# The Test Examples for Approximate Solution of Singular Integro-Differential Equations by Mechanical Quadrature Methods in Classical Hölder spaces

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*Abstract:* In this paper we present the test examples for the for approximative solution of Singular Integro- Differential Equations with kernels of Cauchy type by mechanical quadrature method. The equations are defined on the ellipse. We formulate the theorem about theoretical background of mechanical quadrature method in classical Hölder spaces.

*Singular Integro- differential equations, Mechanical quadrature method, Classical Hölder spaces*

## 1 Introduction

Singular Integro- Differential Equations with Cauchy kernels (SIDE) model many problems in elasticity theory, aerodynamics, mechanics, thermoelasticity, queueing system analysis, etc. [1]-[11]

The general theory of SIDE has been widely investigated in last decades [12]-[18]. It is well known that the exact solution for SIDE can be found in rare spacial cases. Even in these cases, evaluating the solution numerically can be very complicated and laborious.

We note that theoretical background of collocation methods and mechanical quadrature methods for approximate solution of SIDE in Generalized Hölder spaces and classical Hölder spaces has been obtained in [20],[22],[23]. The equations have been defined on the arbitrary smooth closed contours.

In this article we present the test examples for numerical solution of SIDE. The equation are defined on the ellipse. To construct the Riemann function we use the approximative methods.

## 2 The Numerical Schemes of Mechanical Quadrature Methods

In this item we present the numerical schemes of Mechanical Quadrature Methods. We formulate the convergence theorem. The results from this section were obtained in [20], [22], [23].

Let $\Gamma$ be an arbitrary smooth closed contour bounding a simple connected domain $D^+$ in the complex plain, let $z = 0 \in D^+$, and let $D^- = C \setminus \{D^+ \bigcup \Gamma\}$, where $C$ is a full complex plane.

Let $z = \psi(w)$ be the Riemann function which maps conformably the exterior of unit circle $\Gamma_0$ $(= |w| = 1)$ onto $D^-$ so that,

$$\psi(\infty) = \infty, \psi^{'}(\infty) = 1. \qquad (1)$$

We denote by $\Lambda$ the class of contours which satisfies the conditions (1).

We denote $H_\beta(\Gamma)$ the space of functions on $\Gamma$ satisfying the Hölder condition with the exponent[1] $\beta$, $0 < \beta \leq 1$ :

$$H_\beta(\Gamma) = \{g(t) : |g(t^{'}) - g(t^{''})| \leq$$

[1]By $d_1, d_2, \dots,$ we denote the constants.

$$\leq d_1|t' - t''|^{\beta}; t', t'' \in \Gamma\}.$$

The norm on $H_\beta(\Gamma)$ is defined by

$$||\varphi||_\beta = \max_{t\in\Gamma} |\varphi(t)| +$$

$$+ \sup_{t_1 \neq t_2} \frac{\varphi(t_1) - \varphi(t_2)}{|t_1 - t_2|^\beta}$$

$$= ||\varphi||_C + H(\varphi;\beta), \quad t_1, t_2 \in \Gamma, . \quad (2)$$

By $H_\beta^{(q)}(\Gamma)$ we denote the space of $q$ times differentiable functions $g(t)$ such that $g^{(q)} \in H_\beta(\Gamma)$; the norm in $H_\beta^{(q)}(\Gamma)$ is given by the formula

$$||g||_{\beta,q} = \sum_{k=0}^{q} ||g^{(k)}||_C + H(g^{(q)};\beta).$$

In the complex space $H_\beta(\Gamma)$ we will consider the singular integro-differential equations (SIDE)

$$(Mx \equiv) \sum_{r=0}^{q} [\tilde{A}_r(t)x^{(r)}(t) +$$

$$+\tilde{B}_r(t)\frac{1}{\pi i}\int_\Gamma \frac{x^{(r)}(\tau)}{\tau - t}d\tau +$$

$$+\frac{1}{2\pi i}\int_\Gamma h_r(t,\tau)\cdot x^{(r)}(\tau)d\tau] =$$

$$= f(t), \quad t \in \Gamma, \quad (3)$$

where $\tilde{A}_r(t), \tilde{B}_r(t)$ and $h_r(t,\tau)$ $(r = \overline{0,q})$ and $f(t)$ are given functions; $x^{(0)}(t) = x(t)$ is the unknown function; $x^{(r)}(t) = \dfrac{d^r x(t)}{dt^r}$ $(r = \overline{1,q})$; $q$ is a natural number.

We search for the solution of equation (3) in the class of functions, satisfying the condition

$$\frac{1}{2\pi i}\int_\Gamma x(\tau)\tau^{-k-1}d\tau = 0, \quad k = \overline{0, q-1}. \quad (4)$$

We introduce the terminology "the problem (3)-(4)" for the SIDE (3) together with the conditions (4).

We search for the approximate solution of the problem (3)-(4) in the form

$$x_n(t) = \sum_{k=0}^{n} \xi_k^{(n)} t^{k+q} + \sum_{k=-n}^{-1} \xi_k^{(n)} t^k, t \in \Gamma, \quad (5)$$

where $\xi_k^{(n)} = \xi_k$ $(k = \overline{-n,n})$ are unknowns; we note that the function $x_n(t)$, constructed by formula (5) satisfies the conditions (4).

According to the collocation method, we determine the unknowns $\xi_k$ $(k = (\overline{-n,n}))$ from the condition of inversion into zero of the expression

$$Mx_n(t_j) - f(t_j) = 0,$$

in $2n+1$ different points $t_j \in \Gamma$ $(j = 0, 2n)$.

As a result we will obtain the system of linear algebraic equations (SLAE):

$$\sum_{r=0}^{q}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t^{k+q-r}\xi_k +$$

$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\times$$

$$\times t_j^{-k-r}\xi_{-k} + \frac{1}{2\pi i}\cdot\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\times$$

$$\times\int_\Gamma K_r(t_j,\tau)\tau^{k+q-r}d\tau\cdot\xi_k +$$

$$+\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\cdot\frac{1}{2\pi i}\times$$

$$\times\int_\Gamma K_r(t_j,\tau)\tau^{-k-r}d\tau\cdot\xi_{-k}\} =$$

$$= f(t_j), \; j = \overline{0,2n}, \quad (6)$$

where $A_r(t) = \tilde{A}_r(t) + \tilde{B}_r(t)$ $B_r(t) = \tilde{A}_r(t) - \tilde{B}_r(t)$, $r = \overline{0,q}$.

If the problem (3)-(4) is solved by the mechanical quadrature method we will apply as a quadrature formula the following one:

$$\frac{1}{2\pi i}\int_\Gamma g(\tau)\tau^{l+k}d\tau \cong$$

$$\frac{1}{2\pi i}\int_\Gamma U_n(\tau^{l+1}\cdot g(\tau))\tau^{k-1}d\tau,$$

where $k = \overline{0,n}$ for $l = 0,1,2,\ldots$ and $k = \overline{-1,-n}$ for $l = -1,-2,\ldots$; the operator of interpolation $U_n$ is determined by the formula

$$(U_n g)(t) = \sum_{s=0}^{2n} g(t_s) \cdot l_s(t),$$
$$l_j(t) = \prod_{k=0,k\neq j}^{2n} \frac{t-t_k}{t_j-t_k}\left(\frac{t_j}{t}\right)^n \equiv \sum_{k=-n}^{n} \Lambda_k^{(j)} t^k,$$
$$t \in \Gamma, \quad j=\overline{0,2n}.$$

Thus, for the determination of the unknowns $\xi_k$ $(k=\overline{-n,n})$ by the mechanical quadrature method we get the following SLAE:

$$\sum_{r=0}^{q}\{A_r(t_j)\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}t_j^{k+q-r}\xi_k+$$
$$+B_r(t_j)\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}t_j^{-k-r}\cdot\xi_k+$$
$$+\sum_{k=0}^{n}\frac{(k+q)!}{(k+q-r)!}\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{1+k-r}\Lambda_{-k}^{(s)}\xi_k$$
$$\sum_{k=1}^{n}(-1)^r\frac{(k+r-1)!}{(k-1)!}\times$$
$$\times\sum_{s=0}^{2n}K_r(t_j,t_s)t_s^{-1-r}\Lambda_k^{(s)}\xi_{-k}\}=$$
$$=f(t_j), \quad j=\overline{0,2n}. \tag{7}$$

To find the numbers $\Lambda_k^{(s)}$ we will use Viete's theorem.

Let $\dot{H}_\beta^{(q)}(\Gamma)$ be subspace of $H_\beta^{(q)}(\Gamma)$ and elements from $\dot{H}_\beta^{(q)}(\Gamma)$ satisfy the conditions (4). The norm is defined as in $H_\beta^{(q)}(\Gamma)$.

**Theorem 1.** *Let $\Gamma \in \Lambda$ and the following conditions be satisfied:*

1. *the functions $A_k(t)$, $B_k(t)$, $h_k(t,\tau)$, (for both variables) $(k=\overline{0,q})$ and $f(t)$ belong to the space $H_\alpha^{(r)}(\Gamma)$, $0<\alpha<1, r=0,1,\ldots$, $0<\beta<\alpha<1$;*
2. $A_q(t)B_q(t)\neq 0$, $t\in\Gamma$ ;
3. *the index of function $t^q B_q^{-1}(t)A_q(t)$ is equal zero;*
4. *the operator $M : \dot{H}_\beta^{(q)}(\Gamma) \to H_\beta(\Gamma)$ is linearly invertible;*
5. *the points $t_j$ $(j=\overline{0,2n})$ form a system of Fejér knots on $\Gamma$ :*

$$t_j=\psi\left[\exp\left(\frac{2\pi i}{2n+1}(j-n)\right)\right],$$

$$j=\overline{0,2n}, \quad i^2=-1,$$

*Then for values $n\geq n_1$, enough large SLAE of mechanical quadrature method (7) has the unique solution $\xi_k$ $(k=\overline{-n,n})$ and the approximate solutions*

$$x_n(t)=\sum_{k=0}^{n}\xi_k t^{k+q}+\sum_{k=-n}^{-1}\xi_k t^k, \quad t\in\Gamma \tag{8}$$

*converge to the to the exact solution of problem (3)-(4). The following estimate holds:*

$$||x-x_n||_{\beta,q}\leq\frac{d_2+d_3\ln n+d_4\ln^2 n}{n^{r+\alpha-\beta}}H(x^{(r)};\alpha).$$

## 3 Verification of conditions from theorem 1

We can use the analytical methods to verify that the coefficients from (3) belong to $H_\alpha^{(q)}(\Gamma)$. To calculate the index function we use the numerical algorithm from [13].

To construct the Riemann function for Fejér points we use the numerical algorithm from [21].

We present two examples for SIDE The right parts are calculated automatic. The contour $\Gamma$ is an ellipse $R\cos(\varphi)+ir\sin(\varphi)$. The approximative solutions we calculate by formula (5). The programs were written in Pascal.

**Example 1.** $\tilde{A}_r(t)=\frac{t^2}{2}$ $\tilde{B}_r(t)=\frac{2-t^2}{2}$, $K_r(t,\tau)=\frac{t+r}{\tau}$, $r=\overline{0,2}$, exact solution $x(t)=\frac{1}{t}$.

| exact sol. | approx. sol. | Re(Δ) | Im(Δ) |
|---|---|---|---|
| -0.68+0.06i | $-0.68 + 0.06i$ | 2.10E-09 | 3.60E-09 |
| -0.82+0.24i | $-0.82 + 0.24i$ | 3.91E-09 | 1.29E-09 |
| -1.06+0.93i | $-1.06 + 0.93i$ | 2.71E-09 | 2.40E-09 |
| 0.65+1.78i | $0.65 + 1.78i$ | 5.74E-10 | 5.74E-09 |
| 0.95+0.46i | $0.95 + 0.46i$ | 3.88E-09 | 4.41E-09 |
| 0.73+0.13i | $0.73 + 0.13i$ | 1.53E-09 | 1.94E-09 |
| 0.67-0.00i | $0.67 - 0.00i$ | 4.00E-11 | 7.75E-10 |
| 0.73-0.13i | $0.73 - 0.13i$ | 1.68E-09 | 1.79E-09 |
| 0.95-0.46i | $0.95 - 0.46i$ | 4.09E-09 | 3.85E-09 |
| 0.65-1.78i | $0.65 - 1.78i$ | 5.68E-10 | 4.82E-09 |
| -1.06-0.93i | $-1.06 - 0.93i$ | 2.65E-09 | 2.32E-09 |
| -0.82-0.24i | $-0.82 - 0.24i$ | 3.67E-09 | 1.19E-09 |

**Example 2.** $\tilde{A}_r(t) = \frac{t^3 + t}{2}$ $\tilde{B}_r(t) = \frac{t - t^3 - 2}{2}$, $K_r(t, \tau) = \frac{t + r}{\tau}$, $r = \overline{0, 2}$, exact solution $x(t) = \frac{1}{t^2}$.

| exact sol. | approx. sol. | Re(Δ) | Im(Δ) |
|---|---|---|---|
| 0.46-0.08i | $0.46 - 0.08i$ | 1.92E-09 | 3.02E-10 |
| 0.61-0.40i | $0.61 - 0.40i$ | 1.88E-10 | 1.88E-09 |
| 0.26-1.98i | $0.26 - 1.98i$ | 1.26E-09 | 2.61E-10 |
| -2.74+2.30i | -2.74+2.30i | 8.60E-10 | 1.59E-08 |
| 0.69+0.87i | 0.69+0.87i | 3.07E-10 | 8.17E-10 |
| 0.52+0.19i | 0.52+0.19i | 4.96E-10 | 3.13E-10 |
| 0.44-0.00i | 0.44-0.00i | 6.69E-10 | 1.27E-10 |
| 0.52-0.19i | 0.52-0.19i | 2.24E-10 | 1.00E-11 |
| 0.69-0.87i | 0.69-0.87i | 1.62E-09 | 9.60E-10 |
| -2.74-2.30i | -2.74-2.30i | 9.20E-10 | 2.19E-08 |
| 0.26+1.98i | 0.26+1.98i | 1.27E-09 | 3.09E-11 |
| 0.61+0.40i | 0.61+0.40i | 8.14E-10 | 2.47E-09 |

# 4 Conclusion

The test examples were elaborated. We constricted Fejér points numerically.

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**Acknowledgment:** *The research of the first author was supported by SC-STD of ASM grant 07.411.08 INDF and MRDA/CRDF Grant CERIM-1006-06.*