Formal Description and Modeling Topological Relations in 3D Based on Dimension-Extended and Euler-Poincare Characteristics

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Abstract: Formal description and representation of topological relations between 3D spatial features is one of the key issues in developing 3D GIS. The criteria on the description and determination of topological relations are topological invariants, which may be dimension, separations, Euler-Poincare characteristics etc. This paper is concentrated on some basic issues of modeling topological relations using these invariants. With the introduction of the concept of pure k- complex (0≤k≤3), formal description of spatial features in the GIS is given. An approach is proposed to describe the topological relations between 3D spatial features with topological components of k-complex (0≤k≤3), i.e. boundary and interior. The topological relationship set: {disjoint, touches, within, crosses, overlaps, contains, equal} for the pair of spatial features were defined with dimension-extended. The topological relations for the pair of k-complex were examined with dimension-extended 4-intersection prior to calculate of Euler-Poincare characteristics. The model based on dimension-extended and Euler-Poincare characteristics can distinguish the topological relations of 3D spatial features more details than the classic 4/9-intersection model.

Key-Words: Topological Relations, Formal Description, Dimension-Extended, Euler-Poincare Characteristics, 3D Spatial Features, Complex.

1 Introduction

Topological relations is one of the most basic and important relations among the GIS spatial features, and the basis for spatial query, analysis and reasoning [1]. In order to represent the topological relations among 3D spatial features, we need to study and develop the semantic description and formal description of topological relations in 3D spatial features[2]. The representative achievements in the formal research method in topological relations are 4I model, 9I model [3], and RCC model [4]. As for 9I model, many scholars have done a lot of research [5-10]. The topological relations are the invariants in topological transformation and the criteria for differentiation is topological invariants, such as the content invariants put forward by Egenhofer[3], the dimension invariants put forward by Clementini [11] and the sequence invariants by other scholars. Some scholars developed the 4I/9I model by using different topological invariants [2, 8, 11-13]. But we still need to do further research on the formal description and formal model in topological relations of 3D spatial features.

2 Formal Description of Simplicial Complexes

2.1 Formal Description of Simplexes

An n-dimensional simplex σ_n is defined as the convex hull in R_n of any n + 1 points $\alpha_0, \cdots, \alpha_n$ not contained in any (n-1)-dimensional hyper-plane. Thus the points of σ_n£½£$\ddot{\alpha}_0, \cdots, \alpha_n$£ are the linear combinations of the vertices $\alpha_0, \cdots, \alpha_n$ (regarded as n-vectors) of the following form:

$$x \in \sigma_n, x = \sum_{j=0}^{n} x^j \alpha_j, \sum_{j=0}^{n} x^j = 1, x^j \geq 0. \qquad (1)$$

A face of simplex S_n of any dimension n is the simplex determined by any proper subset of the vertices, i.e. the convex hull in R_n of a proper subset of $\{\alpha_0, \cdots, \alpha_n\}$. In particular the faces of dimension n-1 are just the simplexes

σ_j^{n-1}£½£$\ddot{\alpha}_0, \cdots, \hat{\alpha}_i, \cdots, \alpha_n$£, (2)

(where the hat indicates that a symbol is to be considered omitted) . Thus σ_n has exactly n+1 faces of dimension n –1.

For the 3D Euclidean spaces, 0-simplex is a point, 1-simplex is a line and 2- simplex is a triangle . k-simplex $(0 \leq k \leq 3)$ can be represented as equation:

$$\sigma_k = \bigcup_{i=0}^{k} \sigma_i(\cdot) \quad (0 \leq k \leq 3) \qquad (3)$$

In the equation, $\sigma_i(\cdot)$ is the set of $i-$ simplex. 3-simplex is a tetrahedron, represented as a corpora :

$$\sigma_3 = \bigcup_{i=0}^{3} \sigma_i(\cdot). \qquad (4)$$

0- simplex has no direction. We can know from (3) that the corpora of k- simplex $(0 \leq k \leq 3)$ are not sole because of the directional problem. In order for the simplex to be sole and to simplify uniting calculation of a set, several rules are given in the paper as follows:

Rule 1: the direction of 1-simplex is from α_1 to α_2 , which is represented as $\alpha_1\alpha_2$, and $\alpha_1\alpha_2 - \alpha_2\alpha_1 = 0$.

Rule 2: the direction of 2-simplex is counter-clockwise, which can be only ensured by the vertex, $\alpha_1\alpha_2\alpha_3 = \alpha_2\alpha_3\alpha_1 = \alpha_3\alpha_1\alpha_2$ and $\alpha_m\alpha_n\alpha_k - \alpha_k\alpha_n\alpha_m = 0$.

Rule 3: the definition of the direction of 3-simplex is as follows: in the four sides of triangle in tetrahedron, the normal directed towards the outside is positive and at the same time, satisfy the right-hand rule, namely counterclockwise along the normal and $\alpha_1\alpha_2\alpha_3\alpha_4 = \alpha_2\alpha_3\alpha_4\alpha_1 = \alpha_3\alpha_4\alpha_1\alpha_2$ $= \alpha_4\alpha_1\alpha_2\alpha_3$ and $\alpha_m\alpha_n\alpha_k\alpha_l - \alpha_l\alpha_k\alpha_n\alpha_m = 0$.

2.2 Formal Description of Simplicial Complex

A Simplicial complex K is then an arbitrary (finite or countably infinite) collection of simplex, with the following properties:

(1) Together with each simplex in the collection, all of its faces of all dimensions should also be in the collection.

(2) Any two simplexes in the collection that intersect should either coincide (i.e. be the same simplex) or intersect precisely in a common face.

A Simplicial complex is most conveniently given by indicating its vertices $\alpha_0, \cdots, \alpha_n, \cdots,$ together with those (finite) subsets of these that determine simplexes of the complex.

The algebraic boundary of any simplex σ_n£½£$\ddot{\alpha}_0, \cdots, \alpha_n$£ is defined as the following formal linear combination of its faces of dimension n – 1

$$\partial \sigma^n = \sum_{i=0}^{n} (-1)^i \sigma_i^{n-1}, \qquad (5)$$

where σ_i^{n-1}£½£$\ddot{\alpha}_0, \cdots, \hat{\alpha}_i, \cdots, \alpha_n$£ ,the hat over $\hat{\alpha}_i$ indicating its omission. For any complex K the group $C_n(K)$ of n-dimensional integral chains of K is the free Abelian group consisting of all finite formal integral linear combinations of the n-simplexes of K:

$$c \in C_n(K), \quad c = \sum_{\alpha} \lambda_\alpha \sigma_\alpha^n, \quad \sigma_\alpha^n \in K;$$

here α indexes the n-simplexes of K, and the coefficients λ_α are integers. From the defining formula (5) for the algebraic boundary of a simplex we obtain the boundary operator ∂ on the integral n-chains of K:

$$\begin{gathered} \partial : C_n(K) \to C_{n-1}(K), \\ \partial(\sum_{\alpha} \lambda_\alpha \sigma_\alpha^n) = \sum_{\alpha} \lambda_\alpha (\partial \sigma_\alpha^n), \end{gathered} \qquad (6)$$

and thence the following chain complex $C_*(K)$:

$$\cdots \xrightarrow{\partial} C_n(K) \xrightarrow{\partial} C_{n-1}(K) \xrightarrow{\partial}$$

$$C_{n-2}(K) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(K) \xrightarrow{\partial} 0$$

According to the three rules in 1.1, the shared side in the two intersected k- simplex $(0 \leq k \leq 3)$ is opposite in direction and can be counteracted in algebraic addition calculation. ∂C_n is represented as:

$$\begin{aligned} \partial C_n &= \bigcup_{i=0}^{n-1} (\sum_{j=1}^{m} \sigma_i^j(\cdot)) \\ &= \sum_{j=1}^{m} \sigma_0^j(\cdot) + \bigcup_{i=1}^{n-1} (\sum_{j=1}^{m} \sigma_i^j(\cdot)) \end{aligned} \qquad (7)$$

The n- framework ($C_n^{(n)}$) of complex C_n is the uniting set ($\bigcup_{r=0}^{n} \sigma_r$) of all simplexes of C_n whose maximum dimension is n.

The interior (C_n°) of n- complex C_n ,is the set of side in $n-$framework of C_n which does not belong to point set boundary(∂C_n) , namely

$$C_n^{\circ} = C_n^{(n)} - \partial C_n . \tag{8}$$

3 Topological Description of Spatial Feature Based on Simplicial Division

Space related to GIS is measurement space. Neighboring set theory generalized neighboring measurement concept. The topology R , which is from the measurement d in measurement space R_n, becomes the measurement topology defined by d . Obviously, every measurement space is also a topology space. Therefore, the research on 3D spatial data model will enable us to define 3D spatial features and its topological spatial relations in 3D topological space.

If a topological space is homeomorphous with several limited simplexes which are joined together in a Euclidean space, that is, if two simplexes are intersected, the shared part must be a shared side, the topological space is called a simplicial divide. A simplicial divide of topological space X is composed of a simplex C_n and a homeomorphism $h : | C_n | \to X$, represented as $X = (C_n , h)$. So the simplicial divide model of spatial feature is not sole and it depends on the choice of simplex $\boldsymbol{C_n}$ and divide homeomorphism h . however, all simplexes which are decomposed by each model are the whole inlay of spatial feature and they will not tamper with the topological description of spatial feature.

The dimension of simplex $dim(S)$ is the number where the count of nodes subtracts 1, and the dimension of complex $\dim(C_n)$ is the maximum dimension of simplexes including in the complex, namely,

$$dim(S) = n - 1 \square$$

$$dim(C_n) = sup(dim(S_i))(S_i \in C_n)$$

From combination topological theory, we can divide spatial feature by its spatial dimension. According to dimension definition, spatial features in 3D topological space are composed of figures with the dimension equal to or less than 3. In terms of different dimension, the spatial features are divided into 4 feature types, namely point Entity, Line Entity, Surface Entity and Body Entity.

From the definition of 3D spatial features, we use simplex having the same dimension as spatial features to divide space, thus any one spatial feature can be described by pure k-complex $(0 \le k \le 3)$, and its spatial extension and spatial relations can be represented by a series of algebraic elements. The topological characteristics among simplexes reveal the topological relationship between spatial feature and its algebraic elements, which provides strong theoretical basis for the establishment of spatial data model. Here we take the example of pure 3- simplex to describe the steps of spatial features.

(1) It is testified that the n- framework($C_3^{(3)}$) of complex C_3 is uniting set of all C_3 simplexes in which the maximum dimension is 3:

$$C_3^{(3)} \square \bigcup_{r=0}^{3} S_r \square \bigcup_{j=0}^{m} \bigcup_{k=0}^{3} S_k^j(\cdot) \tag{9}$$

In the equation, $S_i^j(\cdot)$ is a k-simplex.

(2) The calculation of boundary of simplex can be divided into 2 steps. Firstly , k-simplex in the C_3 $(0 \le k \le 2)$ is seen as a group of k dimension vector, and we use algebraic operation to obtain the boundary of each dimension algebraic $S_k(\cdot)$; Then, we can obtain the boundary of simplicial complex, which, according to equation (3) ,can be represented as follows:

$$\begin{aligned} \partial C_3 &= \bigcup_{i=0}^{2} (\sum_{j=1}^{m} S_i^j(\cdot)) \\ &= \sum_{j=1}^{m} S_0^j(\cdot) + \sum_{j=1}^{m} S_1^j(\cdot) + \sum_{j=2}^{m} S_2^j(\cdot) \end{aligned} \tag{10}$$

(3) From the defining formula (8) , we can obtain the interior of 3D spatial feature, which is represented as:

$$C_3^{\circ} = C_3^{(3)} - \partial C_3 \tag{11}$$

4 Formal Model of 3D Topological Relations

4.1 Topological Operation Using the Dimension Extended Method

The formal definition of 3D topological spatial features has been given in the above part according to point-set topology. Basing on the method of point set (union, intersection, difference, compensation) and formal definition of spatial features, we study the model of 3D spatial topological relations .

The dimension of the boundary and interior of two complex A and B is defined as the maximum dimension of all faces in ∂C_n. The value ranges from 0 to the maximum dimension of all faces namely the dimension of $\partial A_n \cap \partial B_n$, $\partial A_n \cap B_n^{\circ}$, $A_n^{\circ} \cap \partial B_n$ is between 0 and n-1, and the dimension of $A_n^{\circ} \cap B_n^{\circ}$ is between 0 and n.

We can extend the topological relations of spatial features based on the dimension of simplexes and simplicial complexes, and the definition is as follows:

(1) ***disjoint*** $< A_n, disjoint, B_n > \Leftrightarrow (A_n \cap B_n = \phi)$;

(2) ***touches*** $< A_n, touches, B_n > \Leftrightarrow (dim(A_n^{\circ} \cap B_n^{\circ}) = 0) \wedge (dim(A_n \cap B_n) > 0)$;

(3) ***within*** $< A_n, within, B_n > \Leftrightarrow (dim(A_n^{\circ} \cap B_n^{\circ}) > 0) \wedge (A_n \cap B_n = A_n)$;

(4) ***crosses*** $< A_n, crosses, B_n > \Leftrightarrow (dim(A_n^{\circ} \cap B_n^{\circ}) < sup(dim(A_n^{\circ}), dim(B_n^{\circ}))) \wedge (A_n \cap B_n \neq A_n) \wedge (A_n \cap B_n \neq B_n)$;

(5) ***overlaps*** $< A_n, overlaps, B_n > \Leftrightarrow (dim(A_n^{\circ} \cap B_n^{\circ}) = dim(A_n^{\circ}) = dim(B_n^{\circ})) \wedge (A_n \cap B_n \neq A) \wedge (A_n \cap B_n \neq B)$;

(6) ***contains*** $< A_n, contains, B_n > \Leftrightarrow < B_n, within, A_n >$;

(7) ***equal*** $< A_n, equal, B_n > \Leftrightarrow < A_n = B_n >$

We use dimension extension method to describe the model of the topological spatial relations , which is as follows:

$$I_{dim(A_n,B_n)} = \begin{bmatrix} dim(\partial A_n \cap \partial B_n) & dim(\partial A_n \cap B_n^{\circ}) \\ dim(A_n^{\circ} \cap \partial B_n) & dim(A_n^{\circ} \cap B_n^{\circ}) \end{bmatrix} \quad (12)$$

In the equation, the element $\dim(\cdot)$ is the intersected dimension of the boundary and interior. With the dimension-extended 4I model and the above definition, we can get the topological relations between two spatial features.

4.2 Topological Relations Model Based on Euler-Poincare Characteristics

In the topological relations which are obtained from the model in 2.1, there are still some topological relations among spatial features of non-homeomorphism that can not be differentiated. So we introduce separations as topological invariants, which is an important topological feature of figure set. Generally, separations are obtained by calculating the Euler-Poincare characteristics of figure set. Euler-Poincare characteristics is the invariant used to measure the features of figure structures, and it can represent the separations of the set obtained in the interaction of spatial objects and will not change with the changing of simplicial divide. On the basis of simplicial divide, the Euler-Poincare characteristics $\chi(O_n)$ of a dimension n spatial feature O_n can be defined as:

$$\chi(O_n) = \sum_{i}^{n} (-1)^i a_i \quad (13)$$

In the equation, $a_i (0 \le i \le n)$ is the count of $i-$simplex in O_n.

Based on calculating method of Euler-Poincare characteristics, a Euler-Poincare characteristics model is built to formally describe topological relations , which is represented as :

$$\gamma(A,B) = \begin{bmatrix} \chi(\partial A \cap \partial B) & \chi(\partial A \cap B^{\circ}) \\ \chi(A^{\circ} \cap \partial B) & \chi(A^{\circ} \cap B^{\circ}) \end{bmatrix} \quad (14)$$

In the equation, the value of element $\chi(\cdot)$ is $[0,+\infty]$. Obviously, Euler-Poincare characteristics model is also an extension of 4I model. Different values of the elements have different topological relations between two objects.

5 Conclusion

Topological invariants is the criteria for differentiating topological relations , which will not be changed with spatial revolving, translation, zoom out/in. due to diversity in the form of spatial objects and the complexity in the spatial relations among the objects, there are still lots of problems to be solved in the formal description of spatial features and description model of topological relations . Based on the achievements made by those scholars in the past, this paper does further research and some ideas are put forward at the same time:

(1) in terms of point-set topology, it gives the formal description of simplexes and complexes;

(2) it uses simplexes and complexes as the tool of formal description in 3D spatial features and combines the method of geometric operation and algebraic operation to calculate the topological boundary and interior in topological space.

(3) on the basis of dimension definition of intersection between two spatial boundary and

interior, it presents dimension extended 4I model based on complexes.

(4) in order to make the difference among non-homeomorphism spatial features, it puts forwards 4I model based on Euler-Poincare characteristics.

Compared with the classic 4I/9I model, the spatial topological relations description model based on dimension-extended and separations further describe the topological relations in detail. The research mentioned above will provide the theoretical basis for the formal description of topological relations and establishment of a model which can effectively describe spatial relations.

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