Linear Instability of Wake-Shear Layers in Shallow Water

I. VOLODKO
Department of Engineering Mathematics
Riga Technical University
1 Meza street, Riga
LATVIA

V. KOLISKINA
Department of Engineering Mathematics
Riga Technical University
1 Meza street, Riga
LATVIA

Abstract: - The linear stability analysis of shallow water flows is performed in the present paper. The base flow is modeled as a wake-shear layer and it is described by a two-parameter family of velocity profiles where one parameter characterizes the velocity deficit in the wake while the other parameter is the velocity ratio typical for the analysis of mixing layers. The linear stability problem is solved numerically for different values of the parameters. It is shown that the increase in velocity deficit as well as larger velocity ratios leads to less stable flow, but the velocity deficit is more destabilizing than the velocity ratio. The results of the present study can be applied in practice to control water quality on the sheltered side of islands.

Key-Words: - linear stability analysis, weakly nonlinear theory, shallow water flows

1 Introduction

Shallow water flows are widespread in nature and engineering. By definition, water flow is called shallow if the transverse length scale of the flow is much larger than the water depth. From environmental point of view, the structure of shallow wake flows (i.e., flows behind obstacles such as islands or headlands) is very important. Experimental and theoretical studies have shown that the flow pattern in the wake of obstacles is rather complex. The presence of eddies results in complex flows which can trap sediments and pollutants. Thus, the understanding of the structure of wake flows can help to control water quality and is important in making decisions on where to locate outfall discharges, mud disposals and cooling intakes. In addition, islands can provide shelter for marine culture from the prevailing winds. As a result, poor water quality induced by the eddies behind islands can lead to fish decease and mortality [3].

One of the methods for the analysis of the structure of shallow flows is based on linear stability theory [1]-[5].Theoretical analysis of linear and weakly nonlinear stability of shallow water flows (flows behind obstacles such as islands) is performed in [1]-[4]. Experimental studies of shallow water flows are presented in [6], [7]. Three different flow regimes are identified experimentally in [6], that is, steady bubble, unsteady bubble and vortex street. It is shown in [2]-[4] that these regimes are related to convective or absolute instabilities in shallow wakes.

The linear stability theory provides the marginal stability curve which separates the regions of linear stability and instability. The critical values of the parameters are also calculated by means of the linear stability theory. The development of the most unstable mode in a weakly nonlinear regime (that is, in the case where the value of the parameter of the flow such as the Reynolds number is slightly larger than the critical value) is often described by a relatively simple amplitude evolution equations such as complex Ginzburg-Landau equations. The complex Ginzburg-Landau equation is derived from the Navier-Stokes equations (for the case of a plane Poiseuille flow) in reference [8]. Later (see, for example, [3], [9], [10]) the Ginzburg-Landau equation was derived for other flows (including shallow water flows [3]).

The Ginzburg-Landau model is quite popular in applications [11], [12]. One reason is that it can exhibit a rich variety of solutions depending on the values of the coefficients of the Ginzburg-Landau equation [13]. The model includes such effects as diffusion and nonlinearity. Since the Ginzburg-Landau equation is a scalar equation (at least in the applications described in the present paper), it is a relatively simple task to integrate the equation numerically.

It is well-known that there are also shear layers across the wake [3]. Thus, there is a need to analyze the combined plane wake-shear layer velocity profile. To the authors' knowledge, such a problem has received little attention in the literature. Some results related to the stability of wake-shear flows in deep water are presented in [14]. A family of plane wake-shear layer velocity profiles is analyzed in [15]. It is found in [15] that the presence of shear across the wake forces the neutrally stable modes to be singular.

In the present paper stability characteristics of a family of plane wake-shear layer velocity profiles are presented. The analysis is performed for shallow water flow. Combined effects of shear and bottom friction on the stability of wake flows are discussed.

2 Linear stability analysis

The two-dimensional shallow water equations have the form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{\partial p}{\partial x}+\frac{c_f}{2h}u\sqrt{u^2+v^2}=0, \tag{2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{\partial p}{\partial y}+\frac{c_f}{2h}v\sqrt{u^2+v^2}=0. \tag{3}$$

Here u and v are the velocity components in the x and y directions, respectively, h is water depth, c_f is the friction coefficient, and p is the pressure. The system (1) – (3) can be transformed to the single equation containing only the stream function of the flow, $\psi(x,y,t)$, which is defined by the relations

$$u=\frac{\partial \psi}{\partial y},\ v=-\frac{\partial \psi}{\partial x} \tag{4}$$

Using (1)-(4) we obtain

$$\begin{gathered}(\Delta\psi)_t+\psi_y(\Delta\psi)_x-\psi_x(\Delta\psi)_y+\frac{c_f}{2h}\Delta\psi\sqrt{\psi_x^2+\psi_y^2}\\ +\frac{c_f}{2h\sqrt{\psi_x^2+\psi_y^2}}\left(\psi_y^2\psi_{yy}+2\psi_x\psi_y\psi_{xy}+\psi_x^2\psi_{xx}\right)=0\end{gathered} \tag{5}$$

where $\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

Assume that $\psi_{0y}(y)=u_0(y)$ is the base flow solution. A perturbed solution can be written in the form

$$\psi=\psi_0(y)+\varepsilon\psi_1(x,y,t)+\varepsilon^2\psi_2(x,y,t)+... \tag{6}$$

Substituting (6) into (5) and keeping only the linear terms with respect to ε, we obtain

$$L\psi_1=0, \tag{7}$$

where

$$\begin{gathered}L\psi_1=\psi_{1xxt}+\psi_{1yyt}+\psi_{0y}(\psi_{1xxx}+\psi_{1yyx})-\psi_{0yyy}\psi_{1x}\\ +\frac{c_f}{2h}\left[(\psi_{1xx}+2\psi_{1yy})\psi_{0y}+2\psi_{1y}\psi_{0yy}\right]+B(\psi_{1xx}+\psi_{1yy})\end{gathered}$$

Equation (7) describes the linear stability of the flow given by $u=u_0(y)$. The function $u_0(y)$ in the present study is chosen in the form

$$u_0(y)=1-\frac{f}{\cosh^2 y}+r\tanh y, \tag{8}$$

where f is the wake deficit parameter (i.e, the velocity deficit divided by the mean ambient velocity), r is the velocity ratio (i.e., the velocity difference across the layer divided by mean velocity). The limiting cases $(r=0)$ or $(f=0)$ represent pure wake flow or mixing layer, respectively. The presence of the two parameters in (8) allows one to investigate the effect of shear on the stability of wake flows. The profile (8) is suggested in [15] for the stability analysis of wake-shear layers in deep water flows and is adopted in the present study.

Using the method of normal modes we assume that the function ψ_1 in (7) has the form

$$\psi_1(x,y,t)=\varphi_1(y)\exp[ik(x-ct)] \tag{9}$$

where $\varphi_1(y)$ is the amplitude of the normal perturbation (9). Using (9) and (7) the modified Rayleigh equation for the function $\varphi_1(y)$ is obtained in the form

$$\begin{gathered}\varphi_1''(-ikc+iku_0+2Su_0)+Su_{0y}\varphi_1'\\ +\varphi_1\left(ik^3c-ik^3u_0-iku_{0yy}-Sk^2u_0\right)=0\end{gathered} \tag{10}$$

with the boundary conditions

$$\varphi_1(\pm\infty)=0 \tag{11}$$

Here $S=\frac{c_f b}{2h}$ is the stability parameter (called the bed friction number in [1]) and b is the wake half-width. Problem (10)-(11) is an eigenvalue problem where the eigenvalues are $c=c_r+ic_i$. The base flow (8) is said to be stable if all c_i are negative and unstable if at least one $c_i > 0$.

In order to solve the linear stability problem (10) – (11) we used a pseudospectral collocation method based on Chebyshev polynomials. The Chebyshev polynomials are defined on the interval [-1,1]. Therefore, we map the interval $-\infty < y < +\infty$ onto the interval $-1 < z < 1$ by means of the substitution $z = \frac{2}{\pi}\arctan y$. Then we seek the solution to (10) – (11) in the form

$$\varphi_1(z) = \sum_{k=0}^{N} a_k (1 - z^2) T_k(z), \qquad (12)$$

where $T_k(z)$ is the Chebyshev polynomial of degree k. The function $\varphi_1(z)$ is chosen in the form (12) to simplify the numerical solution of (10) – (11) since the boundary conditions (11) in terms of the variable z will be satisfied automatically. The collocation points z_j are chosen in the form

$$z_j = \cos\frac{\pi j}{N}, \quad j = 0,1,\ldots,N. \qquad (13)$$

Using (12) we evaluate the function $\varphi_1(z)$ and its derivatives at the collocation points (13). As a result, we obtain a generalized eigenvalue problem of the form

$$(A - \lambda B)a = 0 \qquad (14)$$

where A and B are complex-valued matrices and

$a = (a_1 a_2 \ldots a_N)^T$,

the superscript T denotes the transpose.

The analysis performed in [16] shows that the solution in the form (12) has two advantages in comparison with traditional collocation methods [17]. First, the matrix B in (14) is not singular so that there is no need to perform additional transformation of (14) to get rid of the zero rows in the matrix B. Second, the condition number of the matrices in this method is reduced.

Problem (14) is solved numerically by means of IMSL routine DGVCCG. The results of computations are shown in Figs. 1 – 4. Figure 1 plots neutral stability curves for the parameter S versus k for $f = 0.3$ and different values of the velocity ratio r. The region of instability is located below the curves. It is seen from Fig. 1 that as the role of shear across the wake increases (i.e., as the parameter r increases), the flow becomes more unstable – the maximal values of the parameter S also increase. However, the range of the unstable values of the wave number k decreases as the parameter r grows. In addition, the critical wave numbers decrease as the velocity ratio increases (the maxima are shifted towards smaller k values).

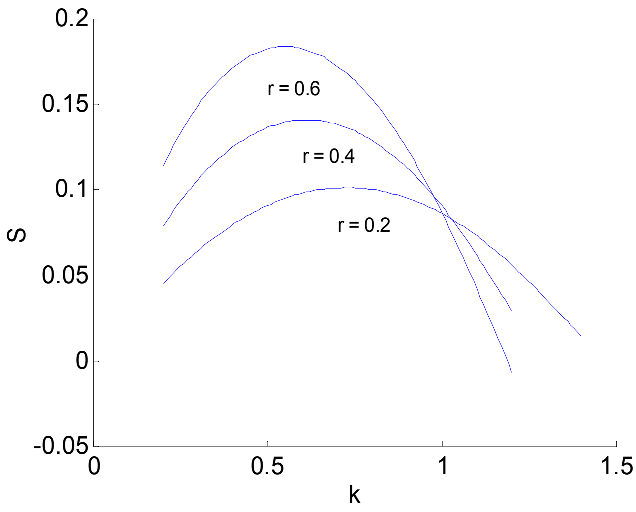

Fig.1. Neutral stability curves for different values of r at $f = 0.3$.

The neutral stability curves for the parameter S versus the wave number k are shown in Fig. 2 for the case $r = 0.2$ and different values of f. The flow becomes more stable as the wake deficit parameter f decreases. For smaller values of f the most unstable mode corresponds to larger wavelength (the critical values of k decrease as f decreases).

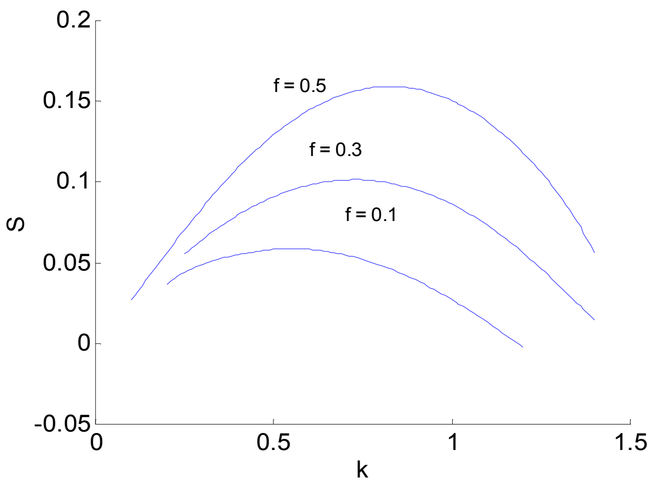

Fig. 2. Neutral stability curves for different values of f at $r = 0.2$.

The critical values of $S\,(S_{cr} = \max_k S)$ are shown in Fig. 3 versus r for different values of f. Destabilizing effect of the velocity ratio on wake stability is clearly seen from the figure. In addition, the critical values of S grow almost linearly as the velocity ratio r increases.

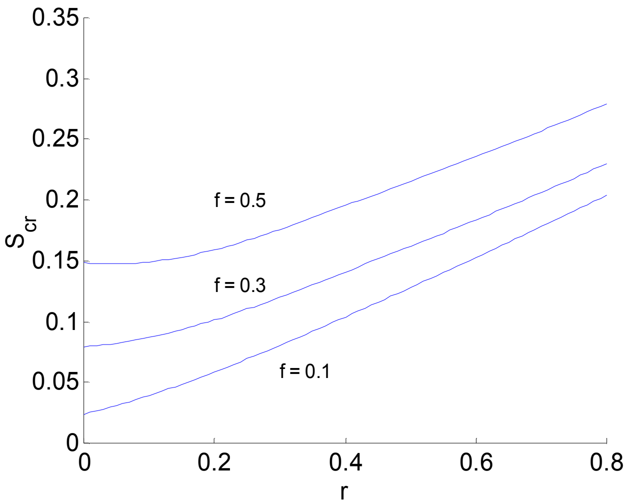

Fig. 3. Critical values of S versus r for three values of f.

The critical values of the stability parameter S versus f for different values of r are shown in Fig. 4. The increase of the wake deficit parameter f has an important impact of the flow instability: the critical values of S grow very quickly as the parameter f increases. The comparison of the stability diagrams in Figs. 3 and 4 shows that the velocity deficit parameter has larger influence on the flow instability. The flow becomes more unstable as both parameters r and f increase. However, the rate at which the flow becomes more unstable is higher for the case where the wake deficit parameter increases for a fixed r in comparison with the case where f is fixed and r increases.

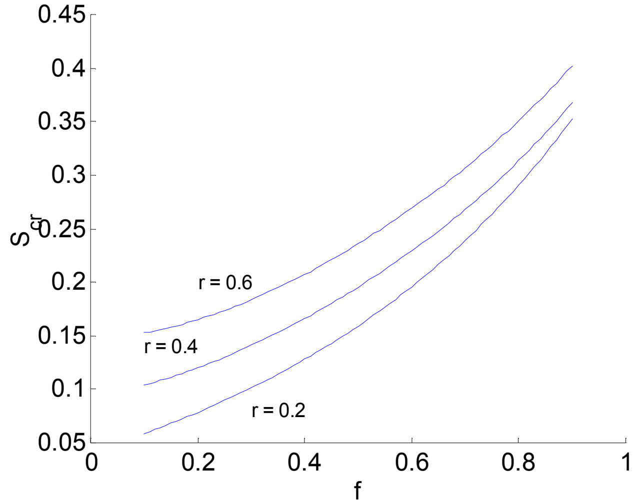

Fig. 4. Critical values of S versus f for three values of r.

3 Discussion

Linear instability analysis of plane wake-shear layer velocity profile is performed in the present paper. The convective instability boundary is found for the two-parameter family of velocity profiles. The parameters are the wake deficit parameter f and the velocity ratio r. Previous analyses of wake flows in shallow water [2]-[4] showed that the wake flow is absolutely unstable for large values of f. This fact is supported by experimental data in [6]. On the other hand, the analysis of mixing layers in [18] for deep water showed the presence of the region of absolute instability for large values of the velocity ratio r. The velocity profiles considered in the present paper simulate the presence of shear across the wake. Thus, it is plausible to assume that the wake-shear layer can also be absolutely unstable in some regions in the parameter space. The analysis of the transition between absolute and convective instability is the topic for future work.

It is shown in [3] that if the parameter S is slightly smaller than the critical value, then one can analyze the development of instability using methods of weakly nonlinear theory. It is shown that in a weakly nonlinear regime the amplitude of the most unstable mode satisfies the complex Ginzburg-Landau equation. The coefficients of the equation are expressed in [3] in terms of integrals containing the parameters of the linear stability problem. The derivation in [3] is obtained for an arbitrary function $u_0(y)$. Thus, one can also apply the theory developed in [3] to plane wake-shear layer velocity profiles (8). The authors are currently working on this problem.

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