Modeling study on a forward recursive construction approach of dynamical equations of mechanism systems with flexible articulated joints

YANG Yuan-ming1,2 Zhao Bing1 Guo Jiansheng1 CHEN Chuan-yao2SONG Tian-xia2
1. Department of Civil Engng., 2.College of Civil Engng.and Mech.
1. Nanyang Institute of Tech., 2. Huazhong Univ. of Sci. and Tech.
1. Nanyang, Henan, 473004, 2. Wuhan, 430074,
P.R.CHINA

Abstract: This paper discusses modeling approach of a forward recursive construction of mechanical system with the flexible hinge joints. A forward recursive construction approach for dynamical equations of articulated flexible multibody systems with open loop construction are presented in the Wittenberg interconnected matrix systems. The motion of the flexible multibody systems with articulated joints has been resolved using the floating reference frames which are defined on the rigid states of flexible bodies. Vibration modes of components and model coordinates are used to express the deformation of the flexible multibody systems with articulated joints. Forward recursive construction approaches are used in kinetic analyzing. Kinetic equations and dynamics equations of the flexible body system with roll hinge and slide hinge are derived. The equations are fairly direct-vision and simple in resolution of the motion, straddling of elastic deformation of component and description of construction of system.

Key words: Flexible articulated joints; A forward recursive construction approach ; Dynamical equations of mechanism systems; Dynamics Analysis

1 Introduce

Dynamics model for Space complex mechanical system is controlled multi-body system. Taking into account the modern structure of the growing and high-speed operation, their components and joints must be considered as flexible body. It is the basis for Digital simulation of mechanical system dynamics that facilitate the mathematical model of flexible multibody system with function of formality and computer automatically. In theory, It is not too difficult that derived a mathematical model of dispersion system according to the basic principles of mechanics, is more complicated.

In recent decades, multibody system dynamics rapid development as the fastest growing area of applied mechanics. On one hand, multi-body system is more and more used in the actual system model as such as robots, spatial structure and dynamics of biological systems, on the other hand, the multi-body system dynamics research activities have contributed to a number of sub-areas of research. Most interested research field of multi-body dynamics is incorporated flexible effect into dynamic control equation[1-3]. For flexible multi-body system, in particular the movement system consisting of the small deformation objects , most of that use the relative described method, and introduce floating coordinates to decompose movement of systematic components, such as all nodes tangent coordinates, secant coordinates, or Trsserand coordinate and Bucken coordinates and so on[4]. Usually ,using the finite element method, the modal components method and other methods discrete elastic body relatively floating coordinates.Modal components method is based on dynamic sub-structure method of vibration analysis field of the modern structure[5], which greatly reduced generalized coordinates of kinetic equation, and can use that static modal recovery mode truncation error, and improve
compute accuracy \cite{6, 7}. When there exist a large component deformation, it would consider using the finite deformation theory to build system model \cite{8}. On the multi-body system dynamics analysis, currently, it has Kane equation, Roberson/Wittenburg system, variational methods, coordinates of the largest number, rotation matrix method and a forward recursive method of dynamic equation. The equation can be derived by Lagrange equation, virtual work principle, principle of virtual power, Gibbs-Appells equations and Newton-Euler equations. For the ways of above, it is existed controversy which is best \cite{1, 8}. On ways of describing multi-body structure, it have the correlative matrix of Wittenburg, access matrix \cite{9}, Huston's array of internal joint rigid \cite{10}, and Kim and Haug's recursive equation \cite{11}, Shabana's recursive projection algorithm \cite{12, 13}, etc. For systems of non-trees, or constraint multi-body system, the method of handling the constraint equations have the pseudo-inverse method, orthogonal-law, singular value decomposition and zero/tangent space law, etc. It is very difficult to assess the merits of these methods of multi-body system dynamics analysis, because every way has each advantage and fault. It need continued study and exploration.

The paper \cite{14} decomposes components campaign by the rigid diaphragm floating coordinates of suitable description of the elastic deformation, and use components modal method to discrete elastic deformation, and establish hinge joints flexible multi-body system dynamics equations of the R/W system \cite{9, 14} starting from D'Alembert's Principle. Equation extends to the non-tree and constraints system. The author uses the singular value decomposition polycondensation to handle closure and constraint conditions, reduce the number of variables of the equation.

The paper \cite{15-16} presents a modeling of joint hinge of the dynamics of flexible multi-body system by CAD, which is hinge joints recursive modeling method. It decomposes components campaign by the rigid diaphragm floating coordinates of suitable description of the elastic deformation, and use components modal method to discrete elastic deformation, and stylized group set to establish dynamic equation that is expressed by hinge's relative coordinates and modal coordinates in flexible multi-body system with Chain, the tree and the closed-loop during the forward recursive process of kinematics. It has the advantages of the methods the traditional absolute coordinates and hinge's relative coordinates and overcome their deficiencies.

In the paper, a forward recursive construction approach for dynamical equations of articulated flexible multibody systems with open loop construction are presented in the Wittenberg interconnected matrix systems. The motion of the flexible multibody systems with articulated joints has been resolved using the floating reference frames which are defined on the rigid states of flexible bodies. Vibration modes of components and model coordinates are used to express the deformation of the flexible multibody systems with articulated joints. Because of independence of forward recursive construction approaches in the chain, the approach is very easy and has been developed as a method of parallel computing.

2 kinematics of Rotating Hinge system

2.1 Description of the object deformation

Considering the i body of flexible system in the small elastic deformation, its elastic deformation $u_p$ can be expressed by elastic modal $\psi_j$ and modal coordinates vector $\eta_i$ thought using the finite element method as following:

$$ u_p = \Psi \eta_i \quad (2.1) $$
Its displacement and angular are write by following forms:

\[ u^d_r = \Psi^d_i \eta_i ; \quad u^r_r = \Psi^r_i \eta_i \]  \hspace{1cm} (2.2)

Where \( o-e \) is system coordinate, \( e = [e^1, e^2, e^3]^T \) is orthogonal coordinate are fixed at it, \( C_i - e \) is floating coordinate, action at the centre of mass \( C_i, e = [e^1, e^2, e^3]^T \) is orthogonal coordinate fixed. \( P_o \) is an arbitrary point at elastic body. The location after deformation is at point \( P \).

The derivative of the formulation (2.2) can be written as following:

\[ \dot{u}^d_r = \dot{\Psi}^d_i \dot{\eta}_i ; \quad \dot{u}^r_r = \dot{\Psi}^r_i \dot{\eta}_i \]  \hspace{1cm} (2.3)

As shown in figure 1, displacement vector of an arbitrary point \( P \) at elastic body is expressed as:

\[ r_p = r_{C_i} + \rho_{P_o} + u^d_r \]  \hspace{1cm} (2.4)

### 2.1 Movement description of elastic-body

According to formulation of displacement vector of the finite element nodes \( P \) at elastic body \( B_i \), and considering formula (2.2), it has the velocity and angular velocity of point \( P \) in the inertial coordinate presented by following forms:

\[ V^p = V^{C_i} + \omega^{C_i} \times (\rho_{P_o} + u^d_r) + \psi^d_r \dot{\eta}_i \]  \hspace{1cm} (2.5)

\[ \omega^p = \omega^{C_i} + \psi^r_i \dot{\eta}_i \]  \hspace{1cm} (2.6)

That \( V^{C_i} \) and \( \omega^{C_i} \) are respectively the centre of mass's speed and relative centroid's ration of elastomer \( B_i \). Acceleration and angular acceleration of node \( P \) of the elastomer \( B_i \) as following:

\[ a^p = a^{C_i} + \alpha^{C_i} \times (\rho_{P_o} + u^d_r) + \alpha^{C_i} \times (\rho_{P_o} + u^d_r) + 2 \omega^{C_i} \times \psi^d_r \ddot{\alpha}_i + \psi^r_i \ddot{\eta}_i \]  \hspace{1cm} (2.7)

\[ \alpha^p = \alpha^{C_i} + \omega^{C_i} \times \psi^r_i \dot{\eta}_i + \psi^r_i \ddot{\eta}_i \]  \hspace{1cm} (2.8)

That \( a^{C_i} \) and \( \alpha^{C_i} \) are respectively speed of the centre of mass and relative centoid's ration at the centre of mass of elastic-body \( B_i \).

### 2.2 The kinematics recurrence relations of joint rotation Hinge tree system

1. Relative movement of adjacent objects

   Considering one pair of adjacent objects \( B_i \) and \( B_j \) of flexible multi-body system, assuming \( B_i \) is the internal objects of \( B_j \), joint hinge of the two objects are recorded as \( a \). For convenience, joint object \( B_i \) of two adjacent objects is recorded \( i^+ (a) \) and the object \( B_j \) is recorded \( i^- (a) \), and \( i^+ (a) \) and \( i^- (a) \) is recorded incorporate \( i^\pm (a) \). For rotational hinge, the joint point \( P \) and \( Q \) of two objects is the coincidence. Now hinge - of two objectst link firmly a vector-based \( P (Q) e_{i^\pm (a)} \). The vector connected to solid center is
expressed as \( e_{i}^{e_{i}}(a) \). As shown in figure 2, elastic rotational and angular velocity of vector-based relative respective objects are respectively following as:

\[
ed_{i}^{\Omega}(\dot{a}) = \dot{G}_{i}^{\Omega} \dot{G}_{i}^{\Omega} e_{i}^{e_{i}}(a) \cdot \dot{\Omega}_{i}^{\Omega} = \dot{\psi}_{i}^{\Omega} \dot{\psi}_{i}^{\Omega} \] (2.9)

\( G_{i}^{\Omega} \) and \( G_{i}^{\Omega}^{(a)} \) are cosine matrix of elastic rotation direction and the cosine matrix of initial direction \( e_{i}^{e_{i}}(a) \) and \( e_{i}^{e_{i}}(a) \) of vector-based. By using angular weight of hinge joints \( a \), \( G_{i}^{\Omega}^{(a)} \) is expressed as:

\[
G_{i}^{\Omega} = \begin{bmatrix}
\alpha_{2} \alpha_{3} & \alpha_{2} \alpha_{3} & -\alpha_{2} \\
\alpha_{2} \alpha_{3} & \alpha_{2} \alpha_{3} & \alpha_{2} \\
\alpha_{2} \alpha_{3} & \alpha_{2} \alpha_{3} & \alpha_{2} \\
\end{bmatrix}_{i}^{(a)}
\] (2.10)

\( c \) and \( s \) of formula (2.10) represent \( \sin \) and \( \cos \) respectively. Considering that deformation of body is small elastic deformation, so \( \sin \approx \varepsilon \), \( \cos \approx 1 \). Formular (2.10) can be written as following forms:

\[
G_{i}^{\Omega} = \begin{bmatrix}
1 & \varepsilon_{3} & -\varepsilon_{2} \\
\varepsilon_{1} \varepsilon_{2} - \varepsilon_{3} & \varepsilon_{1} \varepsilon_{2} + \varepsilon_{3} & \varepsilon_{1} \\
\varepsilon_{1} + \varepsilon_{3} \varepsilon_{2} & \varepsilon_{1} \varepsilon_{2} - \varepsilon_{3} & 1
\end{bmatrix}_{i}^{(a)}
\] (2.11)

The relations of relative rotation between two ends of hinge \( a \) (the rigid relative movement between adjacent objects) can be given by kinematics. The rigid rotation for \( B_{i} \) (\( i \leftarrow (a) \)) objects to objects \( B_{i} \) (\( i \rightarrow (a) \)) is the relative rotation relations between two ends of hinge \( a \), which is given by kinematics. The rigid rotation for \( B_{i} \) (\( i \leftarrow (a) \)) objects to objects \( B_{i} \) (\( i \rightarrow (a) \)) is following form:

\[
ed_{i}^{p}(\dot{a}) = G_{i}^{a} e_{i}^{p}(a) ; \dot{\Omega}_{i}^{a} = p_{a}^{a} \dot{q}_{a} \] (2.11)

Where \( G_{a} \) is direction cosine matrix between two basis vector \( e_{i}^{d}(a) \) and \( e_{i}^{p}(a) \), \( q_{a} \) is generalized coordinates of the hinge \( a \). Parameters \( p_{a} \) is expression basis vector array of the hinge \( a \) at basis vector \( e_{i}^{d}(a) \) and \( e_{i}^{p}(a) \).

Kinematics and geometric relationships of two adjacent objects is written as:

\[
\omega_{i}^{a} - \omega_{i}^{a} = \Omega_{i}^{a} \Omega_{i}^{a} - \Omega_{i}^{a} \Omega_{i}^{a} + \Omega_{i}^{a} a = 1,2,\ldots,n (2.12)
\]

\[
r_{i}^{a} = r_{i}^{a} \ or \ r_{i}^{a} + C_{i}^{a} = r_{i}^{a} + C_{i}^{a} \] (2.13)

Where \( r_{i}^{a} \) and \( r_{i}^{a} \) is displacement vector of hinge \( a \) of two adjacent objects \( i^{a}(a) \) and \( i^{a}(a) \), \( C_{i}^{a} \) and \( C_{i}^{a} \) present displacement vector from the centre of mass \( C_{i}^{a} \) and \( C_{i}^{a} \) of two adjacent objects \( i^{a}(a) \) and \( i^{a}(a) \) to the hinge \( a \). As nodes of the hinge point \( a \) is a coincidence, displacement vector of hinge point \( a \) of two adjacent objects \( i^{a}(a) \) and \( i^{a}(a) \) is equal.

2. The recursive relations of relative movement of objects

1) absolute angular velocity and absolute angular acceleration of body

Now considering absolute angular velocity and absolute angular acceleration of arbitrary body \( B_{i}^{a} \) of the system in inertial reference coordinate. Absolute angular velocity \( \omega_{i} \) of arbitrary body \( B_{i}^{a} \) of the system in inertial reference coordinate is sum of angular velocity \( \omega_{0} \) of the body \( B \) and relative angular velocity of the pair of adjacent objects in the route along the body \( B \) and the body \( B_{i}^{a} \). Introduction of graph theory method, consider Pathway matric \( T_{ij} \) in the route between the body \( B \) and the body \( B_{i}^{a} \), then the absolute angular velocity of arbitrary body \( B_{i}^{a} \) relative to the inertial reference coordinate system can be written as following forms:

\[
\dot{q} = \omega_{0} + \sum_{j=1}^{n} T_{ij} \dot{\omega}_{j}^{a} - T_{ij} \dot{\omega}_{j}^{a} - \dot{\omega}_{ji}^{a} \] (j=1,2,\ldots,n)

Where: \( \omega_{j}^{a} \), \( \Omega_{j}^{a} \) and \( \Omega_{ji}^{a} \) can be
written by formula (2.12) and (2.15) as following: 
$$\Omega_{j'}(a)_{ja} = \psi_{j'}(a)_{ja} + \eta_{j'}(a)_{ja}, \Omega_{ja} = p_{ja}^T \hat{q}_{ja}$$

Therefore the absolute angular velocity of the arbitrary objects $B_i$ relative to the inertial reference system can be written as following formula:

$$\Omega = \omega_0 + \sum_{j=1}^{n} T_{j'}(\omega_{j'}(a)_{ja} + \psi_{j'}(a)_{ja} \hat{u}_{ja} - \psi_{j'}(a)_{ja} \hat{u}_{ja}) + \eta_{j'}(a)_{ja}$$

(j=1,2,...,n) (2.14)

Thinking about formula (2.11), set the angular velocity of rigid movement $\Omega_{ja}$ of object $B_j$ ($i'$ (a)) to object $B_i$ ($i'$ (a)), Define the partial derivative $\hat{\Omega}_{ja}$ that express $\Omega_{ja}$ relative of its Inscribed object, it is called relative angular acceleration of object $i'$ (a) to object $i'$ (a).

$$\hat{\Omega}_{ja} = p_{ja}^T \hat{q}_{ja} + \frac{\partial p_{ja}^T \hat{q}_{ja}}{\partial \eta_{ja}} \hat{u}_{ja} + \omega_{ja} \times \Omega_{ja}$$

(j=1,2,...,n) (2.15)

Which $w_i$ is defined by following formula:

$$w_i = \frac{\partial p_{ja}^T \hat{q}_{ja}}{\partial \eta_{ja}} \hat{u}_{ja}$$

(j,r=1,2,...,n) (2.16)

Differentiating equations (2.14) with respect to time t yields the angular acceleration of the body $B_i$.

$$\hat{\omega} = \hat{\omega}_0 + \sum_{j=1}^{n} T_{j'}(\hat{\psi}_{j'}(a)_{ja} + \hat{w}_ja + \omega_{ja} \times \hat{\Omega}_{ja})$$

(j=1,2,...,n) (2.17)

Where:

$$\hat{\psi}_{j'}(a)_{ja} = \psi_{j'}(a)_{ja} \hat{u}_{ja} + \omega_{j'a} \times \hat{\Omega}_{ja}$$

$$\hat{\psi}_{j'}(a)_{ja} = \psi_{j'}(a)_{ja} \hat{u}_{ja} + \omega_{j'a} \times \hat{\Omega}_{ja}$$

$\omega_{j'(a)}$ is the rotation angular velocity of the fixed coordinate system of the hinge joints $a$.

Formula (2.17) can be expanded as following forms:

$$\hat{\omega} = \hat{\omega}_0 + \sum_{j=1}^{n} T_{j'}(\psi_{j'a} \hat{u}_{ja} + \omega_{j'a} \times \hat{\Omega}_{ja} - \psi_{j'a} \hat{u}_{ja} - \omega_{j'a} \times \hat{\Omega}_{ja})$$

$$\hat{\omega} = \hat{\omega}_0 + \sum_{j=1}^{n} T_{j'}(w_i + \omega_{ja} \times \Omega_{ja})$$

(j=1,2,...,n) (2.18)

Set:

$$\Omega = [\Omega_{ua} \cdots \Omega_{na}]^T, w = [w_i \cdots w_n]^T$$

$$p_a = [p_{ua} \cdots p_{na}]$$

$$q_a = [q_{ua} \cdots q_{na}]$$

$$\eta_a = [\eta_i \cdots \eta_n]^T$$

Formula (2.14) and (2.18) being written into matrix forms:

$$\hat{\omega} = \omega_0 + S^T \hat{\eta}_a - \hat{T} f_i \hat{u}_{ja}$$

(j=1,2,...,n) (2.20)

$$\hat{\omega} = \omega_0 - S^T \hat{\eta}_a - \hat{T} f_i \hat{u}_{ja} - \hat{T} f_i \hat{u}_{ja} - \hat{T} f_i \hat{u}_{ja} - \hat{T} f_i \hat{u}_{ja}$$

(j=1,2,...,n) (2.21)

Which:

$$S^T = S_{ja} \psi_{ja}, \ f = [f_1 \cdots f_n], \ f_i = \eta_{ja} \times p_{ja}$$

$$\xi_i = \omega_{ja} \times \Omega_{ja} - \omega_{ja} \times \Omega_{ja}$$

$$\xi_i = \omega_{ja} \times \Omega_{ja} + \omega_{ja} \times \Omega_{ja}$$

$$\xi_i = \omega_{ja} \times \Omega_{ja} + \omega_{ja} \times \Omega_{ja}$$

(2.22)

2) the the centre of mass velocity and acceleration of the body

Consider the distribution of joints on objects: within the trees system, all joints Correlated with arbitrary object $B_0$, Only with a special hinge become internal linkage as Connectivity with the object $B_0$. When use Rules Labeling, internal hinge is same with the serial number of the object, Credited $O_i$ (figure 7.3). The others joints connected with extranal object of the body $B_i$ is called extranal hinge.

The vector $d_{ja}$ of Inscribed joints from the body $B_{j'(a)}$ to he body $B_{j'(a)}$ is written as form:

$$d_{ja} = -c_{ja} + c_{ja} - c_{ja}$$

(2.23)

or: $d_{ja} = (-\rho_{O_1} + \rho_{O_2}) + (-\rho_{O_2} + \rho_{O_2}) - (-\rho_{O_3} + \rho_{O_3})$

Which $\rho_{O_1} = c_{ja}, \rho_{O_2} = c_{ja}, \rho_{O_3} = c_{ja}, c_{ja}$, $c_{ja}$, $c_{ja}$ is rigid body hinge vector of the hinge linkage point.

So

$$d_{ja} = (-c_{ja} + c_{ja} - c_{ja} + (-\rho_{O_3} + \rho_{O_3}) - (-\rho_{O_3} + \rho_{O_3})$$

(2.24)

Set $r_0$ is displacement vector of the hinge $O_i$, then the vector $r_j$ the centre of mass of arbitrary body Relative hinge $O_0$ is the vector sum of all
pathway matric \( d_{ki\alpha} \) (\( k=1,2,\ldots,n \)) from object \( B_0 \) to object \( B (a) \);

\[
r_i = \sum_{k=1}^{n} d_{ki\alpha} + r_0 + c_0 \quad (c_0 = c_{\alpha\beta}\big|_{\alpha=0}) \quad (2.25)
\]

\[
r_i = \sum_{k=1}^{n} (-c_{\alpha\beta} + c_{\beta\alpha}) + r_0 + c_0 \quad (2.26)
\]

Differentiating equations (2.26) with respect to time \( t \) yields:

\[
\dot{r}_i = \sum_{k=1}^{n} \left[ -(\dot{\rho}_{\alpha\beta} + \dot{u}_{\alpha\beta}^d) + (\dot{\rho}_{\beta\alpha} + \dot{u}_{\beta\alpha}^d) - (\dot{\rho}_{\alpha\beta} + \dot{u}_{\alpha\beta}^d) \right] + \dot{r}_0 + \dot{c}_0 \quad (i=1,2,\ldots,n) \quad (2.27)
\]

Differentiating equations (2.31) with respect to time \( t \) yields:

\[
\dot{r}_i = \sum_{k=1}^{n} \left[ -(\dot{\rho}_{\alpha\beta} + \dot{u}_{\alpha\beta}^d) + (\dot{\rho}_{\beta\alpha} + \dot{u}_{\beta\alpha}^d) - (\dot{\rho}_{\alpha\beta} + \dot{u}_{\alpha\beta}^d) \right] + \dot{r}_0 + \dot{c}_0 \quad (i=1,2,\ldots,n) \quad (2.28)
\]

Introduction of graph theory method, make use of pathway matric \( T \) and correlation matrix \( S \), the formula (2.26), (2.27) and (2.28) can be written by blow expressions:

\[
r_i = (\dot{r}_0 + c_{\alpha\beta})1 - T^T S^T \quad (2.29)
\]

\[
\dot{r}_i = (f \times T^T S^T)\dot{a}_a + (S \times T^T S^T - T^T S^T)\dot{b}_a + \chi \quad (2.30)
\]

\[
\dot{r}_i = (f \times T^T S^T)\dot{a}_a + (S \times T^T S^T - T^T S^T)\dot{b}_a + \chi \quad (2.31)
\]

**Figure 2** the multibody system with arbitrary hinge linkage

Where:

\[
c_{\alpha\beta} = (\rho_{\alpha\beta} + u_{\alpha\beta})_i ; \quad c_0 = c_{\alpha\beta}\big|_{\alpha=0} ; \quad S^e = s_{\alpha\beta} \left[ \psi_{\alpha\beta} \right]_{\alpha\beta}
\]

\[
S^d = s_{\alpha\beta} \left[ \psi_{\alpha\beta} \right]^T ; \quad \chi = (\dot{r}_0 + \dot{c}_0) + \left[ \left( \psi_{\alpha\beta} \right)_{\alpha\beta} - \left( \psi_{\alpha\beta} \right)^T \right]_{\alpha\beta} + \left( \psi_{\alpha\beta} \right)_{\alpha\beta} \left( \psi_{\alpha\beta} \right)^T \quad (2.32)
\]

**3 dynamic equation of Flexible hinge system**

Multi-body system can be divided into tree system and non-tree (including closed hinge) system by structure. Now building dynamics equations of the tree system. Assuming Tree systems are made up of \( n \) components \( B_i \) and \( n \) hinges \( O \) and naught objects whose Movement is known. Assuming that mass of \( B_i \) is \( m_i \), center inertia tensor is \( I_i \), displacement vector of the centre of mass relatively
fixed reference point is \( r_i \), tatatoin angular velocity is \( \omega_i \). Resultant vector of the external force and resultant moment relatively the centre of mass is \( F_i^a, M_i^a \), virtual work of internal force and elastic force is \( \delta P_i \). By D’Alembert’s principle,we have

\[
\sum_{j=1}^{n} (\delta \dot{\mathbf{r}}_j \cdot (m_i \ddot{r}_j - F_j^a) + \delta \omega_j \cdot (I_i \dot{\omega}_j + \epsilon - M_i^a)) - \delta P = 0
\]

\((i = 1, 2, \ldots, n)\) \quad (3.1)

Which

\[
\epsilon_i = \omega_i \times (I_i \cdot \omega_i) \quad (i = 1, 2, \ldots, n)
\]

virtual work of internal force and elactic force is \( \delta P_i \)

\[
\delta P_i = \sum_{j=1}^{n} (\delta V_j \cdot F_j^a + \delta \omega_j \cdot M_j^a) + \sum_{k=1}^{m} \delta \omega_k \cdot F_k^e + \delta V_k
\]

\(
\delta V_j \) and \( \delta \omega_j \) are variation of the relative sliding speed and the relative rotational angular velocity linked external objects of the hinge \( O_j \) to the relative internal objects. \( F_j^a \) and \( M_j^a \) are Resultant vector of the active force action on internal objects and resultant moment relation to hinge \( O_j \). \( \delta V^e_k \) is variation of relative velocity of the \( k \)th force element at force element hinge. \( F_k^e \) is force of external objects to internal objects. \( m \) is number of the force element. \( \delta V_k \) is virtual work of elactic force.

\[
\delta V_k = \{-\delta \dot{x}_k \cdot F_k^a \} K_i \{\delta \dot{x}_k\}
\]

\((3.4)\)

Which \( \{\delta \dot{x}_k\} \) is modal coordinate vector, \( K_i \) is generalized stiffness matrix of the body. Substitution of formula (3.3) and (3.4) into (3.5) gives matrix equation as following:

\[
\delta \dot{\mathbf{r}} \cdot (m_i \ddot{r} - F^e) + \delta \mathbf{\omega} \cdot (I_i \dot{\omega} + \epsilon - M^e) + \delta \mathbf{\omega}^T \cdot F^e + \delta \mathbf{\omega} \cdot M^a) - \delta \omega^e \cdot F^e \cdot \{\delta \eta \}^T K \{\delta \eta\} = 0
\]

\((3.5)\)

The symbol of \( \mathbf{\eta} \) is defined as follows:

\[
m = \text{diag}[m_1 \cdots m_n], \quad I = \text{diag}[I_1 \cdots I_n]
\]

\[
r = [r_1 \cdots r_n]^T, \quad \mathbf{\delta} \dot{\mathbf{r}} = [\mathbf{\delta} \dot{r}_1 \cdots \mathbf{\delta} \dot{r}_n]^T
\]

\[
\omega = [\omega_1 \cdots \omega_n]^T, \quad \mathbf{\delta} \omega = [\mathbf{\delta} \omega_1 \cdots \mathbf{\delta} \omega_n]^T
\]

\[
F^e = [F^e_1 \cdots F^e_n]^T, \quad M^e = [M^e_1 \cdots M^e_n]^T
\]

\[
\epsilon = [\epsilon_1 \cdots \epsilon_n]^T, \quad \delta V = [\delta V_1 \cdots \delta V_n]^T
\]

\(\mathbf{\Omega} = [\mathbf{\Omega}_1 \cdots \mathbf{\Omega}_n]^T, \quad F^a = [F^a_1 \cdots F^a_n]^T
\]

\(M^a = [M^a_1 = M^a_n]^T, \quad \delta \mathbf{\omega}^e = [\delta \mathbf{\omega}^e_1 \cdots \delta \mathbf{\omega}^e_m]^T
\]

\(F^e = [F^e_1 \cdots F^e_m]^T, \quad K = \phi^T K \phi \)

(3.6)

3.2 Dynamics equation of the flexible ratation joints system

Then joints of the multibody system is ratation hinge,set

\[
\delta \mathbf{V} = 0
\]

\((3.7)\)

Due to the multi-body system exist kinematic constraints of the joints between objects and objects, \( \delta \dot{\mathbf{r}}, \delta \mathbf{\omega}, \delta \mathbf{V}, \delta \mathbf{\omega}^e \) in kinetic equations is not independent variational .They must be presented by variation \( \delta q \) and \( \delta \eta \) of generalized velocity.Base on formula of \( \dot{r}, \mathbf{\omega} \) and \( \Omega \) \((2.38), (2.24) \) and \((2.15) \),their variation can be written by following variational form:

\[
\delta \mathbf{\dot{r}} = (f \times T S^T T)^T \delta \mathbf{q} + (S^T \times T S^T T - T S^d)^T \delta \eta
\]

\((3.8)\)

\[
\delta \mathbf{\dot{w}} = -S^T \cdot S^T \cdot S^T \cdot \delta \mathbf{q} + S^T \cdot \delta \eta
\]

\((3.9)\)

\[
\delta \mathbf{\dot{\omega}} = \phi^T \delta \mathbf{q}
\]

\((3.10)\)

\(\delta \mathbf{\omega}^e \) can be obtained by variation of the \( \mathbf{\omega}^e \)

Substitution of \((2.33), (2.20) \) and \((2.11)\) into\((3.8)-(3.10)\) gives

\[
\delta \mathbf{V}^e = -(f \times T S^T T)^T \delta \mathbf{q} + (S^T \times T S^T T - T S^d)^T \mathbf{S} \delta \eta + (S^T \times C^e \cdot S^T \delta \mathbf{q} + (T^T \cdot C^e \cdot \delta \eta)
\]

\((3.11)\)

\(S^e \) and \( C^e \) respectively force elements Correlation matrix and weighted force element vector.

Substitution of \((3.7)-(3.11)\) into\((3.5)\) gives dynamics equation of the flexible ratation joints system

\[
\delta \mathbf{\ddot{q}} \cdot \{f \times T S^T T \cdot \phi \cdot (m \ddot{r} - F^e) - T \ddot{f} \cdot (I \cdot \dot{\omega} + \epsilon - M^e) + p \cdot M^e \} + (f \times T S^T T) S^T + \delta \mathbf{\ddot{q}} \cdot \{(S^e \times T S^T T - T S^d) \cdot (m \ddot{r} - F^e) \cdot S^T \cdot T \}
\]

\((I \cdot \dot{\omega} + \epsilon - M^e) + (S^e \times T S^T T - T S^d) \cdot S^T \cdot T
\]

\(-(S^T \times C^e \cdot S^T - T \times C^e) \cdot T \cdot \delta \mathbf{\dot{r}} - K \cdot \{\delta \eta\} = 0
\]

\((3.12)\)

that

\[
(f \times T S^T T) \cdot (m \ddot{r} - F^e) - T \ddot{f} \cdot (I \cdot \dot{\omega} + \epsilon - M^e) \]

\(+ p \cdot M^e \) + \((f \times T S^T T) S^T = 0
\]

\((3.13)\)

\[
(S^e \times T S^T T - T S^d) \cdot (m \ddot{r} - F^e) \cdot (I \cdot \dot{\omega} + \epsilon - M^e) + (S^e \times T S^T T - T S^d) \cdot S^T \cdot T
\]

\(- (T F ^C e - K \cdot \{\delta \eta\}) = 0
\]

\((3.14)\)
Formula (3.13) and (3.14) is R/W equationa.

4 Conclusions

A forward recursive construction approach for dynamical equations of articulated flexible multibody systems with open loop construction are presented in the Wittenberg interconnected matrix systems. The motion of the flexible multibody systems with articulated joints has been resolved using the floating reference frames which are defined on the rigid states of flexible bodies. Vibration modes of components and model coordinates are used to express the deformation of the flexible multibody systems with articulated joints. Forward recursive construction approaches are used in kinetic analyzing. Kinetic equations and dynamics equations of the flexible body system with roll hinge and slide hinge are derived. The equations are fairly direct-vision and simple in resolution of the motion, straddling of elastic deformation of component and description of construction of system.

References:
[8] Zhu Ming. Several issues of multi-body system dynamics. the latest advances in general mechanics (dynamics, vibration and control) [M]. Beijing: Science Press, 1994:105-113