Investigation on the spectrum of graph $G_l$

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Abstract: Let $G_l$ be the graph obtained from $K_l$ by adhering the root of isomorphic trees $T$ to every vertex of $K_l$. In this paper we study the spectrum of the adjacency matrix $A(G_l)$ for all positive integer $l$ and give some result about the spectrum of the adjacency matrix $A(G_l)$.

Key–Words: Adjacency matrix, complete graph, spectrum

1 Introduction

Let $G$ be a simple undirected graph on $n$ vertices, and let $A(G)$ be a $(0, 1)$-adjacency matrix of $G$. Since $A(G)$ is a real symmetric matrix, all of its eigenvalues are real. Without loss of generality, that they are ordered in non-increasing order, i.e.,

$$\lambda_1(G) \geq \lambda_2(G) \geq ... \geq \lambda_n(G),$$

and call them the spectrum of $G$. The largest eigenvalue $\lambda_1(G)$ is called the spectral radius of $G$.

About the spectrum and the spectral radius of graphs, a great deal of investigation is carried out [1,2,3]. Specially, to the special graphs, for example [4] studied the spectral radius of bicyclic graphs with $n$ vertices and diameter $d$. [5] studied the spectral radius of trees with fixed diameter.

Let $T$ be an unweighted rooted tree of $k$ levels such that in each level the vertices have equal degree. $K_l$ be a complete graph on $l$ vertices. Let $G_l$ be the graph obtained from $K_l$ by adhering the root of isomorphic trees $T$ to every vertex of $K_l$. Similar to the definition of tree’s level, we agree that the complete graph $K_l$ is at level 1, and that $G_l$ has $k$ levels. Thus the vertices in the level $k$ have degree 1.

For $j = 1, 2, 3, ..., k$, Let $n_{k-j+1}$ and $d_{k-j+1}$ be the number of vertices and the degree of them in the level $j$. Observe that $n_k = l$ is the number of vertices in level 1 and $n_k$ the number of vertices in level $k$ (the number of pendant vertices). Then,

$$n_{k-1} = (d_k - l + 1)n_k,$$

$$n_{k-j} = (d_k - j + 1)n_{k-j+1}, j = 2, 3, ..., k - 1$$

Observe that $d_k$ is the degree of vertex of the complete graph $K_l$ in $G_l$, $d_1$ is the degree of the vertices in the level $k$, $n_k = l$. The total number of vertices in the graph $G_l$ is

$$n = \sum_{j=1}^{k-1} n_j + l$$

In general, using the labels $n, n - 1, ..., 1$, in this order, our labeling for the vertices of $G_l$ is:

(1) First, we label the vertices of $K_l$ with clockwise direction.

(2) For one of vertices of level $j (j = 1, 2, ..., k - 1)$, the bigger its labeling is, then the vertex of level $j + 1$ adjacent to it should be labeled first.

(3) Label from level 1 to level $k$ in turn.

Fig.1. graph $G_4$
Above (Fig. 1.) is an example of a such graph $G_4$ for $k = 3, n_1 = 24, n_2 = 8, n_3 = 4, d_1 = 1, d_2 = 4, d_3 = 5$.

[6], [7] studied the spectrum of the adjacency matrix $A(G_l)$ for case $l = 1$ and $l = 2$ respectively. In this paper we will study the spectrum of the adjacency matrix $A(G_l)$ for all positive integer $l$.

## 2 Preliminaries

We introduce the following notations:

1. $0$ is the all zeros matrix, the order of $0$ will be clear from the context in which it is used.
2. $I_m$ is the identity matrix of order $m \times m$.
3. $m_j = \frac{n_j}{n_{j+1}}$, for $j = 1, 2, \ldots, k - 1$.
4. $e_m$ is the all ones column vector of dimension $m$.

For $j = 1, 2, \ldots, k - 1$, $C_j$ is the block diagonal matrix

$$C_j = \begin{pmatrix} e_{m_j} & e_{m_j} & \cdots & e_{m_j} \\ \end{pmatrix}$$

with $n_{j+1}$ diagonal blocks. Thus, the order of $C_j$ is $n_j \times n_{j+1}$.

For example we use these notation with the graph $G_4$ in Fig. 1. $m_1 = \frac{n_1}{n_2} = 3, m_2 = \frac{n_2}{n_3} = 2$, then


$$C_2 = diag\{e_2, e_2, e_2, e_2\},$$

The adjacency matrix $A(G_4)$ in Fig. 1. become

$$A(G_4) = \begin{pmatrix} 0 & C_1 & 0 \\ C_1^T & 0 & C_2 \\ 0 & C_2 & B_4 \end{pmatrix}$$

where $B_4 = A(K_4) = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$

In general, our labeling yields to

$$A(G_l) = \begin{pmatrix} 0 & C_1 \\ C_1^T & 0 & C_2 \\ \vdots & \vdots & \vdots \\ C_2^T & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ C_{l-1}^T & \vdots & \vdots \\ C_{l-1} & B_l \end{pmatrix}$$

where $B_l = A(K_l) = \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{pmatrix}$

Apply the Gaussian elimination procedure we obtained the following lemma:

**Lemma 1.** Let

$$M = \begin{pmatrix} \alpha_1 I_{n_1} & C_1 \\ C_1^T & \alpha_2 I_{n_2} & C_2 \\ C_2^T & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ C_{l-1}^T & \vdots & \vdots \\ C_{l-1} & \alpha_k I_{n_k} + B_l \end{pmatrix}$$

Let

$$\beta_1 = \alpha_1$$

and

$$\beta_j = \alpha_j - \frac{n_{j-1}}{n_j} - \frac{1}{\beta_{j-1}}, j = 2, 3, \ldots, k, \beta_{j-1} \neq 0.$$ 

If $\beta_j \neq 0$ for all $j = 1, 2, \ldots, k - 1$, then

$$detM = \beta_1^{n_1} \beta_2^{n_2} \cdots \beta_{k-1}^{n_{k-1}}(\beta_k - l + 1)(\beta_k + 1)^{k-1}. \quad (1)$$

**Proof.** Apply the Gaussian elimination procedure, without row interchanges, to $M$ to obtain the block upper triangular matrix

$$\begin{pmatrix} \beta_1 I_{n_1} & C_1 \\ \beta_2 I_{n_2} & C_2 \\ \beta_3 I_{n_3} & \vdots \\ \beta_{k-1} I_{n_{k-1}} & C_{k-1} \\ \beta_k I_{n_k} + B_l \end{pmatrix}$$

Hence,

$$detM = \beta_1^{n_1} \beta_2^{n_2} \cdots \beta_{k-1}^{n_{k-1}} det(\beta_k I_{n_k} + B_l),$$

since

$$det(\lambda I - B_l) = (\lambda - l + 1)(\lambda + 1)^{k-1},$$

so

$$detM = \beta_1^{n_1} \beta_2^{n_2} \cdots \beta_{k-1}^{n_{k-1}}(\beta_k - l + 1)(\beta_k + 1)^{k-1}.$$ 

Thus, (1) is proved. #
3 The spectrum of $A(G_l)$

Let

$$D = \begin{pmatrix}
-I_{n_1} & C_1 \\
I_{n_2} & -I_{n_2} \\
& & \ddots \\
& & & -I_{n_k-1} \\
& & & & -I_{n_k}
\end{pmatrix}$$

we can easily see that

$$D(\lambda I + A(G_l))D^{-1} = \lambda I - A(G_l)$$

Let

$$\phi = \{1, 2, ..., k-1\}$$

and

$$\Omega = \{ j \in \phi : n_j > n_{j+1} \}$$

Observe that $n_{k-j} = (d_{k-j+1} - 1)n_{k-j+1}, j = 2, 3, ..., k-1$ and $n_{k-1} = (d_k - t + 1)n_k$. Observe also that if $j \notin \phi - \Omega$ then $n_j = n_{j+1}$ and $C_j$ is the identity matrix of order $n_j$.

**Theorem 1.** Let

$$S_0(\lambda) = 1, S_1(\lambda) = \lambda,$$

$$S_j(\lambda) = \lambda S_{j-1}(\lambda) - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda), \text{ for } j = 2, 3, ..., k-1,$$

$$S_k^-(\lambda) = (\lambda + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda)$$

and

$$S_k^+(\lambda) = (\lambda - t + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda).$$

Then

(i) If $S_j(\lambda) \neq 0$, for $j = 1, 2, ..., k-1$, then

$$\det(\lambda I - A(G_l)) = (S_k^-)(\lambda)^t S_k^+(\lambda) \prod_{j \notin \Omega} S_j^{n_j-1}(\lambda). \quad (2)$$

(ii) The spectrum of $A(G_l)$ is $\sigma(A(G_l)) = \{ \lambda : S_j(\lambda) = 0 \} \cup \{ \lambda : S_k^-(\lambda) = 0 \} \cup \{ \lambda : S_k^+(\lambda) = 0 \}.$

**Proof.** Suppose $S_j(\lambda) \neq 0$ for all $j = 1, 2, ..., k-1$. We apply lemma 1 to $M = \lambda I + A(G_l)$

$$\lambda I + A(G_l) = \begin{pmatrix}
\lambda I_{n_1} & C_1 \\
C_1^T & \lambda I_{n_2} \\
& & \ddots \\
& & & \lambda I_{n_{k-1}} \\
& & & & \lambda I_{n_k} + B_l
\end{pmatrix}$$

We have

$$\beta_1 = \lambda = S_1(\lambda) \neq 0,$$

$$\beta_2 = \frac{\lambda - \frac{n_t}{n_2} S_0(\lambda)}{S_1(\lambda)} = S_2(\lambda) \neq 0$$

Similarly, for $j = 3, 4, ..., k-1, k$

$$\beta_j = \frac{\lambda - \frac{n_{j-1}}{n_j} S_{j-2}(\lambda)}{S_{j-1}(\lambda)} = S_j(\lambda) \neq 0$$

Thus

$$\beta_{k-1} = \frac{S_k^-}{S_{k-1}} + 1$$

$$= \frac{(\lambda + 1)S_{k-1}(\lambda) - \frac{n_{k-1}}{l} S_{k-2}(\lambda)}{S_{k-1}(\lambda)}$$

$$= \frac{S_k^-}{S_{k-1}}$$

Therefore, from Lemma 1,

$$\det(\lambda I - A(G_l)) = \prod_{j \notin \Omega} S_j^{n_j-1}(\lambda) \prod_{j \in \Omega} S_j^{n_j-1}(\lambda)\prod_{j \in \Omega} S_j^{n_j-1}(\lambda)$$

Since $\det(\lambda I - A(G_l)) = \det(\lambda I + A(G_l))$, Thus (i) is proved. Similar to the proof in [7], we can get (ii) by (i).

Let $R_k^+$ and $R_k^-$ be the $k \times k$ symmetric tridiagonal matrices

$$R_k^+ = \begin{pmatrix}
0 & \sqrt{d_1} & & \\
\sqrt{d_1} & 0 & \sqrt{d_3} & \\
& \sqrt{d_3} & \ddots & \sqrt{d_k} \\
& & \ddots & 0 \\
& & & \sqrt{d_k} - l + 1
\end{pmatrix}$$
and \( R_k^- = \frac{0}{\sqrt{d_2 - 1}} \begin{pmatrix} \sqrt{d_2 - 1} & 0 & \sqrt{d_3 - 1} \\ 0 & \ddots & 0 \\ \sqrt{d_3 - 1} & \ddots & \ddots \\ \sqrt{d_k - 1} & 0 & \sqrt{d_k - l + 1} & -1 \end{pmatrix} \)\).

Observe that \( R_k^+ = R_k^- + \text{diag}\{0, 0, \ldots, 0, l\} \).

**Theorem 2.** For \( j = 1, 2, 3, \ldots, k - 1 \), let \( R_j \) be the \( j \times j \) leading principal submatrix of \( R_k^+ \). Then

- \( \det(\lambda I - R_j) = S_j(\lambda), j = 1, 2, \ldots, k - 1 \),
  \( \det(\lambda I - R_k^+) = S_k^-(\lambda) \),
  \( \det(\lambda I - R_k^-) = S_k^+(\lambda) \).

**Proof.** It is well known [8] that the characteristic polynomials \( Q_j \) of the \( j \times j \) leading principal submatrix of the \( k \times k \) symmetric tridiagonal matrix

\[
H = \begin{pmatrix}
  a_1 & b_1 & & & \\
  b_1 & a_2 & b_2 & & \\
  & \ddots & \ddots & \ddots & \\
  & & a_{k-1} & b_{k-1} & \\
  & & & b_{k-1} & a_k
\end{pmatrix}
\]

satisfy the tree-term recursion formula

\[
Q_j(\lambda) = (\lambda - a_j)Q_{j-1}(\lambda) - b_j^2Q_{j-2}(\lambda)
\]

with

\[
Q_0(\lambda) = 1 \quad \text{and} \quad Q_1(\lambda) = \lambda - a_1.
\]

In our case, \( a_1 = a_2 = \ldots = a_{k-1} = 0, a_k = l - 1 \) (or \( a_k = -1 \)) and

\[
b_{k-1} = \sqrt{\frac{n_{k-1}}{n_k}} = \sqrt{d_k - l + 1},
\]

\[
b_j = \sqrt{\frac{n_j}{n_{j+1}}} = \sqrt{d_{j+1} - 1}
\]

for \( j = 1, 2, 3, \ldots, k - 2 \).

For these values, the above recursion formula gives the polynomials \( S_j(\lambda), j = 0, 1, 2, \ldots, k - 1, S_k^+(\lambda) \) and \( S_k^-(\lambda) \).

This completes the proof. \#

**Theorem 3.** Let \( R_j, j = 1, 2, \ldots, k - 1, R_k^+ \) and \( R_k^- \) as above, then

1. \( \sigma(A(G_l)) = (\bigcup_{j \in \Omega} \sigma(R_j)) \cup \sigma(R_k^+) \cup \sigma(R_k^-) \).
2. The multiplicity of each eigenvalue of the matrix \( R_j \), as an eigenvalue of \( A(G_l) \), is at least \( n_j - n_{j+1} \) for \( j \in \Omega, 1 \) for the eigenvalues of \( R_k^+ \) and \( l - 1 \) for the eigenvalues of \( R_k^- \).

**Proof.** (i) is an immediate consequence of Theorem 1 and Theorem 2. From the strict interlacing property[9] for a symmetric tridiagonal matrix with nonzero codiagonal entries, it follows that its eigenvalues are simple. Hence the eigenvalues of \( R_j, j = 1, 2, \ldots, k - 1, R_k^+ \) and \( R_k^- \) are simply. Finally, we use (2) and theorem 2 to obtain(ii). \#

**Theorem 4.** The largest eigenvalue of \( R_k^+ \) is the largest eigenvalue of \( A(G_l) \).

**Proof.** It can be proved by the strict interlacing property immediately. \#

For example, for the graph \( G_4 \) in Fig.1.

\[
R_3^+ = \begin{pmatrix}
  0 & 0 & \sqrt{3} \\
  \sqrt{3} & 0 & \sqrt{2} \\
  \sqrt{2} & \sqrt{2} & 3
\end{pmatrix}
\]

\[
R_3^- = \begin{pmatrix}
  0 & 0 & \sqrt{3} \\
  \sqrt{3} & 0 & \sqrt{2} \\
  \sqrt{2} & \sqrt{2} & -1
\end{pmatrix}
\]

and \( \Omega = \{1, 2\} \). The eigenvalues of \( A(G_4) \) in Fig.1. are the eigenvalues of \( R_1, R_2, R_3^+ \) and \( R_3^- \), they are

- \( R_1 = 0 \)
- \( R_2 = -1.7320, 1.7320 \)
- \( R_3^+ = -2.5139, -0.5720, 2.0860 \)
- \( R_3^- = -1.9459, 1.2521, 3.6938 \)

The spectral radius of \( G_4 \) in Fig.1. is \( \lambda_1(A(G_4)) = 3.6938 \)

**4 Conclusion**

We studied the spectrum of the adjacency matrix \( A(G_l) \) for all positive integer \( l \) with an effective way. Let \( R_j, j = 1, 2, \ldots, k - 1, R_k^+ \) and \( R_k^- \) as in section 3. We found that:

1. \( \sigma(A(G_l)) = (\bigcup_{j \in \Omega} \sigma(R_j)) \cup \sigma(R_k^+) \cup \sigma(R_k^-) \).
2. The multiplicity of each eigenvalue of the matrix \( R_j \), as an eigenvalue of \( A(G_l) \), is at least \( n_j - n_{j+1} \) for \( j \in \Omega, 1 \) for the eigenvalues of \( R_k^+ \) and \( l - 1 \) for the eigenvalues of \( R_k^- \).
3. The spectral radius of \( R_k^+ \) is the spectral radius of \( A(G_l) \).

It is very convenient with conclusions (1), (2), (3) to calculate the spectrum of the adjacency matrix \( A(G_l) \).
References:


