A generalized polyconvex hyperelastic model for anisotropic solids

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Abstract: A generalized polyconvex hyperelastic model for anisotropic solids is presented. The strain energy function is formulated in terms of convex functions of generalized invariants and is given by a series with an arbitrary number of terms. The model addresses solids with orthotropic or transversely isotropic material symmetry as well as fiber-reinforced materials. Special cases of the strain energy function suitable for anisotropic elastomers and soft biological tissues are discussed and illustrated by numerical examples.

Key–Words: Polyconvexity, Anisotropy, Fiber-reinforced materials, Hyperelasticity, Soft biological tissues, Calendered rubber

1 Introduction

In order to model the non-linear mechanical behavior of anisotropic solids undergoing large elastic deformations, appropriate constitutive models are required. There are a number of hyperelastic models which take into account this anisotropy, among those the orthotropic St.Venant-Kirchhoff model [1], and the well known Fung-elastic model [2]. The latter one has been successfully utilized for various soft tissues. However, some of these anisotropic models have been shown to be non-elliptic, a shortcoming which can result in loss of material stability and lead to unphysical behavior [3, 4]. In contrast, polyconvex hyperelastic models a priori ensure the strong ellipticity also called Legendre-Hadamard condition, which furthermore guarantees positive definiteness of the acoustic tensor so that the speed of displacement waves is always real for any direction of propagation. Moreover, in combination with coercivity, polyconvexity guarantees the existence of the global minimizer of the total elastic energy of the body [5] which is of decisive importance in the context of a boundary value problem. To benefit from these positive attributes we propose a generalized approach to formulate polyconvex anisotropic strain energy functions.

2 Material symmetry

Our approach is applicable to two classes of materials. The first class contains those materials that can be described by orthotropic or transversely isotropic material symmetry. For orthotropic materials one introduces orthonormal base vectors $\boldsymbol{l}_i$, $i = 1, 2, 3$, in the principal material directions and defines three structural tensors

$$\begin{aligned} &\mathbf{L}_1 = \boldsymbol{l}_1 \otimes \boldsymbol{l}_1, \;\; \mathbf{L}_2 = \boldsymbol{l}_2 \otimes \boldsymbol{l}_2, \\ &\mathbf{L}_0 = (\mathbf{I} - \mathbf{L}_1 - \mathbf{L}_2) = \boldsymbol{l}_3 \otimes \boldsymbol{l}_3, \end{aligned} \tag{1}$$

where $\mathbf{I}$ denotes the identity tensor of the second order. Transverse isotropy represents a material symmetry with respect to only one preferred direction, denoted here by the unit vector $\boldsymbol{l}_1$. The transversely isotropic symmetry can be described by two structural tensors of the form

$$\mathbf{L}_1 = \boldsymbol{l}_1 \otimes \boldsymbol{l}_1, \;\; \mathbf{L}_0 = \frac{1}{2}\left(\mathbf{I} - \mathbf{L}_1\right). \tag{2}$$

The second class includes fiber-reinforced materials with an isotropic matrix and an arbitrary number n of fiber families that are aligned in certain directions specified by unit vectors $\boldsymbol{l}_i$, $i = 1, 2, ..., n$. In this case, $n + 1$ structural tensors $\mathbf{L}_i$, $i = 0, 1, ..., n$, are defined by

$$\mathbf{L}_i = \boldsymbol{l}_i \otimes \boldsymbol{l}_i, \;\; i = 1, 2, ..., n, \quad \mathbf{L}_0 = \frac{1}{3}\mathbf{I}. \tag{3}$$

With the aid of the above structural tensors one defines the symmetry group of the material

$$\begin{aligned} \mathcal{G} = \big\{ \mathbf{Q} \in \mathbf{O}\mathrm{rth} : \; &\mathbf{Q}\mathbf{L}_i\mathbf{Q}^{\mathrm{T}} = \mathbf{L}_i, \\ &i = 0, ..., n \big\}, \end{aligned} \tag{4}$$

which contains the set of all orthogonal mappings that do not violate the material symmetry. $\mathbf{O}$rth denotes here the set of all orthogonal second-order tensors.

3 Anisotropic polyconvex strain energy functions

A strain energy function $W(\mathbf{F})$ of the deformation gradient $\mathbf{F}$ is said to be polyconvex [5] if there exists a convex function $\breve{W}(\mathbf{F}, \mathrm{adj}\mathbf{F}, \mathrm{det}\mathbf{F})$ such that

$$W(\mathbf{F}) = \breve{W}(\mathbf{F}, \mathrm{adj}\mathbf{F}, \mathrm{det}\mathbf{F}), \tag{5}$$

where $\mathrm{det}\mathbf{F}$ denotes the determinant of the second-order tensor $\mathbf{F}$ and $\mathrm{adj}\mathbf{F} = \mathbf{F}^{-1}\mathrm{det}\mathbf{F}$. According to the classical invariant theory, the anisotropic strain energy function W can be represented by

$$\begin{aligned} &W = \widehat{W}\left(\mathrm{tr}(\mathbf{C}\mathbf{L}_i), \mathrm{tr}(\mathbf{C}^2\mathbf{L}_i), \mathrm{tr}\mathbf{C}^3\right), \\ &i = 0, 1, ..., n, \end{aligned} \tag{6}$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ denotes the right Cauchy-Green tensor. Note that for fiber-reinforced materials (3), the list of invariants in (6) is in general not complete. While the invariants $\mathrm{tr}(\mathbf{C}\mathbf{L}_i)$, $i = 0, 1, ..., n$, and $\mathrm{tr}\mathbf{C}^3$ are convex with respect to $\mathbf{F}$, the terms $\mathrm{tr}(\mathbf{C}^2\mathbf{L}_i)$, $i = 1, 2, ..., n$, are not. However, they can be expressed in terms of invariants convex with respect to $\mathbf{F}$, $\mathrm{adj}\mathbf{F}$ and $\mathrm{det}\mathbf{F}$ [1, 6], which leads to an alternative representation of the strain energy

$$W = \bar{W}\left(I_i, J_i, \mathrm{III}_{\mathbf{C}}\right), \tag{7}$$

where

$$\begin{aligned} &I_i = \mathrm{tr}(\mathbf{C}\mathbf{L}_i), \quad J_i = \mathrm{tr}\left[(\mathrm{cof}\mathbf{C})\mathbf{L}_i\right], \\ &\mathrm{cof}\mathbf{C} = \mathbf{C}^{-\mathrm{T}}\mathrm{det}\mathbf{C}, \quad \mathrm{III}_{\mathbf{C}} = \mathrm{det}\mathbf{C}, \\ &i = 0, 1, ..., n. \end{aligned} \tag{8}$$

In the next step, convex generalized invariants $\tilde{I}_r$ and $\tilde{J}_r$ are formed by means of non-negative weight factors $w_i^{(r)}$ of the principal or fiber directions by

$$\begin{aligned} &\tilde{I}_r = \sum_{i=0}^{n} w_i^{(r)} I_i, \quad \tilde{J}_r = \sum_{i=0}^{n} w_i^{(r)} J_i, \\ &\sum_{i=0}^{n} w_i^{(r)} = 1, \quad r = 1, 2, \ldots, s. \end{aligned} \tag{9}$$

Polyconvex functions can be composed additively in the form $W(\mathbf{F}) = \hat{W}_1(\mathbf{F}) + \hat{W}_2(\mathrm{adj}\mathbf{F}) + \hat{W}_3(\mathrm{det}\mathbf{F})$, where $\hat{W}_1$, $\hat{W}_2$, and $\hat{W}_3$ are convex functions of their arguments [6]. This motivates a series representation

$$\begin{aligned} W &= \frac{1}{4}\sum_{r=1}^{s} \mu_r \left[f_r\left(\tilde{I}_r\right) + g_r\left(\tilde{J}_r\right) \right. \\ &+ \left. h_r\left(\mathrm{III}_{\mathbf{C}}^{1/2}\right)\right] \end{aligned} \tag{10}$$

with an arbitrary number of terms s. Therein, f_r and g_r are convex and monotonically increasing functions, h_r are convex functions of their arguments, and $\mu_r \geq 0$ denote material parameters with the dimension of stress. Calculating the second Piola-Kirchhoff stress tensor $\mathbf{S}$ and enforcing the condition of the energy and stress free natural state, in which $W|_{\mathbf{C}=\mathbf{I}} = 0$, $\mathbf{S}|_{\mathbf{C}=\mathbf{I}} = \mathbf{0}$, we obtain the following requirements for f_r, g_r and h_r

$$\begin{aligned} &f_r(1) = g_r(1) = h_r(1) = 0, \\ &f_r'(1) = g_r'(1) = -\frac{1}{2}h_r'(1) = 1, \\ &r = 1, 2, \ldots, s \end{aligned} \tag{11}$$

similar to those ones imposed on the so-called generalized strain measures.

4 Special cases and applications

Particular forms of the functions f_r, g_r, and h_r have not been discussed yet. A variety of suitable convex functions may be found to account for specific material properties. As an example, we present three useful expressions demonstrating good agreement with experimental data. In order to describe the anisotropic mechanical behavior of calendered rubbers [7], a power function representation of the strain energy (10) was successfully utilized [1]

$$\begin{aligned} W &= \frac{1}{4}\sum_{r=1}^{s} \mu_r \left\{ \frac{1}{\alpha_r}\left(\tilde{I}_r^{\alpha_r} - 1\right) \right. \\ &+ \frac{1}{\beta_r}\left(\tilde{J}_r^{\beta_r} - 1\right) \\ &+ \left. \frac{1}{\gamma_r}\left(\mathrm{III}_{\mathbf{C}}^{-\gamma_r} - 1\right)\right\}, \end{aligned} \tag{12}$$

with material parameters $\alpha_r \geq 1$, $\beta_r \geq 1$, and $\gamma_r > 0$. Similarly, the typical sigmoidal stress-strain curves of this transversely isotropic elastomer may be modeled by Gent-type functions [8]. This motivates the expression

$$\begin{aligned} W &= \frac{1}{4}\sum_{r=1}^{s} \mu_r \left\{ -\alpha_r \ln\left(1 - \frac{\tilde{I}_r - 1}{\alpha_r}\right) \right. \\ &- \beta_r \ln\left(1 - \frac{\tilde{J}_r - 1}{\beta_r}\right) \\ &- \left. \frac{1}{\gamma_r}\left(\mathrm{III}_{\mathbf{C}}^{-\gamma_r} - 1\right)\right\}, \end{aligned} \tag{13}$$

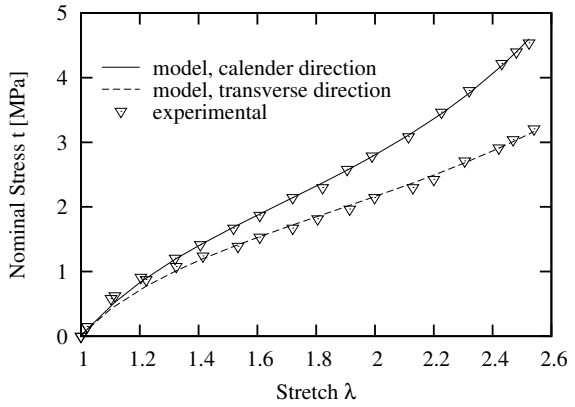

Figure 1: Stress-stretch diagram for polyconvex model with Gent-type terms vs. experimental results on calendered rubber sheets [7] ($s = 1,\ \mu_1 = 4.8135\,\text{MPa},\ \alpha_1 = 4.3347,\ \beta_1 = 96.0502,\ w_1^{(1)} = 0.3997,\ w_0^{(1)} = 1 - w_1^{(1)} = 0.6003$).

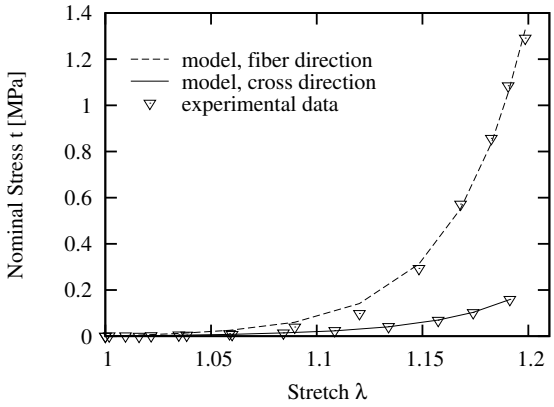

Figure 2: Stress-stretch diagram for polyconvex model with exponential terms vs. experimental results on chemically treated bovine pericardium [4] ($s = 1,\ \mu_1 = 0.0606\,\text{MPa},\ \alpha_1 = 36.9315,\ \beta_1 = 45.6493,\ w_1^{(1)} = 0.1739,\ w_2^{(1)} = 0.4438,\ w_0^{(1)} = 1 - w_1^{(1)} - w_2^{(1)} = 0.3822$).

wherein $\alpha_r \geq 0$, $\beta_r \geq 0$ are the limiting values of $\tilde{I}_r - 1$ and $\tilde{J}_r - 1$, respectively. Figure 1 shows the comparison between the nominal stresses obtained in uniaxial experiments [7] and calculated on the basis of the strain energy (13).

Soft biological tissues are characterized by a typical exponential stress-stretch response. In this case, the exponential representations of f_r and g_r are reasonable [4]:

$$\begin{aligned} W &= \frac{1}{4}\sum_{r=1}^{s}\mu_r\left\{\frac{1}{\alpha_r}\left[e^{\alpha_r(\tilde{I}_r-1)}-1\right]\right. \\ &+ \frac{1}{\beta_r}\left[e^{\beta_r(\tilde{J}_r-1)}-1\right] \\ &+ \left.\frac{1}{\gamma_r}\left[\mathrm{III}_{\mathbf{C}}^{-\gamma_r}-1\right]\right\}, \end{aligned} \tag{14}$$

with material parameters $\alpha_r \geq 0$, $\beta_r \geq 0$, and $\gamma_r > 0$. The exponential model was successfully applied to describe uniaxial tension tests on samples of glutaraldehyde fixed bovine pericardium [4]. The comparison between model and experiment is presented in Figure 2 together with the related material parameters. Note that in both numerical examples, the strain energy series was truncated after the first term ($s = 1$), including only four material constants for the transversely isotropic and five for the orthotropic material. For both rubber and biological tissue, incompressibility was assumed. In this case, the last term in the strain energy function (10) vanishes due to the condition $\mathrm{III}_{\mathbf{C}} = 1$.

5 Summary

A generalized polyconvex hyperelastic model is proposed which offers wide flexibility in application to various anisotropic solids. The model is given by a series with an arbitrary number of terms satisfying a priori the condition of the energy and stress free natural state. The series is composed of convex functions of generalized anisotropic invariants. These functions can be chosen appropriately in order to describe a specific material behavior such as that of elastomers or biological tissues. The model addresses solids with orthotropic and transversely isotropic material symmetry as well as those with arbitrary fiber-reinforced structures which together constitute a significant part of engineering materials and soft biological tissues. Numerical results on the one hand and experimental data on calendered rubber and bovine pericardium on the other hand show good agreement with a small number of material parameters involved.

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