Upon the Solutions Trajectories of an Euler Type Gyroscope

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Abstract: - The paper deals with a study of the solutions trajectories of an Euler type gyroscope, on constant level energy surfaces. Using Mathcad software and the prime integrals of Euler's equations, we generated the possible positions of the angular momentum according to the magnitude of the principal moments of inertia, as a result of the intersection between an ellipsoid and a sphere. Also the stability of the equations solutions was analyzed by help of cross – sections along perpendicular planes determined by the principal axis of inertia for the studied body.

Key-Words: - kinetic energy, angular momentum, trajectory, prime integrals, Euler equations, stationary rotation

1 Introduction

Considering the fixed point motion of a rigid body, not subjected to external forces (Euler type gyroscope) and **K** the angular momentum of the body with respect to the fixed point, Euler used to prove the following theorem:

$$\frac{d\,\boldsymbol{K}}{d\,t} = [\mathbf{K}, \boldsymbol{\Omega}] \tag{1}$$

where $\boldsymbol{\Omega}$ represents the gyroscope angular velocity about the reference system attached to it.

Of course, the angular momentum vector with respect to the space system of reference remains constant, both in magnitude in direction during the entire motion.

The result is a differential equation with respect to **K** or $\boldsymbol{\Omega}$. The equivalent scalar system, taking into account the components of the angular velocity about the principal axis of inertia becomes:

$$\begin{aligned} J_1 \frac{d\Omega_1}{dt} &= (J_2 - J_3)\Omega_2\Omega_3 \\ J_2 \frac{d\Omega_2}{dt} &= (J_3 - J_1)\Omega_3\Omega_1 \\ J_3 \frac{d\Omega_3}{dt} &= (J_1 - J_2)\Omega_1\Omega_2 \end{aligned} \tag{2}$$

where J_1, J_2, J_3 represent the gyroscope principal moments of inertia and Ω_1, Ω_2, Ω_3 the components of the angular velocity with respect to the same axis.

2 Analysis of the prime integrals

As we know, one of the classic solutions to the gyroscope motion equations is based on the determination of two square prime integrals, given by the energy conservation law and by the angular momentum conservation law, due to the absence of the external forces action. The prime integrals are the following:

$$2\,E = \frac{K_1^2}{J_1} + \frac{K_2^2}{J_2} + \frac{K_3^2}{J_3} \tag{3}$$

$$K^2 = K_1^2 + K_2^2 + K_3^2 \tag{4}$$

The first one represents the equation of an ellipsoid whose semi - axis are $\sqrt{2EJ_1}, \sqrt{2EJ_2}, \sqrt{2EJ_3}$ and the second one the equation of a sphere having the radius K. The vector **K** will be at the intersection between the ellipsoid and the sphere.

If we consider a constant energy level, meaning that the ellipsoid is fixed, we may change the sphere radius to find the intersection curves by Mathcad representation.

Assuming there is the following condition between the main moments of inertia of the rigid body $J_1 > J_2 > J_3$, the semi - axis of the ellipsoid will respect the following relation: $\sqrt{2EJ_1} > \sqrt{2EJ_2} > \sqrt{2EJ_3}$. If the sphere radius K is shorter than the minimum semi - axis or longer than the maximum semi - axis, there will be no intersection between the ellipsoid and the sphere, so there are no motions characterized by such values of E and K. If the sphere radius is equal to the minimum semi - axis the intersection consists of two points. The more the value of the radius increases, we obtain two curves around the extremities of the minimum semi - axis. Similarly, if the radius is equal to the maximum semi - axis, we find the two ends of the maximum semi - axis and for lower values of the radius K, we obtain two curves in the vicinity of these ends. Finally, if $K = \sqrt{2EJ_2}$, the intersection consists of two circles which pass through the ends of the middle semi - axis.

3 Mathcad representations of the equation solutions

In order to allow the computer to generate the representations we need to provide some mathematical functions based on the analysis done in the previous paragraph.

Thus, the plane projections of the solutions may be written:

$$X(k_{i,j}) = \sqrt{\frac{J_2^2 - K_j^2}{J_2^2 - |b(q \cdot i)|^2} \cdot |b(q \cdot i)|^2} \quad (5)$$

$$Y(k_{i,j}) = \sqrt{\frac{K_j^2 - |b(q \cdot i)|^2}{J_2^2 - |b(q \cdot i)|^2}} \cdot J_2 \quad (6)$$

where i, j and k are recurrence indexes and b(q) is given by:

$$b(q) = \begin{bmatrix} J_1 & 0 \\ 0 & J_3 \end{bmatrix} \cdot \begin{bmatrix} \cos(q) \\ \sin(q) \end{bmatrix} \quad (7)$$

In order to obtain the ellipsoid we use the following expressions:

$$re^{\langle ke0,i \rangle} = J \cdot e(i \cdot q) \quad (8)$$

$$re^{\langle ke1,i \rangle} = J \cdot RY\left(\frac{\pi}{2}\right) \cdot e(i \cdot q) \quad (9)$$

$$re^{\langle ke2,i \rangle} = J \cdot RX\left(\frac{\pi}{2}\right) \cdot RY\left(\frac{\pi}{2}\right) \cdot e(i \cdot q) \quad (10)$$

where e(i .q) provides the unit vector variation in three dimensional space, expressed:

$$e(i \cdot q) = [0 \quad \sin(i \cdot q) \quad \cos(i \cdot q)]^T \quad (11)$$

while RX, RY are the rotation matrices corresponding to the coordinate axis.

The sphere will be generated by help of the following relation (using (5) and (6)):

$$r^{\langle kj,i \rangle} = \begin{bmatrix} X(k_{i,j}) \cos(q \cdot i) \\ Y(k_{i,j}) \\ X(k_{i,j}) \sin(q \cdot i) \end{bmatrix}^T \quad (12)$$

The intersections between the ellipsoid and the sphere, obtained for a certain interval of the recurrence indexes, were represented in Fig.1 using the functions in (8), (9), (10) and (12), as the possible positions of the angular moment vector.

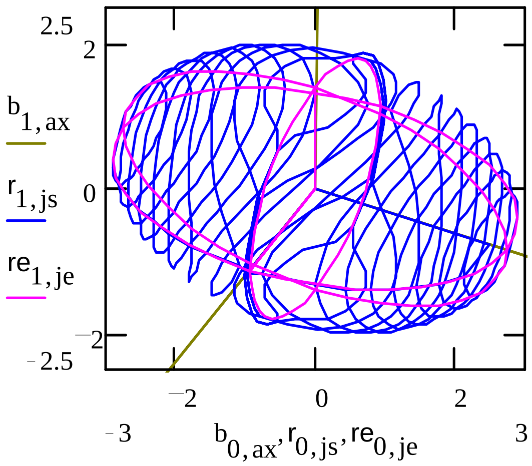

Fig.1

Then the cross-sections with the coordinate planes (determined by the intersection of the principal axis of inertia) were obtained and represented in fig.2, fig.3 and fig.4, to be able to visualize and emphasize the intersection points.

This way, the stability of the solutions can be analyzed and matched with the theoretical approaches.

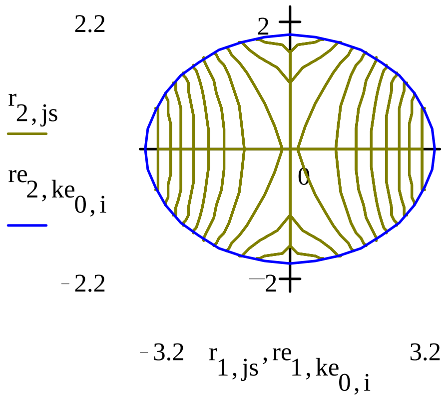

Fig.2

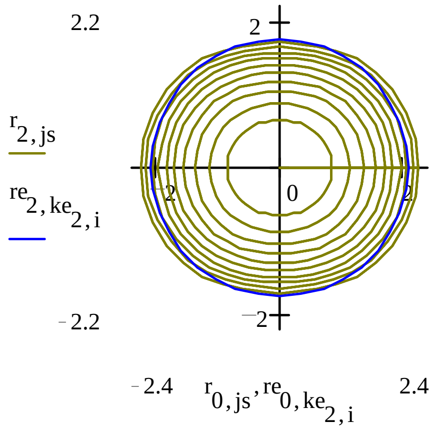

Fig.3

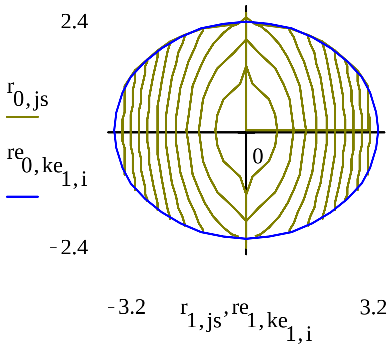

Fig.4

Each of the six ends corresponding to the semi - axis of the ellipsoid is a trajectory of the Euler equation (1), that means a stationary position of the vector angular momentum **K**.

To each position corresponds a constant value of angular velocity vector which is directed towards one of the principal axis of inertia. For such a motion Ω is all the time collinear to **K**, so we can say that the rigid body is in fact rotating with a constant angular velocity around the axis of inertia, whose position remains fixed. This kind of motion is called stationary rotation.

Now, the stability of the stationary solutions of the Euler equation can be discussed based on the representations above. Thus, for a small deviation of the initial condition towards the minimum and maximum axis of inertia, the trajectory will be a closed curve (see fig.3 and 4), while for a small deviation about the middle axis of inertia, the trajectory will be a large curve, which does not remain near the vicinity of the axis (see fig.2).

4 Motion Simulation of Angular Velocity and Momentum

As we already stated in the previous paragraph, for a stationary rotation, the angular velocity directed along one of the principal axis of inertia should remain collinear to the angular momentum.

In order to demonstrate this we can also represent the motions of the vectors angular momentum **K** and angular velocity Ω about the reference system attached to the rigid body. If the magnitude of **K** is different from any value of the ellipsoid semi - axis, the motions will be periodical.

Remember also, another theorem proven by Poinsot and stating that during the motion, the gyroscope ellipsoid of inertia is rolling without sliding on a fixed plane, which is perpendicular to the angular momentum vector determined about the space reference system.

Considering the ellipsoid of inertia rolling on a fixed plane, and analyzing the positions of the studied vectors, angular velocity and angular momentum we can simulate their consecutive locations during the motion.

Thus, the expression used to represent the motion of the angular velocity vector is:

$$\Omega^{\langle k2i,j \rangle} = J \cdot r^{\langle ki,3j \rangle} \quad (13)$$

We get the representation in fig.5, where the positions of the angular velocity are blue.

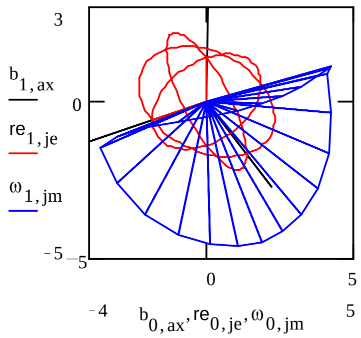

Fig.5

The motion of the angular momentum vector will be expressed by help of

$$K^{\langle k2i,j \rangle} = r^{\langle ki,3j \rangle} \quad (14)$$

In fig.6 the positions of the angular momentum vector are represented during the considered motion using the blue colour.

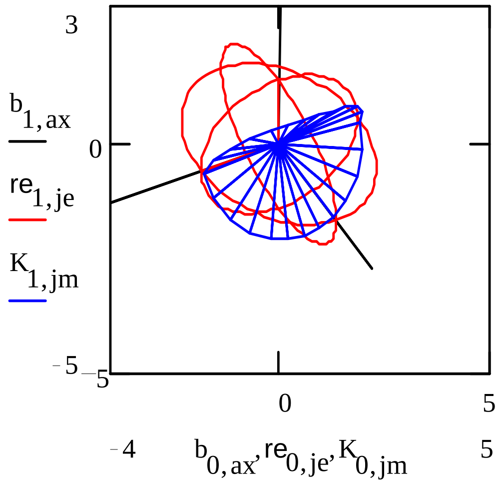

Fig.6

A quick view upon fig.5 and fig.6 proves the statement we were trying to demonstrate, concerning the collinearity of the vectors angular velocity and angular momentum during a stationary rotation.

5 Conclusion

Solving the system of differential equations of the fixed point motion was a challenging task for many scientists starting with Euler. Now using Mathcad software, all the difficult mathematical equations and theories can be proven by representing and simulating the behaviour discovered by theory.

Thus, the solutions of the equations were represented by help of the prime integrals discovered by Euler, using the intersection between their graphical form.

Also, the problem of solutions stability can be visualized by obtaining the cross-sections along three different perpendicular planes (defined by help of the principal axis of inertia).

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