Determining the temperature field for cylinder symmetrical heat conduction problems in unsteady heat conduction in finite space

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Abstract: - This paper proposes to present a new method to calculate unsteady heat conduction for cylinder symmetrical geometry. We will investigate the situation where the temperature field and heat flux created around a heat source placed in finite space are determined. Such a situation arises when we define the temperature field and heat loss around district heating pipes laid underground as well as in the case of geothermal wells. Growing energy prices increasingly justify the use of geothermal energy. It is well-known that geothermal energy, thermal water and its heat can be used directly but if a higher temperature level is needed we must install a heat pump or a boiler plant to meet peak demands. The method presented in this paper can also be used for the energy analysis of geothermal wells.

Key words: - geothermal producing pipe, district heating pipes, heat loss, determining the temperate field, heat flux density, Garbai’s integral equation, heat pump, Laplace transformation

1 Introduction

District heating is widely used in energy supply, transporting the heat through pipes from the heat source to the consumer. In Hungary geothermal energy plays a significant role and there are great possibilities to increase the use of geothermal energy and facilitate the wide-spread installation of heat pumps.

2 Heat loss of geothermal wells and district heating pipes in unsteady state

This paper presents a method to calculate the temperature field around district heating pipes laid underground as well as geothermal wells and the heat loss of thermal water extracted. This enables us to perform the energy analysis of the wells and evaluate their energy goodness at certain working points and for various water yields. We are aware that the steady operation of wells should be ensured but a change of working points may occur or even a natural decrease of yield. In this case we face unsteady heat conduction around the well. (Fig. 1)

In the case of district heating pipes unsteady operation occurs when the pipes are started and the heat loss by far exceeds the heat loss of the steady operation. To cover the subject completely we will also discuss the heat loss in steady state.

2.1 Steady state

In steady state heat conduction occurring in the outside of the insulated pipes is described by equation

$$\dot{Q}_1 = 2\pi\lambda \frac{T_1 - T_2}{\ln \frac{r_2}{r_1}} \quad (1)$$

where r_1 is internal diameter; r_2 is the external radius; T_1 is the temperature on the internal surface of the pipe wall; T_2 is the temperature on the external surface of the pipe wall. If radius r_2 approaches infinity it can be proved that no heat flux takes place. This means that in the case of a heat source placed in infinite space such as thermal water producing well the temperature of the space reaches that of the producing pipe after infinite operation and the gradient of the temperature field becomes zero at the external radius of the producing pipe. Heat flux stops. The heat loss of the producing pipe cannot be modelled for steady state in theory only through approximations.

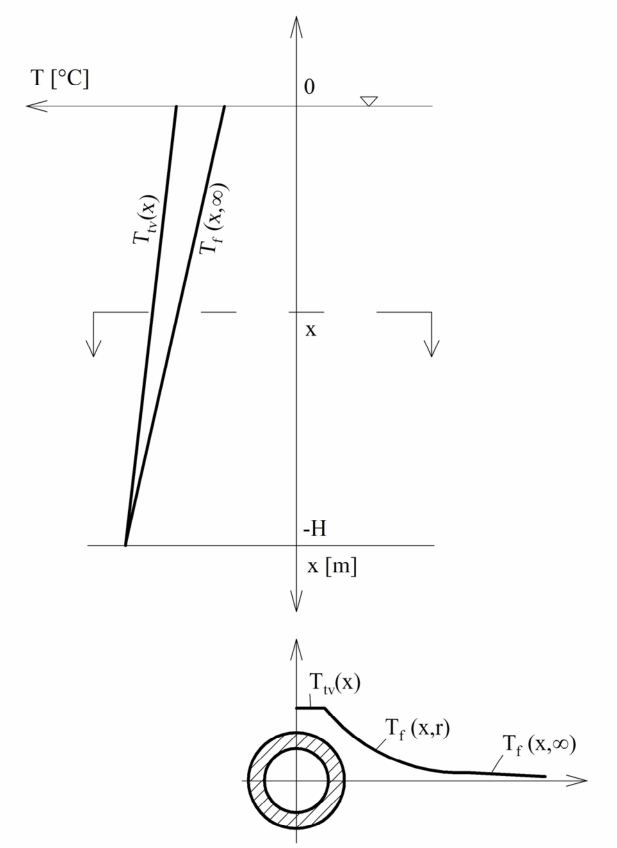

Fig. 1: Temperature relations in the geothermal wells and their environment

2.2 Unsteady state

2.2.1 Determining the temperature field around a circular cylinder in infinite space

Using circular symmetrical heat conduction we have modelled the heat loss of district heating pipes laid horizontally to the surface of the earth and of vertical geothermal wells.

This chapter will present how to determine the temperature field and the heat flux in any x section in the figure of the geothermal well. In that cross section heat flux also means heat loss. We will not deal with integrating the heat flux for the entire length of the pipe. To solve the differential equation for heat conduction we used the method of Laplace transformation. This subject is dealt with in a large number of specialist books that we could not process. Our investigations are based on the works of Carslaw and Jaeger. After this we will present the partly analytical, partly numerical procedure developed by Garbai, which is based on the theory of heat flow equations [2]. We have used Huber's [5] works as well, dealing with the numerical solution of Volterra integral equation.

It was assumed that the initial temperature distribution around the heat source (pipe) is zero in space. At $r = r_0$ spot on the edge of the pipe at • = 0 moment the temperature changes to $T = T_0$, launching the heat conduction process. The Laplace transformed form of the heat conduction equation in circular symmetrical heat conduction is as follows:

$$\frac{d^2\overline{\Theta}}{dr^2} + \frac{1}{r}\cdot\frac{d\overline{\Theta}}{dr} - q^2\overline{\Theta} = 0, \qquad r > r_0, \qquad (2)$$

where $q^2 = \frac{s}{k}$.

If $r \to \infty$ and $\overline{\Theta} = T_0/s$, $r = r_0$, the solution is:

$$\overline{\Theta} = \frac{T_0 K_0(qr)}{sK_0(qr_0)}. \qquad (3)$$

By using the inversion thesis according to Carslaw and Jaeger [11]:

$$J = \frac{T_0}{2pi}\int_{g-i\infty}^{g+i\infty} e^{l\cdot t}\,\frac{K_0(mr)\,dl}{K_0(mr_0)\,l}, \qquad (4)$$

where $m = \sqrt{l/k}$, and K_0 is a modified Bessel function of the second kind, zero order.

If $l = ku^2e^{ip}$, then:

$$2\int_0^\infty e^{-ku^2t}\,\frac{K_0(rue^{\frac{1}{2}p\cdot i})}{K_0(r_0ue^{\frac{1}{2}p\cdot i})}\,\frac{du}{u} = 2\int_0^\infty e^{-ku^2t}\,\frac{J_0(ur)-iY_0(ur)}{J_0(ur_0)-iY_0(ur_0)}\,\frac{du}{u}, \qquad (5)$$

since

$$K_0\left(ze^{\frac{1}{2}\pi \cdot i}\right) = -\frac{1}{2}\pi \cdot i \cdot H_0^2(z) = -\frac{1}{2}\pi \cdot i\left[J_0(z) - iY_0(z)\right].$$

Combining these correlations:

$$\vartheta = T_0 + \frac{2T_0}{\pi}\int_0^{\infty} e^{-\kappa u^2 \tau}\, \frac{J_0(ur)Y_0(ur_0) - Y_0(ur)J_0(ur_0)}{J_0^2(r_0u) + Y_0^2(r_0u)}\,\frac{du}{u}, \tag{6}$$

The asymptotic analysis of the Bessel functions (3) is used for small time units in the Laplace transformed form of the solution:

$$\overline{\Theta} = \frac{T_0}{s}\left(\frac{r_0}{r}\right)^{\frac{1}{2}} e^{-q(r-r_0)}\left\{1 + \frac{(r-r_0)}{8r_0rq} + \frac{(9r_0^2 - 2r_0r - 7r^2)}{128r_0^2r^2q^2}\ldots\right\}.$$

The re-transformed form of which is:

$$\vartheta = \frac{T_0r_0^{1/2}}{r^{1/2}}\,erfc\frac{r-r_0}{2\sqrt{(\kappa\cdot\tau)}} + \frac{T_0(r-r_0)(\kappa\cdot\tau)^{\frac{1}{2}}}{4r_0^{\frac{1}{2}}r^{\frac{1}{2}}}\,ierfc\frac{r-r_0}{2\sqrt{(\kappa\cdot\tau)}} + \frac{T_0(9r_0^2 - 2r_0r - 7r^2)\kappa\cdot\tau}{32r_0^{\frac{3}{2}}r^{\frac{5}{2}}}\,i^2erfc\frac{r-r_0}{2\sqrt{(\kappa\cdot\tau)}} + \ldots, \tag{7}$$

Surface heat flux density is:

$$\dot{q} = -K\left[\frac{\partial\vartheta}{\partial r}\right]_{r=r_0} = \frac{4T_0K}{r_0\pi^2}\int_0^{\infty} e^{-\kappa u^2\tau}\,\frac{du}{u[J_0^2(r_0u) + Y_0^2(r_0u)]}. \tag{8}$$

if $T = \kappa\cdot\tau / r_0^2$, then heat flux for low T values

$$\dot{q} = \frac{KT_0}{r_0}\left\{(\pi\cdot T)^{-\frac{1}{2}} + \frac{1}{2} - \frac{1}{4}\left(\frac{T}{\pi}\right)^{\frac{1}{2}} + \frac{1}{8}T\ldots\right\}, \tag{9}$$

for high T values

$$\dot{q} = \frac{2T_0K}{r_0}\left\{\frac{1}{\ln(4T) - 2\gamma} - \frac{\gamma}{\left[\ln(4T) - 2\gamma\right]^2} - \ldots\right\}, \tag{10}$$

The constant value of Euler in formula (10) is $\gamma = 0{,}57722\ldots$ Its numerical values are included in table (8), compiled by Jaeger and Clarke [11].

In the case of $T \to \infty$ heat loss approaches 0.

2.2.2 Determining heat loss by using the theory of heat flux equations

In the course of our work we will show how heat loss can be calculated based on measurements. The investigation of this problem was first mentioned in bibliography [1], [2] and [3]. We developed so-called heat flow equations that provide a theoretical opportunity to determine the heat flux around the pipe on r radius in circular symmetrical heat conduction if the time-based change of temperature (ϑ) can be measured on a r_0 radius. The following results were obtained:

I. If $r > r_0$ heat flux density can be determined as a convolution integral:

$$j(r,t) = \int_0^t \vartheta(r_0, t-\tau)h(r, r_0, \tau)d\tau \;,$$

where

$$h(r,r_0,t) = \frac{2\kappa K}{\pi}\int_0^{\infty} e^{-\kappa u^2 t}u^2\,\frac{J_1(ru)Y_0(r_0u) - Y_1(ru)J_0(r_0u)}{J_0^2(r_0u) + Y_0^2(r_0u)}\,du,$$

if t > 0, and

$$h(r,r_0,0) = 0\,,\; \kappa = \frac{\lambda}{\rho\cdot c} = a\,.$$

J_0 and Y_0 indicate Bessel functions of the first kind, zero order, while J_1 and Y_1 indicate Bessel of the first and second kind, first order.

II. If $r < r_0$ heat flux density can be determined by the following convolution integral equation of the first order:

$$\int_0^t j(r,\tau)h^*(r,r_0,t-\tau)d\tau = \vartheta(r_0,t),$$

where

$$h^*(r,r_0,t) = \frac{2\kappa}{K\pi}\int_0^{\infty} e^{-\kappa u^2 t}u^2\,\frac{J_1(ru)Y_0(r_0u) - Y_1(ru)J_0(r_0u)}{J_1^2(ru) + Y_1^2(ru)}\,du,$$

if t > 0, and

$$h^*(r,r_0,0) = 0.$$

III. x = x_0, then:

$$h^*(r_0, r_0, 0) = \infty,$$

$$h^*(r_0, r_0, t) = \frac{4k}{Kp^2 r_0}\int_0^{\infty} e^{-ku^2 t} u^2 \frac{du}{u\left[J_1^2(r_0 u) + Y_1^2(r_0 u)\right]},$$

$t > 0$.

These expressions provide a mathematically exact solution to define heat flux density. Their use in practice, however, is hindered by serious computer technology problems. In the following chapter we will present a new integral equation to investigate the problem and to facilitate practical usage.

2.2.3 Determining the heat flux density on the outside of the circular cylinder using Garbai's integral equation

In bibliography [1] Garbai showed that applying the method of Laplace transformation in transient heat conduction around a long circular cylinder at the initial condition of 0 the following integral correlation can be set down between temperature and heat flux density:

$$\int_0^t j(r, t)\frac{e^{-\frac{r_0^2}{4k(t-t)}}}{t-t}dt = \tag{11}$$

$$= \frac{Kr}{2k}\int_0^t J(r_0, t)\frac{e^{-\frac{r^2}{4k(t-t)}}}{(t-t)^2}dt.$$

In the correlation $J(r_0, t)$ is the temperature on the circle of r_0 radius around the cylinder as function of t time, while $j(r, t)$ is the heat flux density on the circle of r_0 radius around the cylinder as function of t time. K is the heat conduction factor of the substance surrounding the cylinder, • the diffusivity factor. (r, r_0) are higher than the radius of the cylinder but otherwise can be any number.

Determining heat flux density means that assuming temperature $J(r_0, t)$ as known function $j(r, t)$ needs to be defined from correlation (11). In terms of mathematics, to determine the heat flux densities is the equivalent of solving a convolution integral equation of the first order. Here we must address the following difficulties:

1. So far in our investigations exact solutions for correlation (10) were only obtained for $r > r_0$ because the nucleus of the equation disappears for $t \to 0 - ra$ therefore the equation cannot be returned to the integral equation of second order, resolvable by the Neumann series.

2. Theoretical function $J(r_0, t)$ is unknown in practice. Temperature occurring at point r_0 was determined through measurements at various t_k times, which produced a set of measurement results $\{J(r_0, t_k)\}$ (k=1,2,… n) and heat flux density can only be determined using them. For simplicity's sake we assume that measured temperatures are equidistically given, i.e. $t_{k+1} - t_k = T$ = constant.

The above shows that a numerical method should be used to investigate integral equation (11).

In the following section Huber's [5] method will be used provided r f r_0. If value T is low, the temperature and heat flux densities will be approached by continuous functions linear by sections as follows:

$$J(r_0, t) = (1-k)J_k + kJ_{k-1} + \frac{J_k - J_{k-1}}{T}t, (J_1 \neq 0)$$

$$j(r, t) = (1-k)\, j_k + kj_{k-1} + \frac{j_k - j_{k-1}}{T}T, \tag{12}$$

if $(k-1)T \le t \le kT$, where k = 1,2,… n.

In the equation J_k means the temperatures measured at point of time kT and j means the approaching markings of prevalent heat flux densities which should be determined from equation (11). n is the number of measured data. It is obvious that $j_0 = J_0 = 0$.

In technical practice transients are followed by a steady state and the temperature thus takes on a steady end value. We can therefore assume that series J_k is limited if $n \to \infty$ (i.e. if $k \to \infty$).

2.2.4 Approach solution of the task by continuous functions linear by sections

Introducing

$$\frac{r_0^2}{4k} = a, \quad \frac{r^2}{4k} = b, \quad \frac{Kr}{2k} = g, \tag{13}$$

equation (11) is as follows:

$$\int_0^t j(r,\tau)\,\frac{e^{-\frac{\alpha}{(t-\tau)}}}{t-\tau}\,d\tau=\gamma\int_0^t \vartheta(r_0,\tau)\,\frac{e^{-\frac{\beta}{(t-\tau)}}}{(t-\tau)^2}\,d\tau\,. \quad (14)$$

Substituting expression (12) in equation (14) we find that at points of time t = KT the following correlation exists:

$$\sum_{l=1}^{k}\int_{(l-1)T}^{lT}\left[(1-l)j_l+lj_{l-1}+\frac{j_l-j_{l-1}}{T}\tau\right]\frac{e^{-\frac{\alpha}{kT-\tau}}}{kT-\tau}d\tau=$$
$$=\gamma\sum_{l=1}^{k}\int_{(l-1)}^{lT}\left[(1-l)\vartheta_l+l\vartheta_{l-1}+\frac{\vartheta_l-\vartheta_{l-1}}{T}\tau\right]\frac{e^{-\frac{\beta}{kT-\tau}}}{(kT-\tau)^2}d\tau, \quad (15)$$

where k = 1,2,… n. Calculating integrals in equation (15) the following recursive equation system is obtained:

$$\sum_{l=1}^{k}\left\{\left[(k-l)(j_l-j_{l-1})+j_l\right]\left[E_1\left(\frac{\alpha}{(k-l+1)T}\right)-E_1\left(\frac{\alpha}{(k-l)T}\right)\right]-\right.$$
$$\left.-(j_l-j_{l-1})\left[(k-l+1)E_2\left(\frac{\alpha}{(k-l+1)T}\right)-(k-l)E_2\left(\frac{\alpha}{(k-l)T}\right)\right]\right\}=$$
$$=\gamma\sum_{l=1}^{k}\left\{\frac{(k-l)(\vartheta_l-\vartheta_{l-1})+\vartheta_l}{\beta}\left(e^{-\frac{\beta}{(k-l+1)T}}-e^{-\frac{\beta}{(k-l)T}}\right)-\right.$$
$$\left.-\frac{\vartheta_l-\vartheta_{l-1}}{T}\left(E_1\left[\frac{\beta}{(k-l+1)T}\right]-E_1\left[\frac{\beta}{(k-l)T}\right]\right)\right\}. \quad (16)$$

k = 1,2,…,n.

In correlation (16) E_1, E_2 mark the exponential integrals.

The approximate value of heat flux density can be determined from equation (16) step by step and in the case of $n\to\infty$ heat flux density can be calculated for the discretionary positive integral of T.

After determining j_1, j_2, …, j_{k-1} and marking the expression by F(k) on the right hand side of equation (16) we find that

$$j_k=\frac{F(k)}{E_1\left(\frac{\alpha}{T}\right)-E_2\left(\frac{\alpha}{T}\right)}-$$
$$-\frac{1}{E_1\left(\frac{\alpha}{T}\right)-E_2\left(\frac{\alpha}{T}\right)}\sum_{l=1}^{k-1}\left\{\left[(k-l)(j_l-j_{l-1})+j_l\right]\left[E_1\left(\frac{\alpha}{(k-l+1)T}\right)-\right.\right.$$
$$\left.-E_1\left(\frac{\alpha}{(k-l)T}\right)\right]-(j_l-j_{l-1})\left[(k-l+1)E_2\left(\frac{\alpha}{(k-l+1)T}\right)-\right.$$
$$\left.\left.-(k-l)E_2\left(\frac{\alpha}{(k-l)T}\right)\right]\right\}-\frac{j_{k-1}E_2\left(\frac{\alpha}{T}\right)}{E_1\left(\frac{\alpha}{T}\right)-E_2\left(\frac{\alpha}{T}\right)}. \quad (17)$$

The obtained recursion can be numerically evaluated by computer. There is an oscillation of results if from a certain k = k_0 value

$$\operatorname{sgn}(j_k-j_{k-l})=-\operatorname{sgn}(j_{k+1}-j_k),\ \text{ha } k > k_0.$$

If oscillation occurs it shows that the used linear approach was not exact. In paper [5] Huber used an elemental geometrical method to show that the oscillation perceived at the approximation of Volterra integral equations can be addressed and thus a better approach is obtained for the solution. The correction in the investigation of heat flux densities can be carried out as follows:

Let's introduce the following markings:

j_k^* - better approach of heat flux density (corrected value),

$\overline{j_k}, \overline{\overline{j_k}}$ - auxiliary volumes calculated from heat flux densities.

Then:

$$j_1^*=\frac{j_0+6j_1+j_2}{8},(j_0=0) \quad (18)$$

Substituting this into the place of j_l in equation (16) we determine values $\overline{j_2}$ and $\overline{j_3}$ for k = 2 and k = 3 with which:

$$j_2^*=\frac{j_1^*+6\overline{j_2}+\overline{j_3}}{8}, \quad (19)$$

Now substituting j_1 and j_2 with corrected values j_1^* and j_2^* in equation (17) we determine auxiliary values $\bar{\bar{j}}_3$ and $\bar{j}_4$ for k = 3 and k = 4 with which:

$$j_3^* = \frac{j_2^* + 6\bar{\bar{j}}_3 + \bar{j}_4}{8}. \qquad (20)$$

Continuing the process the corrected values of heat flux density can be determined for any k. In the event the curve of the theoretical heat flux density is not too high, the method is capable of eliminating the possible oscillation. In rare cases it is sometimes necessary to repeat the correction process (see Huber [5].

The approach method can be used if r ƒ r_0. It can be proven that integral equation (11) has no solution for $r < r_0$.

3 Conclusions, summary

The use of renewable energy plays an ever increasing role in energy supply. One way renewables can be used is to extract the heat of the Earth from the depth of the Earth by geothermal wells. The extracted geothermal heat can be used for heat pumps and district heating. In energy supply the energy analysis of geothermal energy usage and the operation of district heating is an important issue. The core task of energy analysis is to determine heat loss. In unsteady state heat loss may be the multiple of heat loss in steady state. Our paper presented the classical methods of calculating the heat loss around heat transport pipes for unsteady heat conduction. Based on Carslaw and Jaeger we presented the calculation methods for the temperature fields and heat losses around district heating pipes laid horizontally underground and geothermal wells. We described the heat loss calculation method based on heat flow equations developed by Garbai and the numerical solutions for the integral equation. Irrespective of the initial and peripheral conditions, this process, using the presented numerical solution, is able to determine with great accuracy the heat loss of the heat transporting pipe from the unsteady temperatures measured around the pipe in the given cross section. The presented process is based on Huber's method [5].

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