi(0) = 0$, $\varphi(1) = 1$ an *automorphism* of the unit interval.

Note that $N : [0,1] \to [0,1]$ is a strong negation if and only if there is an automorphism φ of the unit interval such that for all $x \in [0,1]$ we have

$$N(x) = \varphi^{-1}(N_s(\varphi(x))).$$

In what follows we assume that T is a t-norm, S is a t-conorm and N is a strong negation.

Clearly, for every t-norm T and strong negation N, the operation S defined by

$$S(x,y) = N(T(N(x),N(y))), \quad x,y \in [0,1] \quad (1)$$

is a t-conorm. In addition, $T(x,y) = N(S(N(x),N(y)))$ $(x,y \in [0,1])$. In this case S and T are called *N-duals*. In case of the standard negation (i.e., when $N(x) = 1-x$ for $x \in [0,1]$)

we simply speak about duals. Obviously, equality (1) expresses the De Morgan's law in the fuzzy case.

Generally, for any t-norm T and t-conorm S we have

$$T_{\mathbf{D}}(x,y) \leq T(x,y) \leq T_{\mathbf{M}}(x,y)$$

and

$$S_{\mathbf{M}}(x,y) \leq S(x,y) \leq S_{\mathbf{D}}(x,y),$$

where $S_{\mathbf{M}}$ is the dual of $T_{\mathbf{M}}$, and $S_{\mathbf{D}}$ is the dual of $T_{\mathbf{D}}$.

These inequalities are important from practical point of view as they establish the boundaries of the possible range of mappings T and S.

Between the four basic t-norms we have these strict inequalities:

$$T_{\mathbf{D}} < T_{\mathbf{P}} < T_{\mathbf{L}} < T_{\mathbf{M}}.$$

3 New Associative and Commutative Operations

3.1 Uninorms and Nullnorms

3.1.1 Uninorms

Uninorms were introduced by Yager and Rybalov [26] as a generalization of t-norms and t-conorms. For uninorms, the neutral element is not forced to be either 0 or 1, but can be any value in the unit interval.

Definition 4. [26] *A uninorm U is a commutative, associative and increasing binary operator with a neutral element $e \in [0,1]$, i.e., for all $x \in [0,1]$ we have $U(x,e) = x$.*

T-norms do not allow low values to be compensated by high values, while t-conorms do not allow high values to be compensated by low values. Uninorms may allow values separated by their neutral element to be aggregated in a compensating way. The structure of uninorms was studied by Fodor *et al.* [15]. For a uninorm U with neutral element $e \in]0,1]$, the binary operator T_U defined by

$$T_U(x,y) = \frac{U(e\,x, e\,y)}{e}$$

is a t-norm; for a uninorm U with neutral element $e \in [0,1[$, the binary operator S_U defined by

$$S_U(x,y) = \frac{U(e+(1-e)x, e+(1-e)y) - e}{1-e}$$

is a t-conorm. The structure of a uninorm with neutral element $e \in]0,1[$ on the squares $[0,e]^2$ and $[e,1]^2$ is therefore closely related to t-norms and t-conorms. For $e \in]0,1[$, we denote by ϕ_e and ψ_e the linear transformations defined by $\phi_e(x) = \frac{x}{e}$ and $\psi_e(x) = \frac{x-e}{1-e}$. To any uninorm U with neutral element $e \in]0,1[$, there corresponds a t-norm T and a t-conorm S such that:

(i) for any $(x,y) \in [0,e]^2$: $U(x,y) = \phi_e^{-1}(T(\phi_e(x), \phi_e(y)))$;

(ii) for any $(x,y) \in [e,1]^2$: $U(x,y) = \psi_e^{-1}(S(\psi_e(x), \psi_e(y)))$.

On the remaining part of the unit square, i.e. on $E = [0,e[\times]e,1] \cup]e,1] \times [0,e[$, it satisfies

$$\min(x,y) \leq U(x,y) \leq \max(x,y),$$

and could therefore partially show a compensating behaviour, i.e. take values strictly between minimum and maximum. Note that any uninorm U is either *conjunctive*, i.e. $U(0,1) = U(1,0) = 0$, or *disjunctive*, i.e. $U(0,1) = U(1,0) = 1$.

3.1.2 Representation of Uninorms

In analogy to the representation of continuous Archimedean t-norms and t-conorms in terms of additive generators, Fodor *et al.* [15] have investigated the existence of uninorms with a similar representation in terms of a single-variable function. This search leads back to Dombi's class of *aggregative operators* [9]. This work is also closely related to that of Klement *et al.* on associative compensatory operators [20]. Consider $e \in]0,1[$ and a strictly increasing continuous $[0,1] \to \overline{\mathbb{R}}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$. The binary operator U defined by

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$, and either $U(0,1) = U(1,0) = 0$ or $U(0,1) = U(1,0) = 1$, is a uninorm with neutral element e. The class of uninorms that can be constructed in this way has been characterized [15].

Consider a uninorm U with neutral element $e \in]0,1[$, then there exists a strictly increasing continuous $[0,1] \to \overline{\mathbb{R}}$ mapping h with $h(0) = -\infty$, $h(e) = 0$ and $h(1) = +\infty$ such that

$$U(x,y) = h^{-1}(h(x) + h(y))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$ if and only if

(i) U is strictly increasing and continuous on $]0,1[^2$;

(ii) there exists an involutive negator N with fixpoint e such that

$$U(x,y) = N(U(N(x), N(y)))$$

for any $(x,y) \in [0,1]^2 \setminus \{(0,1),(1,0)\}$.

The uninorms characterized above are called *representable* uninorms. The mapping h is called an *additive generator* of U. The involutive negator corresponding to a representable uninorm U with additive generator h, as mentioned in condition (ii) above, is denoted N_U and is given by

$$N_U(x) = h^{-1}(-h(x)). \tag{2}$$

Clearly, any representable uninorm comes in a conjunctive and a disjunctive version, i.e. there always exist two representable uninorms that only differ in the points $(0,1)$ and $(1,0)$. Representable uninorms are almost continuous, i.e. continuous except in $(0,1)$ and $(1,0)$, and Archimedean, in the sense that $(\forall x \in]0,e[)(U(x,x) < x)$ and $(\forall x \in]e,1[)(U(x,x) > x)$. Clearly, representable uninorms are not idempotent. The classes $U_{\min}$ and $U_{\max}$ do not contain representable uninorms. A very important fact is that the underlying t-norm and t-conorm of a representable uninorm must be strict and cannot be nilpotent. Moreover, given a strict t-norm T with decreasing additive generator f and a strict t-conorm S with increasing additive generator g, we can always construct a representable uninorm U with desired neutral element $e \in]0,1[$ that has T and S as underlying t-norm and t-conorm. It suffices to consider as additive generator the mapping h defined by

$$h(x) = \begin{cases} -f\left(\dfrac{x}{e}\right) & \text{, if } x \leq e \\ g\left(\dfrac{x-e}{1-e}\right) & \text{, if } x \geq e \end{cases}. \tag{3}$$

On the other hand, the following property indicates that representable uninorms are in some sense also generalizations of nilpotent t-norms and nilpotent t-conorms: $(\forall x \in [0,1])(U(x, N_U(x)) = N_U(e))$. This claim is further supported by studying the residual operators of representable uninorms in [8].

As an example of the representable case, consider the additive generator h defined by $h(x) = \log \frac{x}{1-x}$, then the corresponding conjunctive representable uninorm $\mathcal{U}$ is given by $U(x,y) = 0$ if $(x,y) \in \{(1,0),(0,1)\}$, and

$$U(x,y) = \frac{xy}{(1-x)(1-y)+xy}$$

otherwise, and has as neutral element $\frac{1}{2}$. Note that N_U is the standard negator: $N_U(x) = 1-x$.

The class of representable uninorms contains famous operators, such as the functions for combining certainty factors in the expert systems MYCIN (see [25, 7]) and PROSPECTOR [7]. The MYCIN expert system was one of the first systems capable of reasoning under uncertainty [4]. To that end, certainty factors were introduced as numbers in the interval $[-1,1]$. Essential in the processing of these certainty factors is the modified combining function C proposed by van Melle [4]. The $[-1,1]^2 \to [-1,1]$ mapping C is defined by

$$C(x,y) = \begin{cases} x + y(1-x) & \text{, if } \min(x,y) \geq 0 \\ x + y(1+x) & \text{, if } \max(x,y) \leq 0 \\ \dfrac{x+y}{1-\min(|x|,|y|)} & \text{, otherwise} \end{cases}.$$

The definition of C is not clear in the points $(-1,1)$ and $(1,-1)$, though it is understood that $C(-1,1) = C(1,-1) = -1$. Rescaling the function C to a binary operator on $[0,1]$, we obtain a representable uninorm with neutral element $\frac{1}{2}$ and as underlying t-norm and t-conorm the product and the probabilistic sum. Implicitly, these results are contained in the book of Hájek *et al.* [18], in the context of ordered Abelian groups.

3.1.3 Nullnorms

Definition 5. [5] *A nullnorm V is a commutative, associative and increasing binary operator with an absorbing element $a \in [0,1]$, i.e. $(\forall x \in [0,1])(V(x,a) = a)$, and that satisfies*

$$(\forall x \in [0,a])(V(x,0) = x) \tag{4}$$
$$(\forall x \in [a,1])(V(x,1) = x) \tag{5}$$

The absorbing element a corresponding to a nullnorm V is clearly unique. By definition, the case $a = 0$ leads back to t-norms, while the case $a = 1$ leads back to t-conorms. In the following proposition, we show that the structure of a nullnorm is similar to that of a uninorm. In particular, it can be shown that it is built up from a t-norm, a t-conorm and the absorbing element [5].

Theorem 6. *Consider $a \in [0,1]$. A binary operator V is a nullnorm with absorbing element a if and only if*

(i) *if $a = 0$: V is a t-norm;*

(ii) *if $0 < a < 1$: there exists a t-norm T_V and a t-conorm S_V such that $V(x,y)$ is given by*

$$\begin{cases} \phi_a^{-1}(S_V(\phi_a(x), \phi_a(y))) & \text{, if } (x,y) \in [0,a]^2 \\ \psi_a^{-1}(T_V(\psi_a(x), \psi_a(y))) & \text{, if } (x,y) \in [a,1]^2 \\ a & \text{, elsewhere} \end{cases}; \tag{6}$$

(iii) *if $a = 1$: V is a t-conorm.*

Recall that for any t-norm T and t-conorm S it holds that $T(x,y) \leq \min(x,y) \leq \max(x,y) \leq S(x,y)$, for any $(x,y) \in [0,1]^2$. Hence, for a nullnorm V with absorbing element a it holds that $(\forall(x,y) \in [0,a]^2)$ $(V(x,y) \geq \max(x,y))$ and $(\forall(x,y) \in [a,1]^2)$ $(V(x,y) \leq \min(x,y))$. Clearly, for any nullnorm V with absorbing element a it holds for all $x \in [0,1]$ that

$$V(x,0) = \min(x,a) \quad \text{and} \quad V(x,1) = \max(x,a). \tag{7}$$

Notice that, without the additional conditions (4) and (5), it cannot be shown that a commutative, associative and increasing binary operator V with absorbing element a behaves as a t-conorm and t-norm on the squares $[0,a]^2$ and $[a,1]^2$.

Nullnorms are a generalization of the well-known *median* studied by Fung and Fu [17], which corresponds to the case $T = \min$ and $S = \max$. For a more general treatment of this operator, we refer to [12]. We recall here the characterization of that median as given by Czogała and Drewniak [6]. Firstly, they observe that an idempotent, associative and increasing binary operator O has as absorbing element $a \in [0,1]$ if and only if $O(0,1) = O(1,0) = a$. Then the following theorem can be proven.

Theorem 7. [6] *Consider $a \in [0,1]$. A continuous, idempotent, associative and increasing binary operator O satisfies $O(0,1) = O(1,0) = a$ if and only if it is given by*

$$O(x,y) = \begin{cases} \max(x,y) & , \textit{if } (x,y) \in [0,a]^2 \\ \min(x,y) & , \textit{if } (x,y) \in [a,1]^2 \\ a & , \textit{elsewhere} \end{cases} .$$

Nullnorms are also a special case of the class of T-S aggregation functions introduced and studied by Fodor and Calvo [16].

Definition 8. *Consider a continuous t-norm T and a continuous t-conorm S. A binary operator F is called a T-S aggregation function if it is increasing and commutative, and satisfies the boundary conditions*

$$(\forall x \in [0,1])(F(x,0) = T(F(1,0),x))$$
$$(\forall x \in [0,1])(F(x,1) = S(F(1,0),x)).$$

When T is the algebraic product and S is the probabilistic sum, we recover the class of aggregation functions studied by Mayor and Trillas [24]. Rephrasing a result of Fodor and Calvo, we can state that the class of associative T-S aggregation functions coincides with the class of nullnorms with underlying t-norm T and t-conorm S.

4 Generalized Conjunctions and Disjunctions

4.1 The Role of Commutativity and Associativity

One possible way of simplification of axiom skeletons of t-norms and t-conorms may be not requiring that these operations to have the commutative and the associative properties. Non-commutative and non-associative operations are widely used in mathematics, so, why do we restrict our investigations by keeping these axioms? What are the requirements of the most typical applications?

From theoretical point of view the commutative law is not required, while the associative law is necessary to extend the operation to more than two variables. In applications, like fuzzy logic control, fuzzy expert systems and fuzzy systems modeling fuzzy rule base and fuzzy inference mechanism are used, where the information aggregation is performed by operations. The inference procedures do not always require commutative and associative laws of the operations used in these procedures. These properties are not necessary for conjunction operations used in the simplest fuzzy controllers with two inputs and one output. For rules with greater amount of inputs and outputs these properties are also not required if the sequence of variables in the rules are fixed.

Moreover, the non-commutativity of conjunction may in fact be desirable for rules because it can reflect different influences of the input variables on the output of the system. For example, in fuzzy control, the positions of the input variables the “error” and the “change in error” in rules are usually fixed and these variables have different influences on the output of the system. In the application areas of fuzzy models when the sequence of operands is not fixed, the property of non-commutativity may not be desirable. Later some examples will be given for parametric non-commutative and non-associative operations.

The axiom systems of t-norms and t-conorms are very similar to each other except the neutral element, i.e. the type is characterized by the neutral element. If the neutral element is equal to 1 then the operation is a conjunction type, while if the neutral element is zero the disjunction operation is obtained. By using these properties we introduce the concepts of conjunction and disjunction operations [3].

Definition 9. *Let T be a mapping $T : [0,1] \times [0,1] \to [0,1]$. T is a conjunction operation if $T(x,1) = x$ for all $x \in [0,1]$.*

Definition 10. *Let S be a mapping $S : [0,1] \times [0,1] \to$*

$[0,1]$. *S is a conjunction operation if $S(x,0) = x$ for all $x \in [0,1]$.*

Conjunction and disjunction operations may also be obtained one from another by means of an involutive negation N: $S(x,y) = N(T(N(x), N(y)))$, and $T(x,y) = N(S(N(x), N(y)))$.

It can be seen easily that conjunction and disjunction operations satisfy the following boundary conditions: $T(1,1) = 1$, $T(0,x) = T(x,0) = 0$, $S(0,0) = 0$, $S(1,x) = S(x,1) = 1$. By fixing these conditions, new types of generalized operations are introduced.

Definition 11. *Let T be a mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$. T is a quasi-conjunction operation if $T(0,0) = T(0,1) = T(1,0) = 0$, and $T(1,1) = 1$.*

Definition 12. *Let S be a mapping $S : [0,1] \times [0,1] \rightarrow [0,1]$. S is a quasi-disjunction operation if $S(0,1) = S(1,0) = S(1,1) = 1$, and $S(0,0) = 0$.*

It is easy to see that conjunction and disjunction operations are quasi-conjunctions and quasi-disjunctions, respectively, but the converse is not true.

Omitting $T(1,1) = 1$ and $S(0,0) = 0$ from the definitions further generalization can be obtained.

Definition 13. *Let T be a mapping $T : [0,1] \times [0,1] \rightarrow [0,1]$. T is a pseudo-conjunction operation if $T(0,0) = T(0,1) = T(1,0) = 0$.*

Definition 14. *Let S be a mapping $S : [0,1] \times [0,1] \rightarrow [0,1]$. S is a pseudo-disjunction operation if $S(0,1) = S(1,0) = S(1,1) = 1$.*

Theorem 15. *Assume that T and S are non-decreasing pseudo-conjunctions and pseudo-disjunctions, respectively. Then there exist the absorbing elements 0 and 1 such as $T(x,0) = T(0,x) = 0$ and $S(x,1) = S(1,x) = 1$.*

4.2 A Parametric Family of Quasi-Conjunctions

Let us cite the following result, which is the base of the forthcoming parametric construction, from [3].

Theorem 16. *Suppose T_1, T_2 are quasi-conjunctions, S_1 and S_2 are pseudo disjunctions and h, g_1, g_2 : [0,1] $\rightarrow$ [0,1] are non-decreasing functions such that $g_1(1) = g_2(1) = 1$. Then the following functions*

$$T(x,y) = T_2(T_1(x,y), S_1(g_1(x), g_2(y))) \quad (8)$$

$$T(x,y) = T_2(T_1(x,y), g_1 S_1(x,y)) \quad (9)$$

$$T(x,y) = T_2(T_1(x,y), S_2(h(x), S_1(x,y))) \quad (10)$$

are quasi-conjunctions.

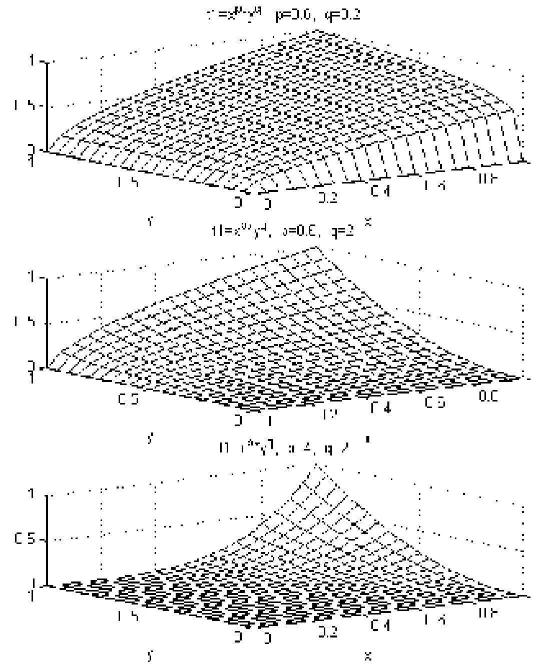

Figure 2: Surfaces of conjunction operation $T(x,y) = x^p y^q$ for different values of p.

By the use of this Theorem the simplest parametric *quasi*-conjunction operations can be obtained as follows:

$$T(x,y) = x^p y^q, \quad (11)$$

$$T(x,y) = \min(x^p, y^q), \quad (12)$$

$$T(x,y) = (xy)^p (x + y - xy)^q \quad (13)$$

where $p, q \geq 0$.

The surfaces of conjunction (11) are is shown in Fig. 2.

4.3 Distance-based Operations

Let e be an arbitrary element of the closed unit interval $[0,1]$ and denote by $d(x,y)$ the distance of two elements x and $y$of $[0,1]$. The idea of definitions of distance-based operators is generated from the reformulation of the definition of the min and max operators as follows

$$\min(x,y) = \begin{cases} x, & \text{if } d(x,0) \leq d(y,0) \\ y, & \text{if } d(x,0) > d(y,0) \end{cases},$$

$$\max(x,y) = \begin{cases} x, & \text{if } d(x,0) \geq d(y,0) \\ y, & \text{if } d(x,0) < d(y,0) \end{cases}$$

Based on this observation the following definitions can be introduced, see [3].

Definition 17. *The* maximum distance minimum operator *with respect to* $e \in [0,1]$ *is defined as*

$$\max_e^{\min}(x,y) = \begin{cases} x, & \text{if } d(x,e) > d(y,e) \\ y, & \text{if } d(x,e) < d(y,e) \\ \min(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{14}$$

Definition 18. *The* maximum distance maximum operator *with respect to* $e \in [0,1]$ *is defined as*

$$\max_e^{\max}(x,y) = \begin{cases} x, & \text{if } d(x,e) > d(y,e) \\ y, & \text{if } d(x,e) < d(y,e) \\ \max(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{15}$$

Definition 19. *The* minimum distance minimum operator *with respect to* $e \in [0,1]$ *is defined as*

$$\min_e^{\min}(x,y) = \begin{cases} x, & \text{if } d(x,e) < d(y,e) \\ y, & \text{if } d(x,e) > d(y,e) \\ \min(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{16}$$

Definition 20. *The* minimum distance maximum operator *with respect to* $e \in [0,1]$ *is defined as*

$$\min_e^{\max}(x,y) = \begin{cases} x, & \text{if } d(x,e) < d(y,e) \\ y, & \text{if } d(x,e) > d(y,e) \\ \max(x,y), & \text{if } d(x,e) = d(y,e) \end{cases}. \tag{17}$$

4.3.1 The Structure of Distance-based Operators

It can be proved by simple computation that if the distance of x and $y$is defined as $d(x,y) = |x-y|$ then the distance-based operators can be expressed by means of the min and max operators as follows.

$$\max_e^{\min} = \begin{cases} \max(x,y), & \text{if } y > 2e - x \\ \min(x,y), & \text{if } y < 2e - x \\ \min(x,y), & \text{if } y = 2e - x \end{cases}, \tag{18}$$

$$\min_e^{\min} = \begin{cases} \min(x,y), & \text{if } y > 2e - x \\ \max(x,y), & \text{if } y < 2e - x \\ \min(x,y), & \text{if } y = 2e - x \end{cases} \tag{19}$$

$$\max_e^{\max} = \begin{cases} \max(x,y), & \text{if } y > 2e - x \\ \min(x,y), & \text{if } y < 2e - x \\ \max(x,y), & \text{if } y = 2e - x \end{cases}, \tag{20}$$

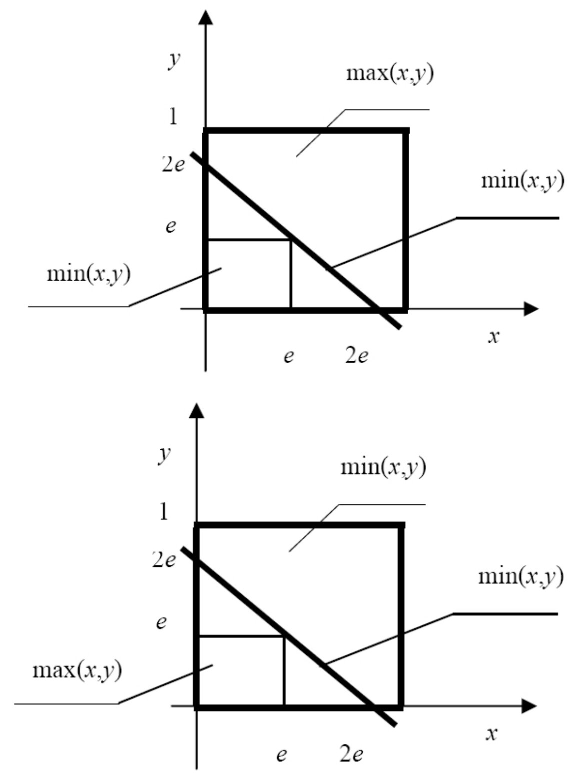

Figure 3: Maximum distance minimum operator (up) and minimum distance minimum operator (down).

$$\min_e^{\max} = \begin{cases} \min(x,y), & \text{if } y > 2e - x \\ \max(x,y), & \text{if } y < 2e - x \\ \max(x,y), & \text{if } y = 2e - x \end{cases} \tag{21}$$

The structures of the $\max_e^{\min}$ and the $\min_e^{\min}$ operators are illustrated in Fig. 3.

4.4 Entropy-based Conjunction and Disjunction Operators

The question of how fuzzy is a fuzzy set has been one of the issues associated with the development of the fuzzy set theory. In accordance with a current terminological trend in the literature, measure of uncertainty is being referred as *measure of fuzziness*, or *fuzzy entropy* [22].

Throughout this part the following notations will be used; X is the universal set, $\boldsymbol{F}(X)$ is the class of all fuzzy subsets of X , $\Re^+$ is the set of non negative real numbers, $\bar{A}$ is the fuzzy complement of $A \in \boldsymbol{F}(X)$ and $|A|$ is the cardinality of A.

Definition 21. *Let* X*be a universal set and* A *is a fuzzy subset of* X *defined as*

$$A = \{(x, \mu_A(x)) \,|\, x \in X\}.$$

The fuzzy entropy is a mapping

$$e: \boldsymbol{F}(X) \quad \rightarrow \quad \Re^{+}$$

which satisfies the following axioms:

AE 1 *$e(A) = 0$ if A is a crisp set.*

AE 2 *If $A \prec B$ then $e(A) \leq e(B)$; where $A \prec B$ means that A is sharper than B.*

AE 3 *$e(A)$ assumes its maximum value if and only if A is maximally fuzzy.*

AE 4 *$e(A) = e(\bar{A}), \forall A \in X$.*

Let e_p be equilibrium of the fuzzy complement C and specify **AE 2** and **AE 3** as follows:

AES 2 *A is sharper than B* in the following sense: $\mu_A(x) \leq \mu_B(x)$ for $\mu_B(x) \leq e_p$ and $\mu_A(x) \geq \mu_B(x)$ for $\mu_B(x) \geq e_p$, for all $x \in X$.

AES 3 A is defined maximally fuzzy when $\mu_A(x) = e_p \; \forall x \in X$.

Let A be a fuzzy subset of X and define the following function $f_A : X \rightarrow$ [0,1] by

$$f_A : x \mapsto \begin{cases} \mu_A(x) & \text{if } \mu_A(x) \leq e_p \\ C(\mu_A(x)) & \text{if } \mu_A(x) > e_p \end{cases} \quad (22)$$

Denote Φ_A the fuzzy set generated by f_A as its membership function.

Theorem 22. *The $g(|\Phi_A|)$ is an entropy, where $g{:}\Re \rightarrow \Re$ is a monotonically increasing real function and $g(0) = 0$.*

Definition 23. *Let A be a fuzzy subset of X. f_A is said to be an elementary fuzzy entropy function if the cardinality of the fuzzy set $\Phi_A = \{(x, f_A(x)) \,|\, x \in X, \; f_A(x) \in [0,1]\}$ is an entropy of A.*

It is obvious that f_A is an elementary entropy function.

Now we introduce some operations based on entropy. For more details we refer to [3].

Definition 24. *Let A and B be two fuzzy subsets of the universe of discourse X and denote φ_A and φ_B their elementary entropy functions, respectively. The minimum entropy conjunction operations is defined as $I_\varphi^* = I_\varphi^*(A,B) = \left\{\left(x, \mu_{I_\varphi^*}(x)\right) | x \in X, \; \mu_{I_\varphi^*}(x) \in [0,1]\right\}$, where*

$$\mu_{I_\varphi^*}: x \mapsto \begin{cases} \mu_A(x), & \textit{if } \varphi_A(x) < \varphi_B(x) \\ \mu_B(x), & \textit{if } \varphi_B(x) < \varphi_A(x) \\ \textit{min}(\mu_A(x), \mu_B(x)), & \textit{if } \varphi_A(x) = \varphi_B(x) \end{cases} \quad (23)$$

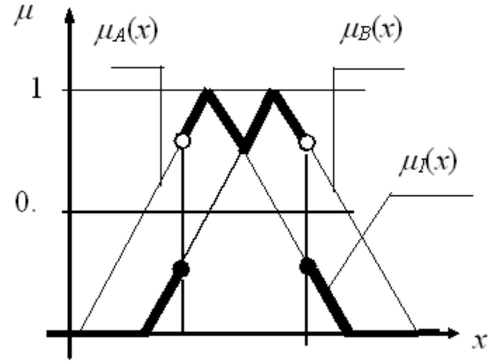

Figure 4: Entropy based conjunction operator

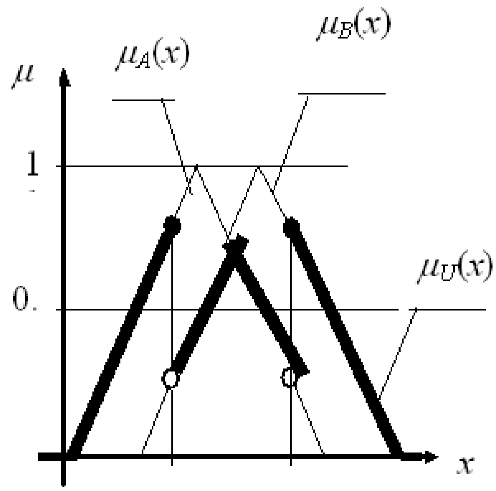

Figure 5: Entropy based disjunction operator

The geometrical representation of the minimum fuzziness generalized intersection can be seen in Fig. 4.

Definition 25. *Let A and Bbe two fuzzy subsets of the universe of discourse X and denote φ_A and φ_B their elementary entropy functions, respectively. The maximum entropy disjunction operation is defined as $U_\varphi^* = U_\varphi^*(A,B) = \left\{\left(x, \mu_{U_\varphi^*}(x)\right) | x \in X, \, \mu_{U_\varphi^*}(x) \in [0,1]\right\}$, where*

$$\mu_{U_\varphi^*}{:}x \mapsto \begin{cases} \mu_A(x), & \textit{if } \varphi_A(x) > \varphi_B(x) \\ \mu_B(x), & \textit{if } \varphi_B(x) > \varphi_A(x) \\ \textit{max}(\mu_A(x), \mu_B(x)), & \textit{if } \varphi_A(x) = \varphi_B(x) \end{cases} . \quad (24)$$

The geometrical representation of the maximum fuzziness operation can be seen in Fig. 5.

Several important properties of these operations as well as their construction can be found in [3]. Now

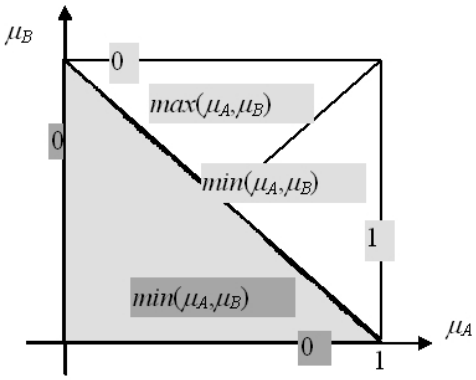

Figure 6: The construction of I^*_φ .

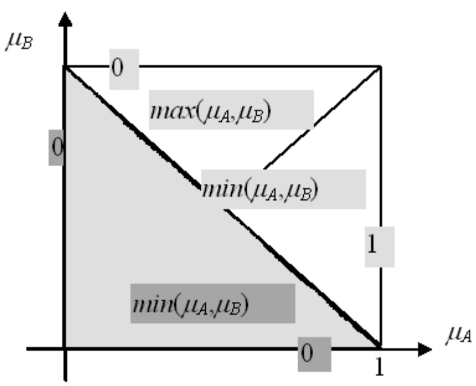

Figure 7: The construction of U^*_φ .

we present only two figures about the construction.

Notice also that I^*_φ is a quasi-conjunction, U^*_φ is a quasi-disjunction operation, and U^*_φ is a commutative semigroup operation on $[0, 1]$ [3].

5 Nonstrict Means

There exists a functional equation characteristic to means: the *bisymmetry* equation. It can also be considered as a generalization of simultaneous commutativity and associativity. This functional equation has been investigated by several authors. For a list of references see Aczél (1966). The equation is given as follows:

$$M[M(x, y), M(u, v)] = M[M(x, u), M(y, v)], \quad (25)$$

where M is a function from $[a, b]^2$ to $[a, b]$ ($a < b$ are real numbers). This is a particular case of the consistent aggregation equation. Without loss of generality, we restrict ourselves to the case $a = 0, b = 1$ in this paper.

This equation is used for characterizing *quasi-arithmetic means*

$$M(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right), \quad (26)$$

and in general, *quasilinear functions*

$$M(x, y) = f^{-1}\left(Af(x) + Bf(y) + C\right), \quad (27)$$

where f is a continuous and strictly monotonic function, f^{-1} is its inverse, and A, B, C are real constants such that $A \neq 0, B \neq 0$.

Note that (27) contains the following particular cases:

a) $A = B = 1/2$, $C = 0$: quasi-arithmetic means;

b) $A + B = 1$, $A, B \geq 0$, $C = 0$: quasilinear means;

c) $A = B = 1$, $C = 0$: strict t-norms.

The following properties of a function $M : [0, 1]^2 \to [0, 1]$ play important role in the sequel. Such an M is called

- *reducible on both sides* if $M(t, z_1) \neq M(t, z_2)$, $M(z_1, t) \neq M(z_2, t)$ hold for $z_1 \neq z_2$;
- *nondecreasing* if $x \leq t$ and $y \leq z$ imply $M(x, y) \leq M(t, z)$;
- *Archimedean* if M is continuous and $\max\{M(x, 0), M(0, x)\} < x < \min\{M(x, 1), M(1, x)\}$ for all $x \in]0, 1[$;
- *internal* if $x < M(x, y) < y$ holds for $x < y$, $x, y \in]0, 1[$.

It has been proved in Aczél (1966) that the quasi-arithmetic mean is the general continuous, on both sides reducible, real solution of the bisymmetry equation, under the additional conditions of idempotency and symmetry. It was also noted that reducibility on both sides can be replaced by internality to have the same representation.

The aim of the present section is to recall the general continuous, idempotent, symmetric and nondecreasing real solutions of the bisymmetry equation originally published in [14].

5.1 Archimedean Bisymmetric Functions

Before determining the general nonstrict solutions, we need an equivalent form of the representation theorem, and another characterization of the quasi-arithmetic means when internality is replaced by the Archimedean property.

Let us denote by $\mathcal{M}_\alpha$ the class of all continuous, idempotent, symmetric, nondecreasing real functions

$M : [0,1]^2 \to [0,1]$ which satisfy the bisymmetry equation and have the boundary condition $M(0,1) = \alpha$ ($\alpha \in [0,1]$ is fixed). Let $\mathcal{M} = \bigcup_{\alpha\in[0,1]} \mathcal{M}_\alpha$. Then any member of $\mathcal{M}$ is called a *nonstrict bisymmetric mean*.

Theorem 26. *A function $M \in \mathcal{M}$ is Archimedean if and only if there exists a continuous and strictly monotonic real function f defined on the unit interval such that representation (26) holds.*

The general solution (26) can be expressed also in the following form that will be useful in the sequel.

Theorem 27. *The general continuous, Archimedean, idempotent, symmetric real solution M of the bisymmetry equation is given by the following form*

(a) $M(x,y) = \varphi^{-1}(\sqrt{\varphi(x)\varphi(y)})$ *if* $M(0,1) = 0$*;*

(b) $M(x,y) = 1 - \varphi^{-1}(\sqrt{\varphi(1-x)\varphi(1-y)})$ *if* $M(0,1) = 1$*;*

(c) $M(x,y) = \varphi^{-1}\left(\dfrac{\varphi(x)+\varphi(y)}{2}\right)$ *if* $0 < M(0,1) < 1$,

where φ is an automorphism of the unit interval.

This theorem says that there are three basic classes of operations satisfying the mentioned properties: the first one contains means isomorphic to the *geometric mean*; the second consists of means isomorphic to the *dual of the geometric mean*; finally, the third class contains means isomorphic to the *arithmetic mean*.

5.2 Characterization of the Class $\mathcal{M}_0$

Suppose that $M \in \mathcal{M}_0$ (that is, $M(0,1) = 0$).

Let φ_i be an automorphism of the unit interval. For typographic reason, we introduce the following short notation when x and y are in $[a_i, b_i] \subseteq [0,1]$:

$$L_i(x,y) := \varphi_i^{-1}\left(\sqrt{\varphi_i\left(\frac{x-a_i}{b_i-a_i}\right)\varphi_i\left(\frac{y-a_i}{b_i-a_i}\right)}\right).$$

We summarize the characterization of the class $\mathcal{M}_0$ in the following theorem.

Theorem 28. *A function M belongs to $\mathcal{M}_0$ if and only if either*

(a) there exists an automorphism φ of the unit interval such that $M(x,y) = \varphi^{-1}(\sqrt{\varphi(x)\varphi(y)})$*,*

or

(b) $M(x,y) = \min(x,y)$*,*

or

(c) there exist an index set K, a family of disjoint subintervals $\{]a_i,b_i[\}$ of $[0,1]$ and for each $i \in K$ an automorphism φ_i of the unit interval such that
$M(x,y) = a_i + (b_i - a_i)L_i(\min(x,b_i), \min(y,b_i))$ *if* $\min(x,y) \in (a_i,b_i)$*, and otherwise*
$M(x,y) = \min(x,y)$*.*

The graphical illustration of this theorem can be seen in Figure 8.

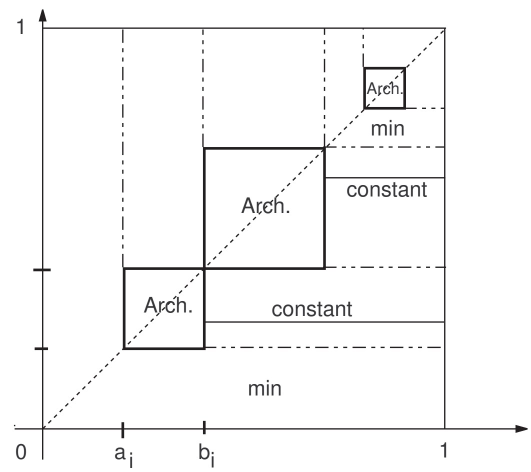

Figure 8: Illustration of Theorem 28

5.3 Characterization of the Class $\mathcal{M}_1$

Turning to the case $M \in \mathcal{M}_1$, we can use the above result for $\mathcal{M}_0$. Indeed, one can easily prove that $M \in \mathcal{M}_1$ if and only if $M^* \in \mathcal{M}_0$ with

$$M^*(x,y) = 1 - M(1-x, 1-y).$$

Therefore, characterization of the class $\mathcal{M}_1$ is immediately obtained by Theorem 28. This is summarized in the following theorem.

Theorem 29. *A function M belongs to $\mathcal{M}_1$ if and only if either*

(a) there exists an automorphism φ of the unit interval such that
$M(x,y) = 1 - \varphi^{-1}(\sqrt{\varphi(1-x)\varphi(1-y)})$*,*

or

(b) $M(x,y) = \max(x,y)$*,*

or

(c) there exist an index set K, a family of disjoint subintervals $\{]a_i,b_i[\}$ of $[0,1]$ and for each $i \in K$ an automorphism φ_i of the unit interval such that
$M(x,y) = a_i + (b_i - a_i)M_i(\max(x,a_i), \max(y,a_i))$ *if* $\max(x,y) \in (a_i,b_i)$*, and otherwise we have*
$M(x,y) = \max(x,y)$*,*

where $M_i(u,v) = 1 - L_i(1-u, 1-v)$ *for* $u, v \in [a_i, b_i]$*.*

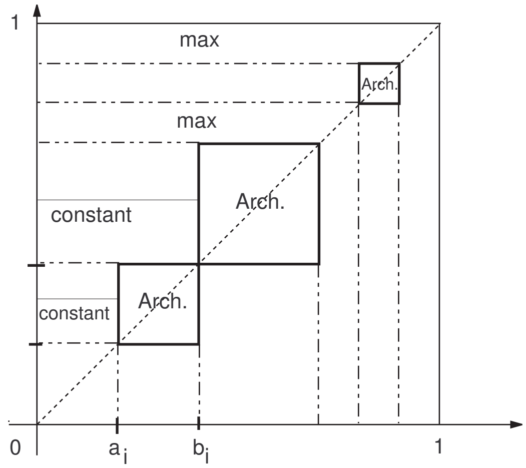

Figure 9: Illustration of Theorem 29

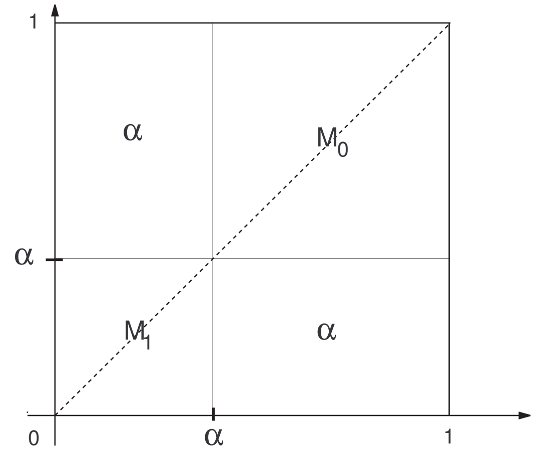

Figure 10: Illustration of Theorem 30

5.4 Characterization of the Class $\mathcal{M}_\alpha$ for $0 < \alpha < 1$

In this section we characterize the class $\mathcal{M}_\alpha$ with $0 < \alpha < 1$.

Define a set $X \subseteq [0, 1]$ by

$$X = \{x \in [0, 1] \mid M(x, 1) = x \text{ or } M(x, 0) = x\}.$$

Obviously, $0, 1 \in X$ since $M(0, 0) = 0$ and $M(1, 1) = 1$. Moreover, X is closed, by continuity of M. Therefore, $Y = [0, 1] \setminus X$ is open and bounded. Thus, there exists an index set K and a family of non-overlapping open intervals $\{]a_i, b_i[\}_{i \in K}$ such that

$$Y = \bigcup_{i \in K}]a_i, b_i[.$$

Since $0 < M(0, 1) = \alpha < 1$, there are two possibilities: either $\alpha \in X$ (that is, we have $M(\alpha, 1) = \alpha$ or $M(\alpha, 0) = \alpha$), or there exists an index $j \in K$ such that $\alpha \in]a_j, b_j[$.

First we state the main theorem when $\alpha \in X$.

Theorem 30. *If $\alpha \in X$ then there exist $M_0 \in \mathcal{M}_0$ and $M_1 \in \mathcal{M}_1$ such that*

$$M(x, y) = \begin{cases} \alpha M_1\left(\frac{x}{\alpha}, \frac{y}{\alpha}\right) & \textit{if } x, y \le \alpha \\ \alpha + (1 - \alpha) M_0\left(\frac{x-\alpha}{1-\alpha}, \frac{y-\alpha}{1-\alpha}\right) & \textit{if } x, y \ge \alpha \\ \alpha & \textit{otherwise.} \end{cases}$$

Let us turn now to the remaining case when there exists an $]a_j, b_j[$ such that $\alpha \in]a_j, b_j[$. Denote this interval simply by $]a, b[$ for short. Notice that we have $M(0, \alpha) < \alpha < M(1, \alpha)$ now.

The main result related to the present subcase can be stated as follows.

Theorem 31. *Suppose α is such that $0 < \alpha < 1$, and $M(0, \alpha) < \alpha < M(1, \alpha)$. Then $M \in \mathcal{M}_\alpha$ if and only if there exist $M_0 \in \mathcal{M}_0$, $M_1 \in \mathcal{M}_1$ and numbers $0 \le a < b \le 1$, such that*

$M(x, y) = a M_0\left(\dfrac{x}{a}, \dfrac{y}{a}\right)$ *if* $x, y \in [0, a]$*;*

$M(x, y) = a + (b - a)\varphi^{-1}\left(\dfrac{\varphi\left(\frac{x-a}{b-a}\right) + \varphi\left(\frac{y-a}{b-a}\right)}{2}\right)$ *if* $x, y \in [a, b]$*;*

$M(x, y) = b + (1 - b) M_1\left(\dfrac{x - b}{1 - b}, \dfrac{y - b}{1 - b}\right)$ *if* $x, y \in [b, 1]$*;*

$M(x, y) = \alpha$ *if* $\min(x, y) \le a$, $\max(x, y) \ge b$*;*

$M(x, y) = a + (b - a)\varphi^{-1}\left(\dfrac{1}{2} \cdot \varphi\left(\dfrac{y - a}{b - a}\right)\right)$ *if* $x \in [0, a]$, $y \in [a, b]$*;*

$M(x, y) = a + (b - a)\varphi^{-1}\left(\dfrac{1}{2} + \dfrac{1}{2} \cdot \varphi\left(\dfrac{y - a}{b - a}\right)\right)$ *if* $x \in [b, 1]$, $y \in [a, b]$.

6 Conclusion

In this paper we summarized some of our contributions to the theory of non-conventional aggregation operators. Further details and another classes of aggregation operators can be found in [3].

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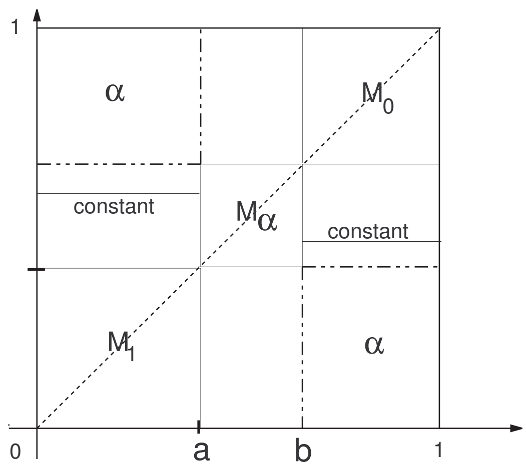

Figure 11: Illustration of Theorem 31

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