HOSVD Based Canonical Form of Polytopic Dynamic Models

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Abstract: This paper defines a canonical form of linear parameter varying models. This canonical form extracts its most important invariant characteristics. The paper also investigates the numerical reconstructability of the canonical form by a tractable uniform method executable in a routine like fashion. In this regard, the paper presents various convergency theorems for given numerical constrains of the reconstruction. The paper also presents an example to study the canonical form and its uniform rather simply executable numerical reconstruction.

Key–Words: LPV model, Polytopic dynamic model, TP model, HOSVD, linear matrix inequalities (LMI)

1 Introduction

Modern control theories mainly focus on analysis and system control design based on LMIs in Linear Parameter Varying (LPV) representation. Currently it seems to be the most usual approach to achieve robust and efficient control results. The typical methods to prepare the system model for LMI based design are usually based on the analytical derivations of the given models. This requires a series of individual and often rather sophisticated analytical solutions. Although we have the analytical solutions, the final result of the LMI design and the implementation of the controller are based on numerical computations, thus the solution is always approximative.

In contrast to the above concept, we propose a uniform, automatic and general approach for preparing the given model upon LMI design so that it is immediately applicable. We introduce this canonical form for quasi LPV model representations. This canonical form extracts various invariant characteristics of the given LPV model.

Furthermore, the tensor product structure of this canonical form offers further creative manipulations of the convex representation of the LPV model to have better observability and controllability.

This canonical form and the method of its numerical reconstruction provides a bridge between the analytical models and the heuristically identified ones.

Since the canonical form is a new concept in control theory and it is a unique representation of LPV models, and also because it offers uniform, tractable and numerical ways to generate convex forms, an important objective is to develop reliable and numerically appealing algorithms to solve a set of LPV control design problems.

This paper focuses attention on how we are capable of numerically reconstructing the canonical form. The paper presents various convergence theorems depending on the different numerical constraints and settings of numerical reconstruction.

The paper also presents numerical examples to show the applicability, efficiency and uniformity of the numerical reconstruction.

2 Nomenclature

This section is devoted to introduce the notations and basic concepts being used in this paper.

- $\{a, b, \ldots\}$: scalar values;
- $\{\mathbf{a}, \mathbf{b}, \ldots\}$: vectors;
- $\{\mathbf{A}, \mathbf{B}, \ldots\}$: matrices;
- $\{\mathcal{A}, \mathcal{B}, \ldots\}$: tensors;
- $\mathbb{R}^{I_1\times\cdots\times I_L}$: vector space of real valued $(I_1\times\cdots\times I_L)$-tensors.
- Subscript defines lower order: for example, an element of matrix $\mathbf{A}$ at row-column number i, j is symbolized as $(\mathbf{A})_{i,j} = a_{i,j}$. The ith column vector of $\mathbf{A}$ is denoted as $\mathbf{A}_i$, or $\mathbf{a}_i$, i.e. $\mathbf{A} = \begin{bmatrix}\mathbf{A}_1 & \mathbf{A}_2 & \cdots\end{bmatrix}$, or $\mathbf{A} = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2 & \cdots\end{bmatrix}$.
- $(\cdot)_{i,j}, \ldots$: are indices;

- $(\cdot)_{I,J},\ldots$: index upper bound: for example: $i = 1\ldots I$, $n = 1,\ldots,N$ or $i_n = 1,\ldots,I_n$.
- $\mathbf{A}_{(n)}$: n-mode matrix of tensor $\mathcal{A} \in \mathbb{R}^{I_1\times\cdots\times I_L}$, $1 \le n \le L$;
- $\mathcal{A} \times_n \mathbf{U}$: n-mode tensor-matrix product;
- $\text{rank}_n(\mathcal{A})$: n-mode rank of tensor $\mathcal{A}$, that is $\text{rank}_n(\mathcal{A}) = \text{rank}(\mathbf{A}_{(n)})$;
- $\mathcal{A} \boxtimes_{l=1}^{L} \mathbf{U}_l$: n-mode multiple product as $\mathcal{A} \times_1 \mathbf{U}_1 \times_2 \cdots \times_L \mathbf{U}_L$;
- $\mathbf{E}_k \in \mathbb{R}^{k\times k}$: k-dimensional identity matrix

Detailed discussion of tensor notations and operations is given in [1].

3 Finite element TP model

Consider the following linear parameter-varying state-space model:

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{pmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix} \tag{1}$$

with input $\mathbf{u}(t)$, output $\mathbf{y}(t)$ and state vector $\mathbf{x}(t)$. The system matrix $\mathbf{S}(\mathbf{p}(t)) \in \mathbb{R}^{I_{N+1}\times I_{N+2}}$ ($I_{N+1} = O$, $I_{N+2} = I$) is a parameter-varying object, where $\mathbf{p}(t) \in \Omega$ is time varying N-dimensional parameter vector, and is an element of the closed hypercube $\Omega = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_N, b_N] \subset \mathbb{R}^N$.The function $\mathbf{p}(t)$ can also include the elements of state-vector $\mathbf{x}(t)$, therefore (1) is considered in the class of nonlinear dynamic state-space models.

Definition 1 (Finite element TP model)**.** *The* $\mathbf{S}(\mathbf{p}(t))$ *of (1) is given for any parameter* $\mathbf{p}(t)$ *as the convex combination of LTI (Linear Time Invariant) system matrices* $\mathbf{S}$ *also called* vertex systems*:*

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{pmatrix} = \left(\mathcal{S} \boxtimes_{n=1}^{N} \mathbf{w}_n^T(p_n(t))\right) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}, \tag{2}$$

where column vector $\mathbf{w}_n(p_n) \in \mathbb{R}^{I_n}$ $n = 1,\ldots,N$ *contains one variable bounded and continuous weighting functions* $w_{n,i_n}(p_n)$, $(i_n = 1..I_n)$. *The weighting function* $w_{n,i_n}(p_n(t))$ *is the* i_n*-th weighting function defined on the n-th dimension of* Ω*, and* $p_n(t)$ *is the n-th element of vector* $\mathbf{p}(t) = (p_1(t),\ldots,p_N(t))$. $I_n < \infty$ *denotes the number of the weighting functions used in the nth dimension of* Ω. *The dimensions of* Ω *are respectively assigned to the elements of the parameter vector* $\mathbf{p}(t)$. *The* $(N+2)$*-dimensional coefficient (system) tensor* $\mathcal{S} \in \mathbb{R}^{I_1\times\cdots\times I_{N+2}}$ *is constructed from LTI vertex systems*

$$\mathbf{S}_{i_1\ldots i_N} = \{S_{i_1\ldots i_N,\alpha,\beta}, 1 \le \alpha \le I_{N+1}, 1 \le \beta \le I_{N+2}\}$$

$\mathbf{S}_{i_1\ldots i_N} \in \mathbb{R}^{I_{N+1}\times I_{N+2}}$. *For further details we refer to [2–5].*

4 HOSVD based canonical form of finite element TP models

Consider such LPV model (1), which can be given in the finite element TP model form (2). Namely, the matrix valued function $\mathbf{S}(\mathbf{p})$ can be given as:

$$\mathbf{S}(\mathbf{p}) = \mathcal{S} \boxtimes_{n=1}^{N} \mathbf{w}_n^T(p_n),$$

where $\mathbf{p} = (p_1,\ldots,p_N) \in \Omega$.

For this model, we can assume that the functions $w_{n,i_n}(p_n)$,$i_n = 1,\ldots,I_n$, $n = 1,\ldots,.N$, are linearly independent (in the means of $\mathfrak{L}_2[a_n, b_n]$) over the intervals $[a_n, b_n]$, respectively. In opposite case we can choose linearly independent functions from $w_{n,i_n}(p_n)$,$i_n = 1,\ldots,I_n$ and we can express the remainder functions with the help of them in linear form This means that the original TP model can be given also with linearly independent functions.

The linearly independent functions $w_{n,i_n}(p_n)$ are determinable by the linear combinations of orthonormal functions (for instance by Gram–Schmidt-type orthogonalization method): thus, one can determine such a system of orthonormal functions for all n as $\varphi_{n,i_n}(p_n)$, $1 \le i_n \le I_n$, respectively defined over the intervals $[a_n, b_n]$, where all $\varphi_{n,k_j}(p_n)$, $1 \le j \le I_n$ are the linear combination of w_{n,i_j}, where i_j is not larger than k_j for all j. The functions w_{n,i_j} can respectively be determined in the same way by functions φ_{n,k_j}. Thus, one can see that if the form (2) of (1) exists then one can determine it in equivalent form as follows

$$\begin{pmatrix} \dot{\mathbf{x}}(t) \\ \mathbf{y}(t) \end{pmatrix} = \left(\mathcal{C} \boxtimes_{n=1}^{N} \varphi_n^T(p_n(t))\right) \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{u}(t) \end{pmatrix}, \tag{3}$$

where tensor $\mathcal{C}$ has constant elements, and column vectors $\varphi_n(p_n(t))$ consists of elements $\varphi_{n,k_n}(p_n(t))$.

Corollary 2. *We can assume, without the loss of generality, that the functions* w_{n,i_n} *in the tensor-product representation of* $\mathbf{S}(\mathbf{p})$ *are given in orthonormal system:*

$$\forall n: \int_{a_n}^{b_n} w_{n,i}(p_n) w_{n,j}(p_n) dp_n = \delta_{i,j}, \quad 1 \le i, j \le I_n,$$

where $\delta_{i,j}$ *is the Kronecker-function* ($\delta_{ij} = 1$, *if* $i = j$ *and* $\delta_{ij} = 0$, *if* $i \ne j$).

Theorem 3 (Higher Order SVD (HOSVD))**.** *Every tensor* $\mathcal{S} \in \mathbb{R}^{I_1\times\cdots\times I_L}$ *can be written as the product [1]*

$$\mathcal{S} = \mathcal{D} \boxtimes_{l=1}^{L} \mathbf{U}_l \tag{4}$$

in which

1. $\mathbf{U}_l = \left[\mathbf{u}_{1,l} \quad \mathbf{u}_{2,l} \quad \ldots \quad \mathbf{u}_{I_l,l}\right]$ *is an orthogonal* $(I_l \times I_l)$*-matrix called l-mode singular matrix.*

2. tensor $\mathcal{D} \in \mathbb{R}^{I_1\times\ldots\times I_L}$ *whose subtensors* $\mathcal{D}_{i_l=\alpha}$ *have the properties of*

(i) all-orthogonality: two subtensors $\mathcal{D}_{i_l=\alpha}$ *and* $\mathcal{D}_{i_l=\beta}$ *are orthogonal for all possible values of* l, α *and* $\beta : \left\langle \mathcal{D}_{i_l=\alpha}, \mathcal{D}_{i_l=\beta}\right\rangle = 0$ *when* $\alpha \neq \beta$,

(ii) ordering: $\left\|\mathcal{D}_{i_l=1}\right\| \geq \left\|\mathcal{D}_{i_l=2}\right\| \geq \cdots \geq \left\|\mathcal{D}_{i_l=I_l}\right\| \geq 0$ *for all possible values of l.*

The Frobenius-norm $\left\|\mathcal{D}_{i_l=i}\right\|$, *symbolized by* $\sigma_i^{(l)}$, *are l-mode singular values of* $\mathcal{D}$ *and the vector* $\mathbf{u}_{i,l}$ *is an ith singular vector.* $\mathcal{D}$ *is termed core tensor.*

Note that the HOSVD uniquely determines tensor $\mathcal{D}$, but the determination of matrices $\mathbf{U}_n$ may not be unique if there are equivalent singular values at least in one dimension.

Theorem 4 (Compact Higher Order SVD (CHOSVD))**.** *For every tensor* $\mathcal{S} \in \mathbb{R}^{I_1\times\cdots\times I_L}$ *the HOSVD is computed via executing SVD on each dimension of* $\mathcal{S}$*. If we discard the zero singular values and the related singular vectors* $\mathbf{u}_{r_l+1}, \ldots, \mathbf{u}_{I_l}$*, where* $r_l = rank_l(\mathcal{S})$*, during the SVD computation of each dimension then we obtain Compact HOSVD as:*

$$\mathcal{S} = \widetilde{\mathcal{D}} \boxtimes_{l=1}^{L} \tilde{\mathbf{U}}_l, \tag{5}$$

which has all the properties as in the previous theorem except the size of $\mathbf{U}_l$ *and* $\mathcal{D}$*. Here* $\tilde{\mathbf{U}}_l$ *has the size of* $I_l \times r_l$ *and* $\widetilde{\mathcal{D}}$ *has the size of* $r_1 \times \ldots \times r_L$.

Let us return to our original task. Having the resulting matrices $\tilde{\mathbf{U}}_n$ by executing the above CHOSVD on the first N-dimension of the system tensor $\mathcal{S} \in \mathbb{R}^{I_1\times\cdots\times I_{N+2}}$ we can determine the following weighting functions:

$$\tilde{\mathbf{w}}_n(p_n) = \tilde{\mathbf{U}}_n^T \mathbf{w}_n(p_n).$$

Then, based on (2) and (5) we arrive at:

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \left(\mathcal{D}_0 \boxtimes_{n=1}^{N} \tilde{\mathbf{w}}_n^T(p_n(t))\right)\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix}, \tag{6}$$

where $\mathcal{D}_0 = \tilde{\mathcal{D}} \boxtimes_{n=N+1}^{N+2} \tilde{\mathbf{U}}_n$ because we do not want to reduce the last two dimensions of the system tensor. Since the number of elements of tensor $\mathcal{D}_0$ in dimension n is equivalent to $r_n = \text{rank}_n(S)$ therefore we have r_n number of functions $\tilde{\mathbf{w}}_{n,i}$ on dimension n ($n = 1..N$) in (6). Observe that, $\mathbf{U}_n$ are orthonormal matrices and $w_{n,i_n}(p_n)$, $1 \leq i_n \leq I_n$ functions are also in orthonormal position (in $\mathfrak{L}_2$ sense) for all $n = 1..N$ (see Corollary 2). Therefore,the components of the function $\tilde{\mathbf{w}}_n(p_n) = (\tilde{w}_{n,1}(p_n), \cdots, \tilde{w}_{n,r_n}(p_n))^T$ are also in orthonormal position for all n, since

$$\int_{a_n}^{b_n} \tilde{\mathbf{w}}_n^T(p_n)\tilde{\mathbf{w}}_n(p_n)dp_n =$$

$$= \tilde{\mathbf{U}}_n^T \left(\int_{a_n}^{b_n} \mathbf{w}_n(p_n)\mathbf{w}_n^T(p_n)dp_n\right)\tilde{\mathbf{U}}_n =$$

$$= \tilde{\mathbf{U}}_n^T \mathbf{E}_{I_n}\tilde{\mathbf{U}}_n = \tilde{\mathbf{U}}_n^T\tilde{\mathbf{U}}_n = \mathbf{E}_{r_n}.$$

Based on the above and Corollary 2 we obtain the following theorem:

Theorem 5. *Consider (1) which have the form of (2). Then we can determine:*

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \left(\mathcal{D}_0 \boxtimes_{n=1}^{N} \mathbf{w}_n(p_n(t))\right)\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix}, \tag{7}$$

via executing CHOSVD on the first N-dimension of $\mathcal{S}$*. The resulting tensor* $\mathcal{D}_0 = \tilde{\mathcal{D}} \boxtimes_{n=N+1}^{N+2} \tilde{\mathbf{U}}_n$ *has the size of* $r_1 \times \ldots \times r_N \times I_{N+1} \times I_{N+2}$*, and the matrices* $\tilde{\mathbf{U}}_k \in \mathbb{R}^{I_k\times r_k}$*,* $k = N+1, N+2$ *are orthogonal.*

The weighting functions have the property of:

1. The r_n *number of weighting functions* $w_{n,i_n}(p_n)$ *contained in vector* $\mathbf{w}_n(p_n)$ *form an orthonormal system. The weighting function* $\mathbf{w}_{i,n}(p_n)$ *is an ith singular function on dimension* $n = 1..N$*.*

Tensor $\mathcal{D}$ *has the properties as:*

2. Tensor $\mathcal{D} \in \mathbb{R}^{r_1\times\ldots\times r_{N+2}}$ *whose subtensors* $\mathcal{D}_{i_n=i}$ *have the properties of*

(i) all-orthogonality: two subtensors $\mathcal{D}_{i_n=i}$ *and* $\mathcal{D}_{i_n=j}$ *are orthogonal for all possible values of* n, i *and* $j : \left\langle \mathcal{D}_{i_n=i}, \mathcal{D}_{i_n=j}\right\rangle = 0$ *when* $i \neq j$,

(ii) ordering: $\left\|\mathcal{D}_{i_n=1}\right\| \geq \left\|\mathcal{D}_{i_n=2}\right\| \geq \ldots \geq \left\|\mathcal{D}_{i_n=r_n}\right\| > 0$ *for all possible values of* $n = 1, \ldots, N+2$.

3. The Frobenius-norm $\left\|\mathcal{D}_{i_n=i}\right\|$, *symbolized by* $\sigma_i^{(n)}$, *are n-mode singular values of* $\mathcal{D}$.

4. $\mathcal{D}$ *is termed core tensor consisting the LTI systems.*

Definition 6. *(HOSVD based canonical form of finite element TP model) We call (**??**) the* HOSVD based canonical form *of (2).*

Remark 7. *If there are equal singular values on any dimensions when CHOSVD is executed, then the canonical form is not unique. Obviously, if the non-zero singular values are different then the sign of the corresponding elements of the singular matrices may systematically vary. This means that the sign of the weighting functions may vary in the same way.*

5 Numerical reconstruction of the HOSVD based canonical form

The main problem investigated in the present paper is the following: if we can calculate the values of the matrix $\mathbf{S}(\mathbf{p}) = \mathcal{D}_0 \boxtimes_{n=1}^{N} \mathbf{w}_n^T(p_n)$ in given points $\mathbf{p} = (p_1, ...p_N)$, then how can we numerically reconstruct the core tensor $\mathcal{D}_0$, the orthonormal matrices $\mathbf{U}_{N+1}$, $\mathbf{U}_{N+2}$ and the functions $\mathbf{w}_n$ playing role in Theorem 5.

Let us divide the intervals $[a_n, b_n]$, $n = 1..N$ into M_n number of disjunct subintervals Ξ_{n,m_n}, $1 \le m_n \le M_n$ so as:

$$\xi_{n,0} = a_n < \xi_{n,1} < \ldots < \xi_{n,M_n} = b_n,$$

$$\begin{aligned} \Xi_{n,m_n} &= [\xi_{n,m_n-1}, \xi_{n,m_n}), 1 \le m_n \le M_N - 1, \\ \Xi_{n,M_N} &= [\xi_{n,M_N-1}, \xi_{n,M_N}]. \end{aligned}$$

Utilizing the above intervals, we can discretize the function $\mathbf{S}(\mathbf{p})$ at given points over the intervals such as let

$$x_{n,m_n} \in \Xi_{n,m_n}, \quad 1 \le m_n \le M_n, 1 \le n \le N \qquad (8)$$

For the sake of brevity let us denote $M_{N+1} = I_{N+1}, M_{N+2} = I_{N+2}$. Let us introduce the discrepancy functions described by the sequences x_{n,m_n} as follows:

$$\Delta_n(s) = \rho_n \left(\sum_{k=1}^{M_n} I(x_{n,k} < s) \right) - (s - a_n),$$

$a_n \le s < b_n, \Delta_n(b_n) = 0$, where I is the indicator function and

$$\rho_n = \frac{b_n - a_n}{M_n}$$

and let us denote

$$\Delta_n = \sup_{a_n \le s < b_n} |\Delta_n(s)|, \quad \Delta_{2,n} = \left(\int_{a_n}^{b_n} \Delta_n^2(s) ds \right)^{1/2}.$$

Let us define a hyper rectangular grid by elements x_{n,m_n}. We define all grid points by N element vector $\mathbf{g}$, (places of observation) whose elements are $\mathbf{g}_{m_1,..,m_N} = \begin{pmatrix} x_{1,m_1} & \cdots & x_{N,m_N} \end{pmatrix}$.

Let us discretize the matrix function $\mathbf{S}(\mathbf{p})$ for all grid points as:

$$\mathbf{B}_{m_1,..,m_N} = \mathbf{S}(\mathbf{g}_{m_1,..,m_N}).$$

Then we construct $N + 2$ dimensional tensor $\mathcal{B}$ from matrices $\mathbf{B}_{m_1,..,m_N}$. Obviously the size of this tensor is $M_1 \times ... \times M_{N+2}$. Further, discretize vector valued functions $\mathbf{w}_n(p_n)$ over the discretization points x_{n,m_n} and construct matrices $\mathbf{W}^{(n)} \in \mathbb{R}^{M_n \times r_n}$ from the discretized values as:

$$\mathbf{W}^{(n)} = \begin{pmatrix} w_{n,1}(x_{n,1}) & w_{n,2}(x_{n,1}) & \cdots & w_{n,r_n}(x_{n,1}) \\ w_{n,1}(x_{n,2}) & w_{n,2}(x_{n,2}) & \cdots & w_{n,r_n}(x_{n,2}) \\ \vdots & & \ddots & \vdots \\ w_{n,1}(x_{n,M_n}) & w_{n,2}(x_{n,M_n}) & \cdots & w_{n,r_n}(x_{n,M_n}) \end{pmatrix} \qquad (9)$$

Let us denote $w_{i,k}^{(n)} = w_{n,i}(x_{n,k}), 1 \le k \le M_n, 1 \le i \le r_n, 1 \le n \le N$, then we can write $\mathbf{W}^{(n)} = \begin{pmatrix} \mathbf{W}_1^{(n)} & \mathbf{W}_2^{(n)} & \ldots & \mathbf{W}_{r_n}^{(n)} \end{pmatrix} \in \mathbb{R}^{M_n \times r_n}, 1 \le n \le N$, where $\mathbf{W}_i^{(n)} = \left(w_{i,1}^{(n)}, \ldots, w_{i,M_n}^{(n)} \right)^T$, $1 \le i \le r_n$ denote the column vectors of the matrix $\mathbf{W}^{(n)}$. Then tensor $\mathcal{B}$ can simply be given by (5) and (6) as

$$\mathcal{B} = \mathcal{D}_0 \times_1 \mathbf{W}^{(1)} \times_2 \cdots \times_N \mathbf{W}^{(N)} \qquad (10)$$

Let us denote the matrices $\varepsilon_n \in \mathbb{R}^{r_n \times r_n}$, $1 \le n \le N$,

$$\varepsilon_n = \left(\varepsilon_{i,j}^{(n)} \right)_{i,j=1}^{r_n},$$

where

$$\varepsilon_{i,j}^{(n)} = \delta_{ij} - \rho_n \sum_{k=1}^{M_n} w_{n,i}(x_{n,k}) w_{n,j}(x_{n,k}), \quad 1 \le i, j \le r_n.$$

The following Lemma gives estimation for upper bound for the quantity $\|\varepsilon_n\|$, which guarantees the convergence $\|\varepsilon_n\| \to 0$ if $\Delta_n \to 0$ and which plays basic role in the formulation of our results.

Further on let us assume that the functions $w_{n,i_n}(p_n)$, $1 \le i_n \le r_n$ $(1 \le n \le N)$ are piece-wise continuously differentiable on the interval $p_n \in [a_n, b_n]$ (at the end points of the interval we understood left and right hand side derivatives). Let us denote

$$K_{0,n} = \max_{1 \le i \le r_n} \max_{a_n \le s \le b_n} \left| w_{n,i}(s) \right|$$

$$K_{1,n} = \max_{1 \le i \le r_n} \int_{a_n}^{b_n} \left| (w_{n,i}(s))' \right| ds,$$

$$K_{2,n} = \max_{1 \le i \le r_n} \left(\int_{a_n}^{b_n} \left[(w_{n,i}(s))' \right]^2 ds \right)^{1/2}.$$

Lemma 8. *The Frobenius-norm of the matrix* $\varepsilon_n = \mathbf{E}_{r_n} - \frac{b_n - a_n}{M_n} \mathbf{W}^{(n)T} \mathbf{W}^{(n)}$ *satisfies the inequalities*

$$\|\varepsilon_n\| \le 2 r_n K_{0,n} K_{1,n} \Delta_n$$

and

$$\|\varepsilon_n\| \le r_n K_{2,n} \Delta_{2,n}.$$

Proof. The statements of Lemma 8 follow from the results of Lemma 10 (Koksma-Hlawka inequality and its $\mathfrak{L}_2$ version in one-dimension case). □

Corollary 9. *If the intervals $[a_n, b_n]$ are divided into equidistant subintervals, i.e. in the case of*

$$\xi_{n,k} = a_n + k\frac{b_n - a_n}{M_n}, \quad 1 \le k \le M_n,$$

then the following inequalities hold

$$\|\varepsilon_n\| \le 2r_n K_{0,n} K_{1,n} \frac{b_n - a_n}{M_n}$$

and

$$\|\varepsilon_n\| \le 2r_n K_{0,n} K_{2,n} \frac{(b_n - a_n)^{3/2}}{M_n}.$$

Lemma 10. *Let $y_1, \dots, y_L$ be arbitrary points from the interval $[a, b]$. If the function $u(t)$ ($t \in [a, b]$) is piecewise continuously differentiable (at the end points of the interval we understood left and right hand side derivatives), then the following inequalities hold*

$$\left| \frac{b-a}{L} \sum_{l=1}^{L} u(y_l) - \int_a^b u(s)ds \right| \le \sup_{a \le s \le b} |\Delta_L(s)| \int_a^b |u'(s)|\, ds,$$

$$\left| \frac{b-a}{L} \sum_{l=1}^{L} u(y_l) - \int_a^b u(s)ds \right| \le$$

$$\le \left(\int_a^b (\Delta_L(s))^2\, ds \right)^{1/2} \left(\int_a^b (u'(s))^2\, ds \right)^{1/2},$$

where $\Delta_L(s) = \frac{b-a}{L} \left(\sum_{l=1}^{L} I(y_l < s) \right) - (s - a), a \le s \le b.$

Proof. Since $u(y) = u(b) - \int_a^b u'(s)I(y < s)ds$ and with integrating by part we get

$$\int_a^b u(s)ds = u(b)b - u(a)a - \int_a^b u'(s)sds,$$

then

$$\frac{b-a}{L} \sum_{l=1}^{L} u(y_l) = (b-a)u(b) - \frac{b-a}{L} \int_a^b u'(s) \sum_{l=1}^{L} I(y_l < s)ds$$

$$= - \int_a^b u'(s)\Delta_L(s)ds + \int_a^b u(s)ds$$

and

$$\left| \frac{b-a}{L} \sum_{i=1}^{L} u(y_l) - \int_a^b u(s)ds \right| \le \left| \int_a^b u'(s)\Delta_L(s)ds \right|.$$

From this relation the first inequality of Lemma 10 immediately follows, and we get the second inequality by the use of Cauchy-Schwartz inequality. □

Proof. of Lemma 8.Since the functions $w_{n,i}(x_n)$, $1 \le i \le r_n$ are orthonormal in $\mathfrak{L}_2$ sense on the interval $[a_n, b_n]$, therefore

$$\int_{a_n}^{b_n} w_{n,i}(s)w_{n,j}(s)ds = \delta_{i,j}.$$

By the first inequality of Lemma 10 it follows

$$\left|\varepsilon_{ij}\right| \le \Delta_n \left| \int_a^b \left(w_{n,i}(s)w_{n,j}(s)\right)'\, ds \right| \le$$

$$\le \Delta_n \int_a^b \left| (w_{n,i}(s))'\, w_{n,j}(s) + w_{n,i}(s) \left(w_{n,j}(s)\right)' \right| ds \le$$

$$\le 2\Delta_n K_{0,n} K_{1,n}$$

We can get easily the second inequality of Lemma 8

$$\begin{aligned} \|\varepsilon_n\| &= \left(\sum_{i,j=1}^{r_n} \varepsilon_{i,j}^2 \right)^{1/2} \le \left(r_n^2 4\Delta_n^2 K_{0,n}^2 K_{1,n}^2 \right)^{1/2} = \\ &= 2r_n \Delta_n K_{0,n} K_{1,n}. \end{aligned}$$

□

Main results: Theorems on the numerical reconstruction.

To simplify the calculations, in the followings we assume that the singular values are all different for each dimension of the discretized tensor.

Based on the previous notations the result of the discretization the discretized tensor can be written in the form $\mathcal{B} = \mathbb{R}^{M_1 \times \cdots \times M_{N+2}}$, where

$$b_{m_1,\dots,m_{N+2}} = \sum_{i_1=1}^{r_1} \cdots \sum_{i_N=1}^{r_N} d_{i_1,\dots,i_N,m_{N+1},m_{N+2}} w_{i_1,m_1}^{(1)} \cdot \dots \cdot w_{i_N,m_N}^{(N)}, \tag{11}$$

$$1 \le m_n \le M_n, 1 \le n \le N + 2.$$

Let us consider the discretizated tensor

$$\mathcal{B} = \mathcal{D}_0 \times_1 \mathbf{W}_2^{(1)} \cdots \times_N \mathbf{W}^{(N)} \in \mathbb{R}^{M_1 \times \cdots \times M_{N+2}}. \tag{12}$$

By the HOSVD decomposition of the discretized tensor (see [1]) it can be rewritten in the form

$$\mathcal{B} = \mathcal{D}^d \times_1 \mathbf{U}^{(1)} \cdots \times_{N+2} \mathbf{U}^{(N+2)} \tag{13}$$

where $\mathcal{D}^d$ is the so-called core tensor, and $\mathbf{U}^{(n)} = \begin{pmatrix} \mathbf{U}_1^{(n)} & \mathbf{U}_2^{(n)} & \dots & \mathbf{U}_{M_n}^{(n)} \end{pmatrix}$ is an $M_n \times M_n$-size orthogonal matrix ($1 \le n \le N+2$).

The subtensors $\mathcal{D}_\alpha^{d,n}$ of the tensor $\mathcal{D}^d \in \mathbb{R}^{M_1 \times \cdots \times M_{N+2}}$ (where the nth index $\alpha = 1, \dots, M_n$ is fixed) has the following properties:

1. Tensors $\mathcal{D}_\alpha^{d,n}$ and $\mathcal{D}_\beta^{d,n}$ are orthogonal for all $1 \le n \le N+2$ and $\alpha \ne \beta$, namely

$$\langle \mathcal{D}_\alpha^{d,n}, \mathcal{D}_\beta^{d,n} \rangle = 0 \tag{14}$$

2. $\sigma_1^{d,n} \ge \cdots \ge \sigma_{M_n}^{d,n}$, where $\sigma_i^{d,n} = \left\| \mathcal{D}_i^{d,n} \right\|$, $1 \le i \le M_n$, $1 \le n \le N+2$.

3. We need the n-mode matrix unfolding of discretized tensor $\mathcal{B}$

$$\mathbf{B}_{(n)}^d = \mathbf{U}^{(n)} \mathbf{D}_{(n)}^d \left(\mathbf{U}^{(n+1)} \otimes \dots \mathbf{U}^{(N+2)} \otimes \right.$$
$$\left. \otimes \mathbf{U}^{(1)} \otimes \cdots \otimes \mathbf{U}^{(n-1)} \right)^T, \tag{15}$$

where $\otimes$ denotes the Kronecker-product.

Let r_n^d be the rank of matrix $\mathbf{B}_{(n)}^d$. Then $r_n^d = \mathrm{rank}(\mathbf{B}_{(n)}^d) = \mathrm{rank}_n(\mathcal{B})$ is the dimension of the linear space spanned by the n-mode vectors. The orthogonal matrix $\mathbf{D}_{(n)}^d$ is the singular matrix from the SVD decomposition with the positive $\sigma_1^{d,n} \ge \cdots \ge \sigma_{r_n^d}^{d,n} > 0$ singular values in its diagonal. The other singular values and the rest of the elements of $\mathbf{D}_{(n)}^d$ are 0. The Frobenius-norm $\sigma_i^{d,n} = \left\| \mathcal{D}_i^{d,n} \right\|$ is equal to the n-mode singular values of tensor $\mathcal{B}$, and vectors $\mathbf{U}_i^{(n)}$, $1 \le i \le M_n$ are n-mode singular vectors. Then it is true for the n-mode singular values

$$\|\mathcal{B}\|^2 = \sum_{i=1}^{r_1} \left(\sigma_i^{d,1} \right)^2 = \cdots = \sum_{i=1}^{r_{N+2}} \left(\sigma_i^{d,N+2} \right)^2 \tag{16}$$

4. The core tensor is:

$$\mathcal{D}^d = \mathcal{B} \times_1 \mathbf{U}^{(1)^T} \times_2 \cdots \times_{N+2} \mathbf{U}^{(N+2)^T} \tag{17}$$

Let $R_n = M_{n+1} \dots M_{N+2} M_1 \dots N_{n-1}$. By definition for the $(i_n, k_{n,m})$th element of matrix $\mathbf{B}_{(n)}^d \in \mathbb{R}^{M_n \times R_n}$ the relation $b_{i_n,k_{n,m}}^{(n)} = b_{m_1,\dots,m_N}$ is true, where

$$k_{n,m} = (m_{n+1} - 1) M_{n+2} \dots M_{N+2} M_1 \dots M_{n-1} + \cdots +$$
$$+ (m_{N+2} - 1) M_1 \dots M_{n-1} + (m_1 - 1) M_2 \dots M_{n-1} +$$
$$+ \cdots + (m_{n-2} - 1) M_{n-1} + m_{n-1}$$
$$m = (m_1, \dots m_{N+2}), 1 \le m_n \le M_n, 1 \le n \le N+2.$$

Note that the value of $k_{n,m}$ only depends on the values of $m_1, \dots, m_{n-1}, m_{n+1}, \dots, m_N$, and does not depend on the actual value of index i_n.

Lemma 11. *For every matrix* $\mathbf{U} \in \mathbb{R}^{M_n \times L_n}$ *with rank*$\mathbf{U} = L_n$*.and for every tensor* $\mathcal{A} \in \mathbb{R}^{L_1 \times \cdots \times L_N}$ *the following relations hold*

$$\mathrm{rank}_k(\mathcal{A} \times_n \mathbf{U}) = \mathrm{rank}_k \mathcal{A}, \quad 1 \le k \le N.$$

Proof. Let us denote $r_k = \mathrm{rank}_k \mathcal{A}$ and

$$\mathbf{A}_{(k)}(i_1, ..., i_{k-1}, i_{k+1}, ..., i_N) = \begin{pmatrix} a_{i_1,\dots,i_{k-1},1,i_{k+1},\dots,i_N} \\ \vdots \\ a_{i_1,\dots,i_{k-1},L_k,i_{k+1},\dots,i_N} \end{pmatrix},$$

$1 \le k \le N$. Firstly we prove that $\mathrm{rank}_n(\mathcal{A} \times_n U) = r_n$. By the definition of n-mode product we can write

$$(\mathcal{A} \times_n \mathbf{U})_{i_1,\dots,i_{n-1},j_n,i_{n+1},\dots,i_N} = \sum_{i_n=1}^{L_n} a_{i_1,\dots,i_N} u_{j_n,i_n},$$

therefore the M_n-dimensional n-mode vectors of the tensor $(\mathcal{A} \times_n \mathbf{U})$ can be given in the form

$$\sum_{i_n=1}^{L_n} a_{i_1,\dots,i_n,\dots,i_N} u_{i_n} = \mathbf{U} \mathbf{A}_{(n)}(i_1, ..., i_{n-1}, i_{n+1}, ..., i_N),$$

where u_i, $1 \le i \le L_n$ denote the column vectors of the matrix $\mathbf{U}$. Since $r_n = \mathrm{rank}_n \mathcal{A}$ and the vectors $\mathbf{A}_{(n)}(i_1, ..., i_{n-1}, i_{n+1}, ..., i_N)$, $1 \le i_j \le L_j$, $j \ne n$ from the right side of the last equation are the n-mode vectors of the tensor $\mathcal{A}$, then we can select from them exactly r_n linearly independent column vectors. From the condition $\mathrm{rank}\, \mathbf{U} = L_n \ge r_n$ it follows that the rank of linear space spanned by the n-mode vectors is exactly r_n.

Proof of the case of $\mathrm{rank}_k(\mathcal{A} \times_n \mathbf{U}) = \mathrm{rank}_k \mathcal{A}$, $1 \le k \le N, k \ne n$. Let us denote the $L_k \times L_n$ matrix

$$\mathbf{A}_{(k,n)} = \left[\mathbf{A}_{(k)}(i_1, ..., i_{k-1}, i_{k+1}, ..., i_{n-1}, j, i_{n+1}, ...i_N) \right],$$

$j = 1, ..., L_n$. It can be seen that the k-mode vectors of the tensor $(\mathcal{A} \times_n \mathbf{U})$ correspond to the column vectors of the matrices $\mathbf{A}_{(k,n)} \mathbf{U}^T$, $n = 1, ..., L_N$. Since $\mathrm{rank}_k \mathcal{A}$, thus the set of vectors $\mathbf{A}_{(k)}(i_1, ..., i_{n-1}, i_{n+1}, ..., i_N)$, $1 \le i_j \le L_j$, $j \ne n$ consists exactly r_k linearly independent vectors. Thus, from the condition $\mathrm{rank}\, \mathbf{U} = L_n \ge r_n$ follows that rank of linear space generated by the column vectors of the matrices $\mathbf{A}_{(k,n)} \mathbf{U}^T$, $n = 1, ..., L_N$ is exactly r_k also. □

By the definition of the matrices $\varepsilon_n, 1 \le n \le N$ we have $\mathbf{W}^{(n)^T}\mathbf{W}^{(n)} = E_{r_n} - \varepsilon_n$. If $\|\varepsilon_n\| < 1, 1 \le n \le N$, then by the well-known result of matrix theory it follows that

$$\mathrm{rank}(E_{r_n} - \varepsilon_n) = \mathrm{rank}(\mathbf{W}^{(n)T}\mathbf{W}^{(n)}) = r_n, 1 \le n \le N,$$

thus using Lemma 11 we get $r_k^d = \mathrm{rank}_k \mathcal{B} = r_k$ As a consequence of this relation we have the following theorem.

Theorem 12. *If $\|\varepsilon_n\| < 1, 1 \le n \le N$, then $r_k^d = \mathrm{rank}_k \mathcal{B} = \mathrm{rank}_k \mathcal{D}^d = r_k, 1 \le n \le N$.*

Note that during the proof of the following theorem, it results that $r_n^d = r_n, 1 \le n \le N$, if $\Delta = \min_{1\le n\le N} \Delta_n$ is small enough.

Consider $r_1 \times \cdots \times r_{N+2}$-size reduced version $\widetilde{\mathcal{D}}^d = (\mathcal{D}^d_{m_1,\dots,m_{N+2}}, 1 \le m_n \le r_n, 1 \le n \le N+2)$ of the $M_1 \times \cdots \times M_{N+2}$-size tensor $\mathcal{D}^d$.

It follows from (12) that $\sigma^{d,n}_{r_n+1} = \cdots = \sigma^{d,n}_{M_n}$, thus instead of $\mathcal{D}^d$ the analysis of tensor $\widetilde{\mathcal{D}}^d$ is enough.

Theorem 13. *If $\Delta \to 0$ then $\sqrt{\rho}\widetilde{\mathcal{D}}^d \to \mathcal{D}$ and $\mathbf{U}^{(n)} \to \mathbf{U}_n, n = N+1, N+2$, where $\rho = \prod_{n=1}^{N} \rho_n = \prod_{n=1}^{N} \frac{b_n - a_n}{M_n}$*

Proof. Taking into consideration the result of the discretization in two different ways (12) and (13), we get

$$\mathcal{B} = \mathcal{D}_0 \boxtimes_{n=1}^N \mathbf{W}^{(n)} = \mathcal{D}^d \boxtimes_{n=1}^{N+2} \mathbf{U}^{(n)}. \quad (18)$$

Then, by applying the rule of n-mode multiplication of tensor by matrices, it results

$$\mathcal{D}_0 \boxtimes_{n=1}^N \mathbf{U}^{(n)^T}\mathbf{W}^{(n)} = \mathcal{D}^d \boxtimes_{n=1}^{N+2} \mathbf{U}^{(n)^T}\mathbf{U}^{(n)}. \quad (19)$$

Here $\mathbf{U}^{(n)}$ is an $M_n \times M_n$ orthogonal matrix, then

$$\mathbf{U}^{(n)^T}\mathbf{U}^{(n)} = \mathbf{E}_{M_n},$$

so from (**??**) it follows

$$\mathcal{D} \times_{N+1} \mathbf{H}^{(N+1)} \times_{N+2} \mathbf{H}^{(N+2)} \boxtimes_{n=1}^N \mathbf{H}^{(n)} = \sqrt{\rho}\mathcal{D}^d, \quad (20)$$

where $\mathbf{H}^{(n)} = \sqrt{\rho_n}\mathbf{U}^{(n)^T}\mathbf{W}^{(n)} \in \mathbb{R}^{M_n \times r_n}, 1 \le n \le N$ and $\mathbf{H}^{(n)} = \mathbf{U}^{(n)^T}\mathbf{U}_n \in \mathbb{R}^{I_n \times I_n}, n = N+1, N+2$. It is evident that

$$\mathbf{H}^{(n)^T}\mathbf{H}^{(n)} = \rho_n \mathbf{W}^{(n)^T}\mathbf{W}^{(n)} = \mathbf{E}_{r_n} - \varepsilon_n, 1 \le n \le N, \quad (21)$$

where $\|\varepsilon_n\| \to 0$, if $\Delta \to 0$, thus $\mathrm{rank}(\mathbf{H}^{(n)}) = r_n$ if Δ is small enough. We note that the matrices $\mathbf{H}^{(N+1)}$ and $\mathbf{H}^{(N+2)}$ are orthogonal, i.e.

$$\mathbf{U}^{(n)^T}\mathbf{U}_n = \mathbf{E}_{I_n}, n = N+1, N+2, \quad (22)$$

which immediately follows from the property of $\mathbf{U}^{(n)}$ and $\mathbf{U}_n$.

Let us denote the column vectors of matrix $\mathbf{H}^{(n)}$ by $\mathbf{H}_j^{(n)}, 1 \le j \le r_n$. The reduced $r_n \times r_n$ matrix derived from $\mathbf{H}^{(n)}, 1 \le n \le N$ is denoted by $\widetilde{\mathbf{H}}^{(n)}$ (we select the first r_n row of matrix $\mathbf{H}^{(n)}$). From the (18) it follows that the linear spaces spanned by the linearly independent vectors $\mathbf{W}_1^{(n)}, \dots, \mathbf{W}_{r_n}^{(n)}$ (if Δ is small enough) and $\mathbf{U}_1^{(n)}, \dots, \mathbf{U}_{r_n}^{(n)}$, respectively, are the same, therefore $\mathbf{W}_i^{(n)}$ and $\mathbf{U}_j^{(n)}$ are orthogonal, if $1 \le i \le r_n, r_n + 1 \le j \le M_n$, therefore

$$\mathbf{H}^{(n)^T}\mathbf{H}^{(n)} = \widetilde{\mathbf{H}}^{(n)^T}\widetilde{\mathbf{H}}^{(n)}. \quad (23)$$

According to (21) and (23), the column vectors $\tilde{\mathbf{H}}_j^{(n)}, 1 \le j \le r_n$ are asymptotically orthonormal, namely $\tilde{\mathbf{H}}_j^{(n)^T}\tilde{\mathbf{H}}_j^{(n)} \to \delta_{ij}, \Delta \to 0, 1 \le i, j \le r_n$, thus

$$\widetilde{\mathbf{H}}^{(n)^T}\widetilde{\mathbf{H}}^{(n)} \to \mathbf{E}_{I_n}, \quad \Delta \to 0. \quad (24)$$

Somehow let us increase the discretization points $a_n \le x_{n,1} < \cdots < x_{n,M_n} \le b_n, n = 1, \dots N$ with the property $\Delta \to 0$. Then the elements of matrix $\widetilde{\mathbf{H}}^{(n)}$ are still bounded, so a partial series can be selected and a matrix $\mathbf{G_n}$, $\mathbf{G}_n^T\mathbf{G} = \mathbf{E}_{r_n}$ can be given independently from the discretization points such way that the convergence $\widetilde{\mathbf{H}}^{(n)} \to \mathbf{G}_n$ is valid for the partial series. So, based on the partial series and an appropriate tensor $\widetilde{\mathcal{D}}$, $\sqrt{\rho}\widetilde{\mathcal{D}}^d \to \mathcal{D} \times_1 \mathbf{G}_1 \cdots \times_{N+2} \mathbf{G}_{N+2} = \widetilde{\mathcal{D}}$. Here $\widetilde{\mathcal{D}}$ satisfies the properties of a core-tensor because it is given as the limit value of core tensors. By rewriting this equation we get

$$\mathcal{D} \times_1 \mathbf{G}_1 \cdots \times_{N+2} \mathbf{G}_{N+2} = \widetilde{\mathcal{D}} \quad (25)$$

As the construction of $\mathcal{D}$ by (**??**) is unique, then the matrices $\mathbf{G}_n, 1 \le n \le N$ and tensor $\widetilde{\mathcal{D}}$ are also uniquely. Then it is evident that $\mathbf{G}_n = \mathbf{E}_{r_n}, \widetilde{\mathcal{D}} = \mathcal{D}$ and

$$\widetilde{\mathbf{H}}^{(n)} \to \mathbf{E}_{r_n}, \quad \sqrt{\rho}\widetilde{\mathcal{D}}^d \to \widetilde{\mathcal{D}}, \quad \Delta \to 0 \quad (26)$$

and

$$\mathbf{U}^{(n)} \to \mathbf{U}_n, n = N+1, N+2 \quad (27)$$

□

Let us denote the elements of matrix $\mathbf{U}^{(n)}$ by $\mathbf{U}_{i,k}^{(n)}$ and introduce similarly to $v_{n,i}(x)$ the step functions $u_{n,i}(x), 1 \le i \le r_n$ such way that by definition the intervals $\Xi_{n,k}$ is

$$u_{n,i}(x) = \frac{1}{\sqrt{\rho}}\mathbf{U}_{i,k}^{(n)} I(x \in \Xi_{n,k}), 1 \le k \le M_n \quad (28)$$

$$(v_{n,i}(x) = w_{i,k}^{(n)} I(x \in \Xi_{n,k})).$$

Theorem 14. *If* $\Delta \to 0$*, then*

$$\int_{a_n}^{b_n} (w_{n,i}(x) - u_{n,i}(x))^2 \mathrm{d}x \to 0, \quad 1 \le i \le r_n, 1 \le n \le N \tag{29}$$

Proof. It is trivial that

$$\int_{a_n}^{b_n} (w_{n,i}(x) - u_{n,i}(x))^2 \mathrm{d}x \le$$
$$2 \int_{a_n}^{b_n} (w_{n,i}(x) - v_{n,i}(x))^2 \mathrm{d}x + 2 \int_{a_n}^{b_n} (v_{n,i}(x) - u_{n,i}(x))^2 \mathrm{d}x,$$

so it is enough to analyze the second integral on the right. Then from Lemma 8 we get

$$\int_{a_n}^{b_n} (v_{n,i}(x) - u_{n,i}(x))^2 \mathrm{d}x = \sum_{k=1}^{M_n} \left(w_{n,i}^{(n)} - \frac{1}{\sqrt{\rho}} U_{i,k}^{(n)} \right)^2 \rho_n =$$

$$= \sum_{k=1}^{M_n} \left[(w_{i.k}^{(n)})^2 \rho_n - 2\sqrt{\rho_n} w_{i,k}^{(n)} U_{i,k}^{(n)} + (U_{i,k}^{(n)})^2 \right] \to 0, \Delta \to 0.$$

□

6 Conclusion

This paper defined the HOSVD canonical form of the LPV models and investigated its numerical reconstructability. In this regard the paper presented various convergency theorems.

Acknowledgements: This research was supported by the Hungarian National Found OTKA No. T 049838 and T 042896. Dr. Baranyi was supported by János Bolyai Postdoctoral Scholarship.

References:

[1] L. De Lathauwer, B. De Moor, and J. Vandewalle, "A multilinear singular value decomposition," SIAM *Journal on Matrix Analysis and Applications*, vol. 21, no. 4, pp. 1253–1278, 2000.

[2] P. Baranyi, D. Tikk, Y. Yam, and R. J. Patton, "From differential equations to PDC controller design via numerical transformation," *Computers in Industry, Elsevier Science*, vol. 51, pp. 281–297, 2003.

[3] P. Baranyi, "TP model transformation as a way to LMI based controller design," *IEEE Transaction on Industrial Electronics*, vol. 51, no. 2, pp. 387–400, 2004.

[4] ——, "Tensor-product model-based control of two-dimensional aeroelastic system," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 2, pp. 391–400, 2006.

[5] ——, "Output feedback control of 2-D aeroelastic system," *Journal of Guidance, Control, and Dynamics*, vol. 29, no. 3, pp. 762–767, 2006.