Convergence and Stability of a Numerical Algorithm for the Neutrons Transport Equation

NIKOS MASTORAKIS
Head of the Department of Computer Science
Military Inst. of University Education / Hellenic Naval Academy
Terma Hatzikyriakou 18539,
Piraeus, GREECE
http://www.wseas.org/mastorakis

OLGA MARTIN
Department of Mathematics
University “Politehnica” of Bucharest
Splaiul Independentei 313
Bucharest, ROMANIA
omartin_ro@yahoo.com

Abstract: - This paper describes a numerical procedure to solve the homogeneous boundary problem for a stationary transport equation. The stability and convergence of the proposed finite differences scheme is proved. The error value that corresponds to the numerical solution is also obtained using the Lax theorem.

Key-Words: integral-differential equation, finite differences scheme, integral identity method.

1 Introduction

Recently, computer science has led to the development of numerical algorithms for solving the mathematical physics problems. This has raised fundamental questions about the convergence and the stability of these algorithms. An unstable scheme will be sensitive to the rounded errors that appear in the calculating process and thus, the numerical solution will differ from the exact solution.

There is a large literature on the numerical solution of a neutrons transport equation, [1], [5], [6], [8], [11], [13]. The recent studies use the methods of Ritz and Galerkin, the method of least squares, the method of finite elements and Nyström method.

The algorithm proposed by us in the work, replaces the solution of an integral-differential equation with homogeneous boundary conditions by the solution of a diffusion equation with non-homogeneous boundary conditions. For this new problem we study the stability and the convergence rate of a differences scheme based on the integral identity method.

2 Problem formulation

In the stationary case, we consider a transport equation of the form

\[ \frac{\mu}{\mu'} \frac{d\varphi(x, \mu)}{dx} + \varphi(x, \mu) = \frac{1}{\alpha} \int_{-\alpha}^{\alpha} \varphi(x, \mu')d\mu' + f(x, \mu) \]

where

\[ \alpha > 0, \forall (x, \mu) \in D_1 \times D_2 = [0, H] \times [-1, 1], \]

\[ D_2 = D_2' \cup D_2'' = [-1,0] \cup [0,1]. \]

The boundary conditions are

\[ \varphi(0, \mu) = 0 \text{ if } \mu > 0 \]
\[ \varphi(\mu, \mu) = 0 \text{ if } \mu < 0 \] (2)

Here \( \varphi \) is the density of neutrons, which migrate in a direction defined by the angle \( \gamma \) against Ox axis and we denote \( \mu = \cos \gamma \). Let us consider the radioactive source \( f \) as an even function with respect to \( \mu \).

Using the notations:

\[ \varphi^+ = \varphi(x, \mu) \text{ if } \mu > 0; \quad \varphi^- = \varphi(x, \mu) \text{ if } \mu < 0 \]

and substituting \( \mu'' = -\mu' \), we get

\[ \int_{-1}^{0} \varphi(x, \mu')d\mu' = \int_{0}^{1} \varphi(x, -\mu')d\mu' = \int_{0}^{1} \varphi^- d\mu' . \]

Then the conditions (2) become
\[ \varphi^+(0, \mu) = 0, \varphi^-(H, \mu) = 0 \] (4)

and the equation (1) can be rewritten in the form

\[ \mu \frac{\partial \varphi^+}{\partial x} + \varphi^+ = \frac{1}{\alpha} \int (\varphi^+ + \varphi^-)d\mu' + f^+ + \frac{1}{\mu} \frac{\partial \varphi^-}{\partial x} + \varphi^- = \frac{1}{\alpha} \int (\varphi^+ + \varphi^-)d\mu' + f^- \] (5)

Adding and subtracting the equations (5) and introducing the notations:

\[ u = \frac{1}{2} (\varphi^+ + \varphi^-), \quad v = \frac{1}{2} (\varphi^+ - \varphi^-) \]

\[ g = \frac{1}{2} (f^+ + f^-), \quad r = \frac{1}{2} (f^+ - f^-) = 0 \] (6)

we obtain the following system

\[ \mu \frac{\partial v}{\partial x} + u = \frac{2}{\alpha} \int ud\mu + g \] (a)

\[ \mu \frac{\partial u}{\partial x} + v = 0 \] (b)

The boundary conditions become

\[ u + v = 0 \quad \text{for} \quad x = 0, \]
\[ u - v = 0 \quad \text{for} \quad x = H. \] (8)

Now, we find \( v \) from the second equation of (7) and using the first equation, we rewrite the problem (7)-(8) in the following form

\[ -\mu^2 \frac{\partial^2 u}{\partial x^2} + u = \frac{2}{\alpha} \int ud\mu + g \] (9)

\[ \left( u - \mu \frac{\partial u}{\partial x} \right)_{x = 0} = \left( u + \mu \frac{\partial u}{\partial x} \right)_{x = H} = 0 \] (10)

In order to get a solution of the problem (9)-(10), we consider on \( x \) axis two points systems:

- a principal system: \( \{x_k\} = \Delta'_1, \ k \in \{0, 1, \ldots, N\} \), with \( x_0 = 0, x_N = H \) and \( h = x_{k+1} - x_k \);
- a secondary system, \( \{x_{k+1/2}\} = \Delta'_1, \ k \in \{0, 1, \ldots, N - 1\} \), which verifies the inequalities: \( x_{k-1/2} < x_k < x_{k+1/2} \), where

\[ x_{k+1/2} = (x_k + x_{k+1}) / 2 \]

and

\[ 0 = x_0 < x_1/2 < \ldots < x_{N-1/2} < x_N = H. \]

Besides, let \( \Delta_2 = \{\mu_l\}, l \in \{0, 1, \ldots, L\} \) be a partition of the interval \( D_2^* = [0,1] \) and \( \tau = \mu_{l+1} - \mu_l \), \( l \in \{0, 1, \ldots, L-1\} \).

Further on, we consider \( H = 1 \). For every value \( \mu_l \in \Delta_2 \), the problem (9)-(10) becomes:

\[ -\mu_l^2 \frac{d^2u(x, \mu_l)}{dx^2} + u(x, \mu_l) = F(x, \mu_l) \] (11)

where

\[ F(x, \mu_l) = S(x) + g(x, \mu_l), \quad S(x) = \frac{2}{\alpha} \int u(x, \mu) d\mu \]

and

\[ \left\{ u(x, \mu_l) - \mu_l \frac{du(x, \mu_l)}{dx} \right\}_{x = 0} = 0 \]

\[ \left\{ u(x, \mu_l) + \mu_l \frac{du(x, \mu_l)}{dx} \right\}_{x = 1} = 0 \] (12)

Now (11)-(12) is a boundary problem for a one-dimensional diffusion equation (11). Integrating (11) with respect to \( x \) on the intervals: \( (x_{k-1/2}, x_{k+1/2}) \), we obtain

\[ -J_{k+1/2} + J_{k-1/2} + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - F)dx = 0 \] (14)

where

\[ J_{k\pm1/2} = J(x_{k\pm1/2}), \quad J(x, \mu_l) = \mu_l^2 \frac{du(x, \mu_l)}{dx} \]

We find \( J_{k\pm1/2} \) integrating (11) on the interval \( (x_{k\pm1/2}, x) \), where \( x \in (x_{k-1}, x_k) \). We get

\[ \mu_l^2 \frac{du(x, \mu_l)}{dx} = J_{k-1/2} + \int_{x_{k-1/2}}^{x} (u - F)dx \] (15)

Then, dividing (15) by \( \mu_l^2 \) and integrating on \( (x_{k-1}, x_k) \) we have

\[ u_k - u_{k-1} = J_{k-1/2} \int_{x_{k-1}/2}^{x} \frac{dx}{\mu_l^2} + \int_{x_{k-1}/2}^{x} \frac{dx}{\mu_l^2} \int_{x_{k-1}/2}^{x} (u - F)dx \] (16)

where \( u_k = u(x_k, \mu_l) \). Finally, we get
Proceedings of the 2nd IASME / WSEAS International Conference on Continuum Mechanics (CM’07), Portoroz, Slovenia, May 15-17, 2007

\[ J_{k-1/2} = \frac{\mu_i^2}{h} \left[ u_k - u_{k-1} - \frac{1}{\mu_i^2} \int_{x_{k-1/2}}^{x_k} (u - F) d\xi \right] \]

In a similar manner, we obtain the exact relation for \( J_{k+1/2} \) replacing \( k \) by \( k+1 \). Consequently, the equality (14) becomes:

\[ \mu_i^2 \left( \frac{u_{k+1} - u_k}{h} + \frac{u_k - u_{k-1}}{h} \right) + \int_{x_{k-1/2}}^{x_{k+1/2}} (u - F) d\xi = \]

\[ = -\frac{1}{h} \int_{x_k}^{x_{k+1/2}} (u - F) d\xi + \frac{1}{h} \int_{x_{k-1/2}}^{x_k} (u - F) d\xi \]

where \( k \in \{1, 2, \ldots, N-1\} \).

The formula (16) is named the fundamental identity for the finite differences equations.

Let us now define the operator \( A \) on the set \( U \) of the solutions of equation (11) that satisfy the boundary conditions (12)

\[(Au)_k = -\frac{\mu_i^2}{\Delta x_k} \left( \frac{u_{k+1} - u_k}{h} + \frac{u_k - u_{k-1}}{h} \right) + \frac{1}{\Delta x_k} \left[ \frac{1}{h} \int_{x_k}^{x_{k+1/2}} u d\xi - \frac{1}{h} \int_{x_{k-1/2}}^{x_k} u d\xi \right] \]

\[(F)_k = \frac{1}{\Delta x_k} \int_{x_{k-1/2}}^{x_{k+1/2}} F dx + \frac{1}{\Delta x_k} \left[ \frac{1}{h} \int_{x_k}^{x_{k+1/2}} F d\xi - \frac{1}{h} \int_{x_{k-1/2}}^{x_k} F d\xi \right] \]

where \( \Delta x_k = x_{k+1/2} - x_{k-1/2} = h, \ k \in \{1, 2, \ldots, n-1\} \).

Writing now (18) for \( k = 1, 2, \ldots, n - 1 \), we get the system

\[ Au = F \]

To study the approximations of the equation (20), we consider the space \( \Phi \) of the reticulated functions

\[ u^h = (u^h_0, u^h_1, u^h_2, \ldots, u^h_{n-1}, u^h_n) \]

that are defined in the points \( x_0, x_1, \ldots, x_n \).

In order to can use the functions \( F \) and \( u \), which can to have the discontinuous points in any points \( x_i \) that belong to the principal system, we define the value of the function in the node \( x_i \) of the form

\[ (F^h)_k = \frac{1}{\Delta x_k} \int_{x_{k-1/2}}^{x_{k+1/2}} F dx = \frac{1}{h} \int_{x_{k-1/2}}^{x_{k+1/2}} F dx \]

if the step is constant. In the following, we write the approximate form of the equation (20)

\[ A^h u^h = F^h \]

Where

\[ (A^h u^h)_k = -\frac{\mu_i^2}{\Delta x_k} \left( \frac{u_{k+1} - u_k}{h} - \frac{u_k - u_{k-1}}{h} \right) + \]

\[ + \frac{1}{\Delta x_k} u^h_k \]

\[ + \frac{1}{\Delta x_k} \left[ \frac{1}{h} \int_{x_k}^{x_{k+1/2}} u d\xi - \frac{1}{h} \int_{x_{k-1/2}}^{x_k} u d\xi \right] \]

\[ k = 1, 2, \ldots, n-1 \quad \text{and} \quad \Delta x_k = h. \]

Let us now consider in \( \Phi \) the scalar product of the form

\[ (a, b) = \sum_{k=1}^{n-1} \Delta x_k a_k b_k \]

and the norm

\[ \|a\|^2 = \sum_{k=1}^{n-1} \Delta x_k a_k^2 \]

3 Stability and convergence

Let us now show that the difference scheme is stably.

We suppose that \( F \in C([0,1]) \) (the function and its derivates until two order are continuous everywhere in \([0,1]\) except a finite number of first-order discontinuous that belong to the set \( \{x_1, \ldots, x_n\} \)) and approximate solution of equation (23). \( u^h \in \Phi \) is continuous on \([0,1]\).

Using the finite differences method, the boundary conditions (12) become

\[ u_0 = \mu_l \frac{u_1 - u_0}{h} \Rightarrow u_0 = \frac{\mu_l}{h + \mu_l} u_1 \]

\[ u_n = -\mu_l \frac{u_n - u_{n-1}}{h} \Rightarrow u_n = -\frac{\mu_l}{h + \mu_l} u_{n-1} \]

By using (24), (26) and (27), we get

\[ (u^h, F^h) = (u^h, A^h u^h) = \]
(31)

we obtain

(33)

Taking (28) into account, it can be shown that the following inequalities

Hence

If \( \frac{1}{\alpha} < \frac{(r+1)\mu_1^r}{2} \leq \frac{r+1}{2} \), we obtain

for every \( \mu_1 \) fixed into \( \Delta_2 \). Finally, from the definition

we get

and the stability of differences scheme is proved.

Definition

The problem

is an approximation of the \( n \)-order with respect to the solution \( u \) of the equation (20), if there are the constants \( h_1 \) and \( M_1 \) such that for \( h < h_1 \) we have

where \( (u)_h \) is the vector with the \( n \)-1 dimension from \( \Phi \) with the components \( u(x_k) \). Using the equations (20) and (22) we get

Therefore, we get

Finally, from the definition

we get

and the stability of differences scheme is proved.
Let us now define

\[
(\varepsilon^h)_k = \frac{1}{h} \left( \frac{x_{k+1/2}}{x_{k-1/2}} \int_{x_{k-1/2}}^{x_{k+1/2}} u(x, \mu_j)dx - u(x_k, \mu_j)h \right)
\]

\[
(\eta^h)_k = -\frac{1}{h^2} \int_{x_{k-1/2}}^{x_{k+1/2}} u(t, \mu_j)dt + \frac{1}{h^2} \int_{x_{k-1/2}}^{x_{k+1/2}} u(t, \mu_j)dt + \frac{1}{h^2} \int_{x_{k-1/2}}^{x_{k+1/2}} F(t, \mu_j)dt
\]

\[
(\theta^h)_k = -\frac{1}{h^2} \int_{x_{k-1/2}}^{x_{k+1/2}} F(t, \mu_j)dt + \frac{1}{h^2} \int_{x_{k-1/2}}^{x_{k+1/2}} F(t, \mu_j)dt
\]

Since the functions \( F \) and \( u \) have its derivatives continuous until order 2 and the functions \( u \) have its derivates discontinuous that belong to the set \( \{x_1, \ldots, x_n\} \) and using the Taylor formula in the vicinity of the nodes \( x_k \), we get

\[
\left| (\varepsilon^h)_k \right| \leq N_1h; \left| (\eta^h)_k \right| \leq N_2h; \left| (\theta^h)_k \right| \leq N_3h
\]

Let us now define

\[
(\omega^h)_k = \max \left\{ (\varepsilon^h)_k, (\eta^h)_k, (\theta^h)_k \right\}
\]

where \( N = \max(N_1, N_2, N_3) \). By using the definition (25), the square of the norm verifies the inequality

\[
\|\omega^h\|^2_{\phi} = \sum_{k=1}^{n-1} (\omega^h)^2_k h \leq N^2h^3
\]

Consequently,

\[
\|\omega^h\|_{\phi} \leq Mh^{3/2}
\]  

(37)

Thus it has been show that the differences scheme (22) approximates the initial problem (20) with the order 3/2 with respect to the solution \( u \).

With the help of the following theorem,[5], we shall estimate the speed of the convergence of the approximate solution \( u^h \) to the exact solution \( u \).

**Theorem (Lax).** If

1. the differences scheme (22) approximates the initial problem (20) with the order \( n \) with respect to the solution \( u \);
2. \( A^h \) is a linear operator;
3. the difference scheme is stably in accordance with (29), then the solution of the approximate problem is convergent to the exact solution and the evaluation of the convergent speed is defined by the following inequality

\[
\| (u)_h - u^h \| \leq MC_ih^n
\]  

(38)

Using the above theorem to our problem, the value of the error is

\[
\| (u)_h - u^h \|_{\phi} \leq 3\| \omega^h \|_{\phi} C_1 \leq 3C_1Mh^{3/2} = K_ih^{3/2}
\]

for \( \mu_i \) fixed. Let us now consider \( K = \max K_i \) and finally, the estimation of the error is the following

\[
\| (u)_h - u^h \| \leq Kh^{3/2}
\]  

(39)

**4 Conclusion**

In this paper a numerical procedure for the solution the homogeneous boundary problem for a stationary transport equation has been presented. The finite differences scheme based on the integral identity method has convergence and stability. Error analysis is presented using the Lax theorem.

**References:**


[8]. Martin O., *Une Méthode de Résolution de*
l’Équation du Transfert des Neutrons, 


