The Design of Axisymmetric Ducts for Incompressible flow with Blockage Effects.

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Abstract

In this paper a numerical algorithm is described for solving the boundary value problem associated with axisymmetric, inviscid, incompressible and irrotational flow with a circumferentially arranged cascade of aerofoils placed in the duct. The algorithm is capable of calculating the duct wall geometries from prescribed wall velocity distributions. The equations modeling the flow are derived using the stream function ψ(x,y) and the function ϕ(x,y) as independent variables where for irrotational flow ϕ(x,y) can be recognized as the velocity potential function, for rotational flow ϕ(x,y) ceases being the velocity potential function but does remain orthogonal to the stream lines, the x and y are the usual axial and radial coordinates in cylindrical polar coordinates respectively. The technique described is capable of tackling the so-called inverse problem where the velocity wall distributions are prescribed from which the duct geometry is calculated, as well as the direct problem where the velocity distribution on the pressure and suction surfaces are calculated from prescribed geometries. The two different cases outlined in this paper are boundary value problems with Neumann and Dirichlet boundary conditions respectively with results for the Neumann boundary condition only included. The axial velocity and the swirl velocity are prescribed such that no vorticity is transported through the duct. The governing linear elliptic second order partial differential is coupled with a set of quasi-linear hyperbolic first order partial differential equations with characteristics parallel to the ϕ and ψ axes, the numerical solution is thus obtained iteratively using finite differences to approximate the derivatives. The presence of the blades has a bearing on the rate of mass flow and thus alters the usual equation of continuity.

1. Introduction

Designers of annular ducts require numerical techniques for calculating wall geometries from a prescribed velocity distribution. The objective of the prescribed velocity is typically to avoid boundary layer separation see for example Curle (2). At inlet a constant axial velocity is prescribed (along with an appropriate swirl component of velocity) such that the flow is irrotational.

This paper describes a numerical algorithm for solving the boundary value problem that arises when the independent variables are ϕ and ψ where ϕ may be identified as the velocity potential function (for irrotational flow only), for flow with vorticity, ϕ ceases being the velocity potential function but does remain orthogonal to ψ which may be identified as the stream function. The dependent variable *y*, is the radial coordinate and *x* the axial coordinate. The numerical technique is based on the finite difference scheme on a uniform rectangular mesh.

2. The Design Plane

Defining
$w = u - iv = qe^{-i\vartheta}$ and $z = x + iy$

then using the Cauchy-Riemann equations the identity

$$\frac{\partial w}{\partial \bar{z}} = \frac{1}{2}(\eta - i\omega_\alpha)$$

is easily verified where

$$\eta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

and

$$\omega_\alpha = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

In application to steady plane flow, with rectilinear coordinates *x, y* and velocity components u, v in the *x, y* directions respectively q is the flow speed, ϑ is the flow direction measured from the x axis, ω_α is the component of vorticity normal to the plane and η is the rate of expansion or dilation. If η is zero everywhere apart from at isolated singularities e.g. point sources the velocity components can be derived from a stream function ψ level lines of which coincide with the streamlines. If ω_α is zero everywhere except at point vortices then the velocity components can also be derived from a velocity potential ϕ, level lines of which are orthogonal to the streamlines.

3. The First Auxiliary Flow.

Consider a flow of complex conjugate velocity $w^{(1)}$ where

$$w^{(1)} = \frac{\partial \psi}{\partial n}\, e^{-i\vartheta} \qquad (3.1)$$

with ψ real and $\frac{\partial \psi}{\partial n}$ its derivative in a direction $(\vartheta + \frac{\pi}{2})$ from the x-axis. This auxiliary flow and the actual flow (of complex conjugate velocity w) clearly share the direction ϑ and taking $\frac{\partial \psi}{\partial s}$, the derivative in the direction ϑ, to vanish over either flow field then

$$\frac{\partial \psi}{\partial s} = \cos\vartheta \frac{\partial \psi}{\partial x} + \sin\vartheta \frac{\partial \psi}{\partial y} \qquad (3.2)$$
$$= 0$$

while

$$\frac{\partial \psi}{\partial n} = -\sin\vartheta \frac{\partial \psi}{\partial x} + \cos\vartheta \frac{\partial \psi}{\partial y}$$
$$= -\cos ec\vartheta \frac{\partial \psi}{\partial x} = \sec\vartheta \frac{\partial \psi}{\partial y}$$

and substituting in definition (3.1)

$$w^{(1)} = \frac{\partial \psi}{\partial y} + i\frac{\partial \psi}{\partial x}$$

so that

$$\frac{\partial w^{(1)}}{\partial \bar{z}} = 2i\frac{\partial^2 \psi}{\partial z \partial \bar{z}} \qquad (3.3)$$

Certain observations can be made on this auxiliary flow characterized so far, by equation (3.2) and (3.3). From equation (3.3) it has zero rate of expansion and a vorticity given by

$$w^{(1)} = -\nabla^2 \psi$$

Level lines of $\psi(x,y)$ define its stream line pattern and also that of the actual flow, but the distribution of ψ across the stream has not yet been allocated.

4. The Second Auxiliary Flow

Next consider a flow of complex conjugate velocity $w^{(2)}$, where

$$w^{(2)} = \frac{\partial \varphi}{\partial s}\, e^{-i\vartheta} \qquad (4.1)$$

with $\phi(x,y)$ real. This flow also shares direction and streamline pattern with the actual flow but in order to establish a family of curves orthogonal to the streamlines, this time $\frac{\partial \varphi}{\partial n}$ is taken to vanish over the flow field, i.e.,

$$\frac{\partial \varphi}{\partial n} = -\sin\vartheta \frac{\partial \varphi}{\partial x} + \cos\vartheta \frac{\partial \varphi}{\partial y} = 0$$

so that

$$\frac{\partial \varphi}{\partial s} = \cos\vartheta \frac{\partial \varphi}{\partial x} + \sin\vartheta \frac{\partial \varphi}{\partial y}$$
$$\therefore \frac{\partial \varphi}{\partial s} = \sec\vartheta \frac{\partial \varphi}{\partial x} = \cos ec\vartheta \frac{\partial \varphi}{\partial y}$$

and substituting in definition (4.1)

$$w^{(2)} = \frac{\partial \varphi}{\partial x} - i\frac{\partial \varphi}{\partial y}$$
$$\Rightarrow \frac{\partial w^{(2)}}{\partial \bar{z}} = 2\frac{\partial^2 \varphi}{\partial z \partial \bar{z}}$$

This second auxiliary flow therefore has zero vorticity but a rate of expansion given by

$$\eta^{(2)} = \nabla^2 \varphi$$

level lines of $\phi(x,y)$ define a family of curves orthogonal to the streamline pattern common to both auxiliary flows and the actual flow but the distribution of ϕ along the stream has yet to be allocated.

5. Intrinsic Flow Equations

The differential operator identity

$$\frac{\partial}{\partial s} + i\frac{\partial}{\partial n} = 2e^{-i\vartheta}\frac{\partial}{\partial \bar{z}}$$

is easily verified and when applied to the function *log(w)* there follows after some simple manipulation

$$\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial n}\right)\log(\mathrm{w}) = \frac{1}{q}(\eta - i\omega_\alpha) \qquad (5.1)$$

Applying equation (5.1) to the actual flow and each subsidiary flow gives

$$\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial n}\right)(\log(\mathrm{q}) - i\vartheta) = \frac{1}{q}(\eta - i\omega_\alpha) \qquad (5.2)$$

$$\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial n}\right)(\log(\Psi) + i\vartheta) = -i\Psi\nabla^2\psi \qquad (5.3)$$

$$\left(\frac{\partial}{\partial s} + i\frac{\partial}{\partial n}\right)(\log(\Phi) + i\vartheta) = -\Phi\nabla^2\varphi \qquad (5.4)$$

where Φ and Ψ represent the reciprocals of $\frac{\partial \phi}{\partial s}$

and $\frac{\partial \psi}{\partial n}$ respectively. The system of implicit flow equations comprises the real and imaginary parts of equation (5.2), the real part of equation (5.3) and the imaginary part of (5.4)

$$\frac{\partial}{\partial s}(\log(q)) + \frac{\partial \vartheta}{\partial n} = \frac{\eta}{q} \qquad (5.5)$$

$$\frac{\partial}{\partial n}(\log(q)) - \frac{\partial \vartheta}{\partial s} = -\frac{\omega_\alpha}{q} \qquad (5.6)$$

$$\frac{\partial}{\partial s}(\log(\Psi)) - \frac{\partial \vartheta}{\partial n} = 0 \qquad (5.7)$$

$$\frac{\partial}{\partial s}(\log(\Phi)) + \frac{\partial \vartheta}{\partial s} = 0 \qquad (5.8)$$

6. The Fundamental Design Plane Equations

Eliminating ϑ between equations (5.5) and (5.6) and again between equations (5.7) and (5.8) gives

$$\frac{\partial}{\partial s}(\log(q\Psi)) = \frac{\eta}{q}$$

$$\frac{\partial}{\partial n}(\log(q\Phi)) = -\frac{\omega_\alpha}{q}$$

substituting $A = q\Psi$ and $B = q\Phi$. The last pair of equations can be written as

$$\frac{\partial}{\partial \varphi}(\log(A)) = \frac{\eta}{q^2}B \qquad (6.1)$$

$$\frac{\partial}{\partial \psi}(\log(B)) = -\frac{\omega_\alpha}{q^2}A \qquad (6.2)$$

whilst equations equation (5.7) and (5.8) similarly become

$$\frac{A}{B}\frac{\partial}{\partial \varphi}(\log(\frac{q}{A})) = -\frac{\partial \vartheta}{\partial \psi} \qquad (6.3)$$

and

$$\frac{B}{A}\frac{\partial}{\partial \psi}(\log(\frac{q}{B})) = \frac{\partial \vartheta}{\partial \varphi} \qquad (6.4)$$

eliminating ϑ between equations (6.3) and (6.4) gives

$$\frac{\partial}{\partial \phi}\left[\frac{A}{B}\frac{\partial}{\partial \varphi}\log\left(\frac{q}{A}\right)\right] + \frac{\partial}{\partial \psi}\left[\frac{B}{A}\frac{\partial}{\partial \psi}\log\left(\frac{q}{B}\right)\right] = 0 \qquad (6.5)$$

Regarding temporarily η, ω_α and q as known functions of ϕ and ψ the system (6.1) and (6.2) is quasi-linear hyperbolic with characteristics parallel to the ϕ and ψ axes which maps the physical flow field into an infinite strip in the (ϕ, ψ) plane. Bearing in mind the freedom available in the stream wise variation of ϕ and the cross stream variation of ψ, suitable values of A can be prescribed along one ϕ characteristic and those of B can be prescribed along one ψ characteristic.

Regarding similarly A and B as known functions of ϕ and ψ equation (6.5) is linear elliptic and although boundary conditions for it will depend on the particular application, the Dirichlet choice involves the prescription of q over the flow field. Numerical coupling of the two schemes yields the solution in the Design Plane.

7. Physical Coordinates

From elementary geometric considerations and definitions given previously

$$dz = e^{i\vartheta}(ds + idn)$$

$$= \frac{e^{i\vartheta}}{q}\left(Bd\varphi + iAd\psi\right) \qquad (7.1)$$

Thus $qds=Bd\varphi$ and $qdn=Ad\psi$. So that when ϑ, q, A and B are known in the (ϕ, ψ) plane the physical coordinates x and y can be calculated. Alternatives to equations (6.3), (6.4) and (6.5) which are more convenient in some applications can be obtained using the values of

$$\frac{\partial z}{\partial \varphi} \text{ and } \frac{\partial z}{\partial \psi}$$

given by equation (7.1), so that

$$\frac{\partial x}{\partial \varphi} = \frac{B}{q}\cos\vartheta, \quad \frac{\partial y}{\partial \varphi} = \frac{B}{q}\sin\vartheta,$$

$$\frac{\partial x}{\partial \psi} = -\frac{A}{q}\sin\vartheta, \quad \frac{\partial y}{\partial \psi} = \frac{A}{q}\cos\vartheta.$$

hence

$$\frac{\partial x}{\partial \varphi} = \frac{B}{A}\frac{\partial y}{\partial \psi} \qquad (7.2)$$

and

$$\frac{\partial x}{\partial \psi} = -\frac{A}{B}\frac{\partial y}{\partial \varphi} \qquad (7.3)$$

hence eliminating x in (7.2) and (7.3) yields

$$\frac{\partial}{\partial \varphi}\left(\frac{A}{B}\frac{\partial y}{\partial \varphi}\right) + \frac{\partial}{\partial \psi}\left(\frac{B}{A}\frac{\partial y}{\partial \psi}\right) = 0 \qquad (7.4)$$

Equation (7.4) may be used to replace equation (6.5) in the design system previously described and for use in equations (6.1) and (6.2)

$$\frac{1}{q^2} = \frac{1}{A^2}\left(\frac{\partial y}{\partial \psi}\right)^2 + \frac{1}{B^2}\left(\frac{\partial y}{\partial \varphi}\right)^2$$

this time completion of the physical coordinates is provided from equations (7.2) and (7.3) by

$$dx = \frac{B}{A}\frac{\partial y}{\partial \psi}d\varphi - \frac{A}{B}\frac{\partial y}{\partial \varphi}d\psi$$

The Dirichlet boundary condition involves the prescription of y on the boundaries of the design plane, whilst the Neumann the prescription of

$\frac{\partial y}{\partial \varphi}$ or $\frac{\partial y}{\partial \psi}$ (depending on which bounding surface is being considered). The technique can easily be extended to cope with the so-called Cauchy and Robin boundary conditions. An analytic treatment of equation (7.4) can be found in Pavlika (4), Cousins (1) and Klier (3).

8. The Numerical Algorithm in the Design Plane

Rewriting the partial differential equation that y satisfies as:

$$\frac{\partial}{\partial \phi}\left(a\frac{\partial y}{\partial \phi}\right)+\frac{\partial}{\partial \psi}\left(b\frac{\partial y}{\partial \psi}\right)=\mathrm{c}$$

where $a \equiv a(y,\phi,\psi)$, $b \equiv b(y,\phi,\psi)$ and $c \equiv c(y,\phi,\psi)$, where a, b and c are function of y, ϕ and ψ. For problems posed in the design plane $c=0$, the a and b will vary depending on whether the flow field is irrotational or swirl free etc. Writing in finite difference form using central differences (with $c\neq 0$) gives:

$$\frac{\partial}{\partial \phi}\left(a\frac{\partial y}{\partial \phi}\right)_{i,j}=\frac{1}{2(\Delta\phi)^2}\begin{bmatrix}(a_{i+1,j}+a_{i,j})y_{i+1,j}-4a_{i,j}y_{i,j}\\+(a_{i-1,j}+a_{i,j})y_{-1i,j}\end{bmatrix}$$

and

$$\frac{\partial}{\partial \psi}\left(b\frac{\partial y}{\partial \psi}\right)_{i,j}=\frac{1}{2(\Delta\psi)^2}\begin{bmatrix}(b_{i,j+1}+b_{i,j})y_{i,j+1}-4b_{i,j}y_{i,j}\\+(b_{i,j-1}+b_{i,j})y_{i,j-1}\end{bmatrix}$$

Thus at the point $(i\Delta\phi, j\Delta\psi)$ (to be denoted by (i,j) from now on in this paper), the equation is represented by a computational molecule as:

$$\begin{aligned}&+N_{i,j}y_{i,j-1}\\ W_{i,j}y_{i-1,j}-C_{i,j}y_{i,j}&+E_{i,j}y_{i+1,j}=R_{i,j} \qquad (8.1)\\ &+S_{i,j}y_{i,j+1}\end{aligned}$$

Where the N, S, E and W and R may be identified as

$$W_{i,j}=(\Delta\psi)^2(a_{i,j}+a_{i-1,j})$$
$$E_{i,j}=(\Delta\psi)^2(a_{i+1,j}+a_{i,j})$$
$$N_{i,j}=(\Delta\varphi)^2(b_{i,j-1}+b_{i,j})$$
$$S_{i,j}=(\Delta\varphi)^2(b_{i,j+1}+b_{i,j})$$
$$C_{i,j}=4((\Delta\varphi)^2a_{i,j}+(\Delta\psi)^2b_{i,j}))$$
$$R_{i,j}=2(\Delta\varphi)^2(\Delta\psi)^2c_{i,j}$$

9. The Difference Equations

Equation (8.1) applies for $i=1$ to M; $j=1$ to N on a uniform mesh as described in Pavlika (4), with special consideration at $j=1$ and $j=N$, so that with Dirichlet boundary conditions, say for $j=N$

$$\begin{aligned}&+N_{i,N}y_{i,N-1}\\ W_{i,N}y_{i-1,N}-C_{i,N}y_{i,N}&+E_{i,N}y_{i+1,N}=R_{i,j}-\\ &\qquad\qquad S_{i,N}y_{i,N+1}\end{aligned}$$

with $y_{i,N+1}$ prescribed as the Dirichlet data for $0\leq i\leq M$. For $j=2$ to $N-1$

$$\begin{aligned}&+N_{i,j}y_{i,j-1}\\ W_{i,j}y_{i-1,j}-C_{i,j}y_{i,j}&+E_{i,j}y_{i+1,j}=R_{i,j}\\ &+S_{i,j}y_{i,j+1}\end{aligned}$$

and for $j=1$

$$\begin{aligned}W_{i,1}y_{i-1,1}-C_{i,1}y_{i,1}+E_{i,1}y_{i+1,1}&=R_{i,1}-N_{i,1}y_{i,0}\\ &+S_{i,1}y_{i,2}\end{aligned}$$

similarly $y_{i,0}$ prescribed as the Dirichlet data for $0\leq i\leq M$..

10. Vector form of the Difference Equations

The above equations can be written more conveniently in matrix-vector form as:

$$\begin{bmatrix}W_{i,1}&0&0&.&.\\0&W_{i,2}&0&&\\&0&W_{i,3}&.&.\\&&&.&.\\&&&&W_{i,N}\end{bmatrix}\begin{bmatrix}y_{i-1,1}\\y_{i-1,2}\\.\\.\\y_{i-1,N}\end{bmatrix}.+$$

$$\begin{bmatrix}-C_{i,1}&S_{i,1}&0&.&.\\N_{i,2}&-C_{i,2}&S_{i,2}&0&.\\&N_{i,3}&-C_{i,3}&.&.\\&&&.&.\\&&&&-C_{i,N}\end{bmatrix}\begin{bmatrix}y_{i,1}\\y_{i,2}\\.\\.\\y_{i,N}\end{bmatrix}.+$$

$$\begin{bmatrix}E_{i,1}&0&0&.&.\\0&E_{i,2}&0&&\\&0&E_{i,3}&.&.\\&&&.&.\\&&&&E_{i,N}\end{bmatrix}\begin{bmatrix}y_{i+1,1}\\y_{i+1,2}\\.\\.\\y_{i+1,N}\end{bmatrix}.=$$

$$\begin{bmatrix}R_{i,1}-N_{i,1}y_{i,0}\\R_{i,2}\\.\\.\\R_{i,N}-S_{i,N}y_{i,N+1}\end{bmatrix}=R^{(i)}, say. \qquad (10.1)$$

11. Direct Solution of the Difference Equations

The matrix-vector equation (equation (10.1)) can be written as

$$W^{(i)}\underline{Y}^{(i-1)} + A^{(i)}\underline{Y}^{(i)} + E^{(i)}\underline{Y}^{(i+1)} = \underline{R}^{(i)} \qquad (11.1)$$

With diagonal matrices $W^{(i)}$ and $E^{(i)}$ and tridiagonal matrix $A^{(i)}$ all of order *(NxN)*, and column vectors $\underline{Y}^{(i)}$ and $\underline{R}^{(i)}$ of order N. To solve the vector recurrence relation a speculation is made that the $\underline{Y}^{(i-1)}$ vector can be related linearly to the $\underline{Y}^{(i)}$ vector as follows:

$$\underline{Y}^{(i-1)} = B^{(i)}\underline{Y}^{(i)} + \underline{K}^{(i)} \qquad (11.2)$$

where the $B^{(i)}$ and the $\underline{K}^{(i)}$ are at present unknown matrices and column vectors respectively. Substituting (11.2) into (11.1) gives

$$(W^{(i)}B^{(i)} + A^{(i)})\underline{Y}^{(i)} = \underline{R}^{(i)} - W^{(i)}\,\underline{K}^{(1)} - E^{(i)}\underline{Y}^{(i+1)}$$
$$\Rightarrow \underline{Y}^{(i)} = -(W^{(i)}B^{(i)} + A^{(i)})^{-1}\,E^{(i)}\underline{Y}^{(i+1)}$$
$$+ (W^{(i)}B^{(i)} + A^{(i)})^{-1}\,\underline{(R}^{(i)} - W^{(i)}\,\underline{K}^{(i)})$$

but

$$\underline{Y}^{(i)} = B^{(i+1)}\,\underline{Y}^{(i+1)} + \underline{K}^{(i+1)}$$

Thus equating coefficients implies

$$B^{(i+1)} = -(W^{(i)}B^{(i)} + A^{(i)})^{-1}\,E^{(i)} \qquad (11.3)$$

and

$$\underline{K}^{(i+1)} = (W^{(i)}B^{(i)} + A^{(i)})^{-1}\,\underline{(R}^{(i)} - W^{(i)}\,\underline{K}^{(i)})$$

For $i=0$ this gives

$$\underline{Y}^{(0)} = B^{(1)}\,\underline{Y}^{(1)} + \underline{K}^{(1)} \qquad (11.4)$$

To determine the $\underline{K}^{(1)}$, if the first iterate $B^{(1)} = \underline{0}$ then $\underline{K}^{(1)} = \underline{Y}^{(0)}$

The matrix and vector sequences are now defined by equations (11.3) and (11.4) for $i=1$ *to* M. The $\underline{Y}^{(i)}$ vectors are now calculated starting from right to left (as $\underline{Y}^{(M+1)}$ is known) using

$$\underline{Y}^{(M)} = B^{(M+1)}\,\underline{Y}^{(M+1)} + \underline{K}^{(M+1)}$$

The diagonal matrices $W^{(i)}$ and $E^{(i)}$ have elements

$W^{(i)} = W_{i,j}$ and $E^{(i)} = E_{i,j}$

The tridiagonal matrix A has entries

$A_{j,j} = -C_{i,j} \quad j = 1$ *to* N

$A_{j,j+1} = S_{i,j}\ , \quad A_{j,+1,j} = N_{i,j+1},\ j = 1$ *to* $N\text{-}1$

12. The Blockage Effect: Deriving the Additional Flow Equation due to Circumferentially arranged Aerofoils

In deriving the additional flow equation the effect of the circumferentially arranged blades placed in the duct must be considered. The blades effect the rate of mass flow η, considering figure 12.1, with $k\equiv k(x,y)$ representing the blockage effect, the mass flow into and out of the fluid element is:

Face A:

$$\left(2u_y + \frac{\partial u_y}{\partial x}\delta x\right)\delta x\left(2k + \frac{\partial k}{\partial x}\delta x\right) + O((\delta x)^2)$$

Face B:

$$-\left(2u_x + 2\frac{\partial u_x}{\partial x}\delta x + \frac{\partial u_x}{\partial y}\delta y\right)*$$
$$\delta y\left(2k + 2\frac{\partial k}{\partial x}\delta x + \frac{\partial k}{\partial y}\delta y\right)$$
$$+O((\delta x)^2) + O((\delta y)^2)$$

Face C:

$$-\left(2u_y + \frac{\partial u_y}{\partial x}\delta x + 2\frac{\partial u_y}{\partial y}\delta y\right)*$$
$$\delta x\left(2k + 2\frac{\partial k}{\partial y}\delta y + \frac{\partial k}{\partial x}\delta x\right)$$
$$+O((\delta x)^2) + O((\delta y)^2)$$

Face D:

$$\left(2u_x + \frac{\partial u_x}{\partial y}\delta y\right)\delta y\left(2k + \frac{\partial k}{\partial y}\delta y\right)$$
$$+O((\delta y)^2)$$

summing these terms, using the principle of conservation of mass and taking the limit as $\delta x \to 0, \delta y \to 0$ gives:

$$\lim_{\substack{\delta x \to 0 \\ \delta y \to 0}}\left(\frac{massflow}{4\delta x\delta y}\right) = \frac{\partial(ku_x)}{\partial x} + \frac{\partial(ku_y)}{\partial y} = 0$$

which may be identified as the continuity equation for two dimensional compressible flow with the density term being replaced by the blockage factor k. This will be given in cylindrical coordinates in section 16.

13. The Blockage Function *k(x,y)*

The blockage function $k(x,y)$ is defined to be of the form

$$k(x,y) = 1 - \frac{\lambda(x)}{T(y)}$$

where the function $\lambda(x)$ represents the contour shape of the aerofoil and the term $T(y)$ is a scaling factor given be $T(y) = \frac{2\pi}{N}y$, where N is the number of blades (arbitrary). If the axial span of the aerofoil is x_1 then the function $\lambda(x)$ is defined to have a maximum at $x_1/5$ and $\lambda(x_1/5)=x_1/10$. Furthermore $\lambda(x)$ is chosen to vanish at $x=0$ and $x=x_1$. Choosing $\lambda(x)$ to be of the form $\lambda(x)=cx^{\alpha}(x^{\beta}-x_1^{\beta})$, where c is a constant.

with $\beta = 1(arbitrary) \Rightarrow \alpha = \frac{1}{4}$, applying the conditions mentioned above gives

$$c=-\frac{1}{8}\left(\frac{5}{x_1}\right)^{1/4}$$

hence $\lambda(x)=-\frac{1}{8}\left(\frac{5x}{x_1}\right)^{1/4}(x-x_1)$.

14. The Boundary Conditions

Initially the Neumann boundary condition will be analysed. In this case the vector of unknown y values is extended to include the $j=0$ row (for the top boundary) and $j=N+1$ for the bottom boundary, (as shown in Pavlika (5)). The difference scheme is now applied over this extended set i.e. the scheme is centered on the point j=0, (and $j=N+1$ for the bottom boundary). Considering for the moment only having a Neumann condition on the top boundary, the centering the scheme on $j=0$ will involve the value of y at $j=-1$, this term is expressed in terms of the value of y at $j=1$ using the known normal derivative, such that:

$$\frac{y_{i,-1}-y_{i,1}}{2\Delta\varphi} \approx \left(\frac{\partial y}{\partial \varphi}\right)_{i,0} = known \text{ exp}ression$$

so at the mesh point $(i,0)$ $(i=1,2,\ldots M)$ the finite difference scheme gives

$$W_{i,0}y_{i-1,0}-C_{i,0}y_{i,0}+E_{i,0}y_{i+1,0}=R_{i,0}-N_{i,0}y_{i,-1} \\ +S_{i,0}y_{i,1}$$

Applying the boundary condition gives

$$W_{i,0}y_{i-1,0}-C_{i,0}y_{i,0}+E_{i,0}y_{i+1,0} \\ +(S_{i,0}+N_{i,0})y_{i,1} \\ =R_{i,0}-2\Delta\psi N_{i,0}\left(\frac{\partial y}{\partial \psi}\right)_{i,0}$$

Using

$$\left(\frac{\partial y}{\partial \psi}\right)_{i,0}=\sqrt{A_{i,0}^2\left(\frac{1}{q_{i,0}^2}-\frac{1}{B_{i,0}^2}\left(\frac{\partial y}{\partial \varphi}\right)_{i,0}^2\right)}$$

the normal derivative is now known in terms of the prescribed speed which in this case is along the top boundary. The matrix-vector equations become

$$\begin{bmatrix} W_{i,0} & 0 & 0 & . & . \\ 0 & W_{i,1} & 0 & & \\ & 0 & W_{i,2} & 0 & . \\ & & & . & . \\ & & & & W_{i,N} \end{bmatrix}\begin{bmatrix} y_{i-1,0} \\ y_{i-1,1} \\ . \\ . \\ y_{i-1,N} \end{bmatrix}.+$$

$$\begin{bmatrix} -C_{i,0} & (S_{i,0}+N_{i,0}) & 0 & . & . \\ N_{i,1} & -C_{i,1} & S_{i,1} & 0 & . \\ & N_{i,2} & -C_{i,2} & . & . \\ & & & . & . \\ & & & & -C_{i,N} \end{bmatrix}\begin{bmatrix} y_{i,0} \\ y_{i,1} \\ . \\ . \\ y_{i,N} \end{bmatrix}.$$

$$+$$

$$\begin{bmatrix} E_{i,0} & 0 & 0 & . & . \\ 0 & E_{i,1} & 0 & & \\ & 0 & E_{i,2} & . & . \\ & & & . & . \\ & & & & E_{i,N} \end{bmatrix}\begin{bmatrix} y_{i+1,0} \\ y_{i+1,1} \\ . \\ . \\ y_{i+1,N} \end{bmatrix}.=$$

$$\begin{bmatrix} R_{i,0}-2\Delta\psi N_{i,0}\left(\frac{\partial y}{\partial \psi}\right)_{i,0} \\ R_{i,1} \\ . \\ . \\ R_{i,N}-S_{i,N}y_{i,N+1} \end{bmatrix}$$

Similar analysis can be performed if the bottom boundary is to have a Neumann boundary condition as described in Pavlika (4). The technique can also be applied to the case of Robin boundary conditions.

15. Axisymmetric Flow in the Absence of Body Forces

Here numerical solutions to inviscid irrotational flow with a free vortex swirl velocity profile are derived. The axial velocity component $u_x(y)$ at inlet will be chosen to be of the form $u_x(y)=\alpha$, where α is a constant and the swirl velocity $u_\alpha(y)$, will be of the form $u_\alpha(y)=\frac{l}{y}$ where the 1 is a constant representing the so-called free vortex term. For the case when body forces exist, for example when the effect of the blades are being considered the force is resolved into a component perpendicular do the flow direction, modeling the guiding action of the blades and into a component parallel to the flow direction, modeling viscous effects. These two cases are discussed in Pavlika (5).

16. The general flow equations in the physical plane(y, α, x).

Here equations are given for the general case of axisymmetric, inviscid and rotational flow in the absence of body forces so that the algorithm can accommodate a larger class of flow problems as shown in Pavlika (5). Adopting cylindrical polar coordinates with y being the radial coordinate, α the circumferential and x the axial coordinate, defining velocity components u_y , u_α and u_x with corresponding vorticity components ω_y , ω_α, ω_x in the direction of increasing y, α and x respectively, then the equation of motion with unit density becomes:

$$\frac{D\underline{u}}{Dt} = -\,\underline{\nabla}.\underline{p} \qquad (16.1)$$

Where $\frac{D}{Dt}$ is the material derivative. Equation (16.1) can be written using well known vector identities as:

$$\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} - \frac{u_\alpha^2}{y} = -\frac{\partial p}{\partial y}$$
$$\frac{\partial u_\alpha}{\partial t} + u_x\frac{\partial u_\alpha}{\partial x} + u_y\frac{\partial u_\alpha}{\partial y} - \frac{u_\alpha u_y}{y} = 0$$
$$\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} = -\frac{\partial p}{\partial x} \qquad (16.2)$$

Furthermore

$$\frac{\partial \underline{u}}{\partial t} + (\underline{u}.\underline{\nabla})\underline{u} = -\,\underline{\nabla}.\underline{p}$$

can be written (once again using an appropriate vector identity as)

$\frac{\partial \underline{u}}{\partial t} + (\underline{\omega}\wedge\underline{u}) = -\underline{\nabla}(\underline{p} + \frac{1}{2}q^2)$. Thus for steady flow Crocco's form of the equation of motion is obtained, i.e.

$$(\underline{u}\wedge\underline{\omega}) = \underline{\nabla}H \qquad (16.3)$$

where H is the total head defined by $H = p + \frac{1}{2}q^2$.

Calculating the cross product on the left hand side of equation 16.3, gives

$$\frac{\partial H}{\partial y} = u_\alpha\omega_x - u_x\omega_\alpha$$
$$0 = u_x\omega_y - u_y\omega_x$$
$$\frac{\partial H}{\partial x} = u_y\omega_\alpha - u_\alpha\omega_x$$

(16.4)

In addition for axisymmetric flow the vorticity vector $\underline{\omega}$ becomes

$$\underline{\omega} = \underline{\nabla}\wedge\underline{u} = \left\{-\frac{\partial u_\alpha}{\partial x}\right\}\underline{y} + \left\{\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right\}\underline{\alpha} + \left\{\frac{1}{y}\frac{\partial(yu_\alpha)}{\partial y}\right\}\underline{x} \qquad (16.5)$$

The equation of continuity becomes

$$\underline{\nabla}.\underline{u} = \frac{\partial(yu_x)}{\partial x} + \frac{\partial(yu_y)}{\partial y} = 0$$

17. The Design Plane counterparts

In order to compute numerical solutions in the design plane, expressions are required for the terms A, B and ω_α, thus

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = -\frac{1}{y}\left(u_x\frac{\partial y}{\partial x} + u_y\right)$$
$$= -q\frac{\partial}{\partial s}(\log(y))$$

or

$$\eta = -\frac{q^2}{B}\frac{\partial}{\partial\varphi}(\log(y)),$$

but

$$\eta = \frac{q^2}{B}\frac{\partial}{\partial\varphi}(\log(A))$$

thus $Ay = f(\psi)$, that is $\frac{\partial\psi}{\partial n} = \frac{yq}{f(\psi)}$. The arbitrary function $f(\psi)$ represents the freedom in the cross stream distribution of ψ and choosing $f(\psi)$ to be unity everywhere ψ can be identified as the usual Stokes stream function given by

$$\frac{\partial\psi}{\partial x} = -yu_y;\ \frac{\partial\psi}{\partial y} = yu_x$$

Equation (16.5), (circumferential component) gives

$$0 = u_x\frac{\partial(yu_\alpha)}{\partial x} + u_y\frac{\partial(yu_\alpha)}{\partial y}$$

Referring to the meridional plane figure 17.1, it may be deduced that

$$u_x = q\frac{\partial x}{\partial s}; u_y = q\frac{\partial y}{\partial s}$$

$$\Rightarrow \frac{\partial}{\partial s}(yu_\alpha) = 0$$

$$\therefore yu_\alpha = C(\psi)$$

where $q = \frac{ds}{dt}$. In terms of $C(\psi)$ the vorticity vector (equation (16.5)) becomes

$$\underline{\omega} = \underline{\nabla} \wedge \underline{u} = \left\{-\frac{1}{y}\frac{\partial C}{\partial x}\right\}\underline{y} + \left\{\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right\}\underline{\alpha} + \left\{\frac{1}{y}\frac{\partial C}{\partial y}\right\}\underline{x}$$

$= \omega_y\underline{y} + \omega_\alpha\underline{\alpha} + \omega_x\underline{x}$, by definition.

An expression for ω_α is required as this appears in the expression for B, so using the radial component of equation (16.4) gives

$$\omega_\alpha = \frac{u_x}{u_\alpha}\left(\frac{1}{y}\frac{\partial C}{\partial y}\right) - \frac{1}{u_x}\frac{\partial H}{\partial y}$$

using the Stokes' stream function this becomes

$$\omega_\alpha = \frac{C(\psi)}{y}\left(\frac{dC}{d\psi}\right) - y\frac{dH}{d\psi}$$

which is the required expression to be used in calculation of B according to definition (6.2). If far upstream the flow is assumed to be cylindrical so that all quantities are independent of x, then with unit density the equation of motion and the Stokes' Stream function give:

$$u_y = 0; \frac{\partial p}{\partial x} = 0; \frac{\partial p}{\partial y} = \frac{u_\alpha^2}{y}; \frac{\partial \psi}{\partial x} = 0; \frac{\partial \psi}{\partial y} = yu_x$$

giving

$$\omega_\alpha = \frac{C(\psi)}{y}\left(\frac{dC}{d\psi}\right) - \frac{y}{2}\frac{d}{d\psi}(u_x^2 + u_\alpha^2) - \frac{u_\alpha^2}{u_x y}$$

With $u_x(y) = \alpha$ and $u_\alpha(y) = \frac{l}{y}$ as previously defined. Once $\frac{dH}{d\psi}$ has been calculated upstream it takes this value throughout the (ϕ, ψ) since as is self evident the expression is independent of ϕ. This last expression for ω_α is required in the calculation of B and numerical coupling with equation (7.4) gives the numerical solution in the design plane.

18. Downstream Conditions

Downstream a cylindrical flow condition as discussed below will be prescribed. Defining the pressure function $H(\psi)$ and the function $C(\psi)$ as

$$H(\psi) = \frac{1}{2}(u_x^2 + u_\alpha^2) + \frac{p}{\rho} \text{ and } C(\psi) = yu_\alpha$$

for cylindrical flow radial equilibrium (from equation (16.2) radial component gives

$$\frac{1}{\rho}\frac{dp}{dy} = \frac{u_\alpha^2}{y}$$

Integrating gives

$$\frac{1}{\rho}(p - p_{y-inner}) = \int_{y-inner} \frac{u_\alpha^2}{y}dy = \int_{y-inner} \frac{C^2(\psi)}{y^3}dy$$

Which gives $H(\psi)$ as

$$H(\psi) = \frac{1}{2}(u_x^2 + u_\alpha^2) + \frac{p_{y-inner}}{\rho} + \int_{y-inner} \frac{C^2(\psi)}{y^3}dy$$

Now $\int_{y-inner} \frac{C^2(\psi)}{y^3}dy = -\frac{1}{2}\int_{y-inner} C^2 d(1/y^2)$

$$= -\frac{1}{2}\left[\frac{C^2}{y^2} - \left(\frac{C^2}{y^2}\right)_{y-inner}\right] + \frac{1}{2}\int_{y-inner} \frac{1}{y^2}\frac{dC^2}{dy}dy$$

Therefore

$$H(\psi) = \frac{1}{2}u_x^2 + \frac{p_{y-inner}}{\rho} + \frac{1}{2}(u_\alpha^2)_{y-inner} + \int_{\psi=0} \frac{1}{y^2}\frac{dC^2}{d\psi}d\psi$$

Suppose $u_{x,1} = u_{x,1}(\psi)$ and $u_{\alpha,1} = u_{\alpha,1}(\psi)$, where the subscript 1 denotes upstream conditions, then $u_{x,2} = u_{x,2}(\psi)$ and $u_{\alpha,2} = u_{\alpha,2}(\psi)$ are required as functions of ψ, where the subscript 2 similarly denoting downstream conditions, so that

$$\frac{1}{2}u_{x,2}^2 = H(\psi) - \frac{p_{2,inner}}{\rho} - \frac{1}{2}(u_{\alpha,2}^2)_{inner} - \frac{1}{2}\int_{\psi=0} \frac{1}{y_1^2}\frac{dC^2}{d\psi}d\psi \tag{18.1}$$

and $\int_{\psi=0} \frac{d\psi}{u_{x,2}}d\psi = \frac{1}{2}\left(y_2^2 - y_{2,inner}^2\right)$

Furthermore $C(\psi) = y_1u_{\alpha,1} = y_2u_{\alpha,2}$, and equation (18.1) now gives

$$\frac{1}{2}u^2_{x,2} = \frac{1}{2}u^2_{x,1} + \frac{p_{1,inner}}{\rho} - \frac{p_{2,inner}}{\rho} +$$
$$\frac{1}{2}((u^2_{\alpha,1})_{inner} - (u^2_{\alpha,2})_{inner})$$
$$+\frac{1}{2}\int_{\psi=0}\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right)d(C^2)$$

or

$$u^2_{x,2} = u^2_{x,1} + K + \int_{\psi=0}\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right)d(C^2) \quad (18.2)$$

where

$$K = 2\left(\frac{p_{1,inner}}{\rho} - \frac{p_{2,inner}}{\rho}\right) + (u^2_{\alpha,1})_{inner} - (u^2_{\alpha,2})_{inner}$$

and $y_2^2 = y^2_{2,inner} + 2\int_{\psi=0}\frac{d\psi}{u_{x,2}}$ (18.3)

with $u_{x,2}$ in this case given by (18.2).

19. Calculation procedure

The calculation of the downstream radii $y_2(\psi)$ follow from equation (18.3) with $u_{x,2}$ given by equation (18.2), which can be written as

$u^2_{x,2} = g(\psi) + K$, where (19.1)

$$g(\psi) = u^2_{x,1} + \int_{\psi=0}\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right)\frac{d(C^2)}{d\psi}d\psi$$

In order to calculate the $(n+1)^{th}$ iterate it is known that:

$$\frac{\partial}{\partial K}(y^2_{2,outer}) = 2\int_{\psi=0}\frac{\partial}{\partial K}\left(\frac{d\psi}{\sqrt{g(\psi)+K}}\right)$$
$$= -\int_{\psi=0}^{\Psi}\frac{d\psi}{\left(u^3_{x,2}\right)^{(n)}}$$

but

$$\left(\frac{\partial}{\partial K}(y^2_{2,outer})\right)^{(n)} = \frac{\left(y^2_{2,outer}\right)^{(n+1)} - \left(y^2_{2,outer}\right)^{(n)}}{K^{(n+1)} - K^{(n)}}$$

(19.2)

from which as can be seen from equation (19.2) the $K^{(n)}$ must be calculated iteratively with $K^{(0)}=0$, Once the $K^{(n+1)}$ has been calculated it is introduced into equation (19.1), giving rise to a new $(u^2_{x,2})^{(n+1)}$ which in turn gives a new $(y^2_{x,2})^{(n+1)}$ from equation (18.3) and the process repeated until some convergence criteria is satisfied.

20. Prescription of the speed distribution, cubic in arclength, s along the duct

In this paper the Neumann boundary condition will be prescribed on the top wall boundary so that it is the speed q that is given as a function of arclength along the top boundary. The function chosen to give a the q distribution is chosen to be cubic given by the following piecewise continuous function

$q(s) = q_u$, for $s < s_1$

$q(s) = as^3 + bs^2 + cs + d$, for $s_1 < s < s_2$

$q(s) = q_d$, for $s > s_2$

where the constants a, b, c and d are determined such that the cubic $q(s)$ distribution satisfies the following conditions:

i) $q(s) = q_u$ for $s=s_1$

ii) $q(s) = q_d$ for $s=s_2$

iii) $\frac{dq}{ds} = \beta(\neq 0)$, at $s=s_3$, where

$s_3 \in (s_1, s_2)$ and $\beta = \varepsilon\left(\frac{q_d - q_u}{s_2 - s_1}\right)$, with

$\varepsilon \neq 0, s_1 \neq s_2$.

Choosing $s_3 = \frac{1}{2}(s_1 + s_2)$ for example gives a symmetrical speed distribution, ε is an arbitrary scaling multiplier

iv) $\frac{d^2q}{ds^2} = 0$ at $s=s_3$

application of these conditions with ε, q_u, q_d, s_1 and s_2 known gives:

$$a = \frac{q_u - q_d - \beta(s_1 - s_2)}{s_1^3 - s_2^3 - 3s_3(s_1^2 - s_2^2) + 3s_3^2(s_1 - s_2)}$$

knowing a the values of b, c and d follow by back substitution, whence $b=-3as_3$, $c= 3a\,s_3^2\beta$ and

$d = q_u - as_1^3 - bs_1^2 - cs_1$

The inner radius is prescribed using this piecewise continuous function giving rise to a radius distribution as a function of the axial coordinate.

21. Conclusions

As shown, geometries have been produced subject to given upstream and downstream conditions with prescribed Neumann boundary conditions. In this case the flow is chosen to be irrotational by defining the inlet axial velocity profile to be of the form $u_x(y) = \alpha$ where α is a constant and the swirl velocity profile chosen to be of the form $u_\alpha(y) = \frac{l}{y}$, where l is a constant, defining the so-called free vortex whirl. The downstream conditions are defined such that cylindrical flow is present, even though it was the speed that was prescribed the algorithm can accommodate the case when Dirichlet conditions are prescribed. The number of blades has been varied and the geometries produced are shown in figures 21.1, 21.2 and 21.3 respectively. Further examples of the algorithm with a combination of boundary condition are given in Pavlika (5). It was found that at most four iterations were required to achieve an acceptable level of convergence, with the technique accelerated using Aitken's Method.

22. References

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[5] Pavlika, V., *Vector Field Methods and the Hydrodynamic Design of Annular Ducts,* Ph.D thesis, University of North London, Chapter VIII, 1995.

23. Figures

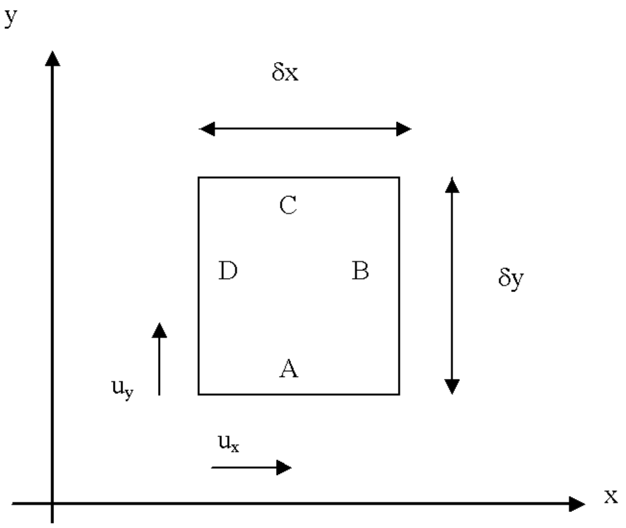

Fig 12.1. A Fluid Element

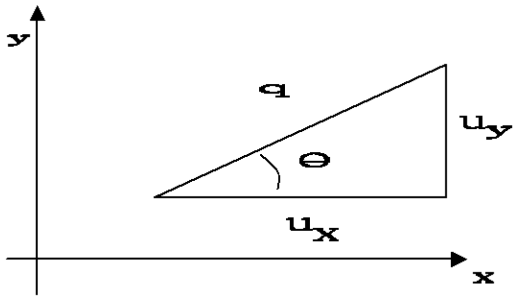

Fig 17.1. The meridional plane.

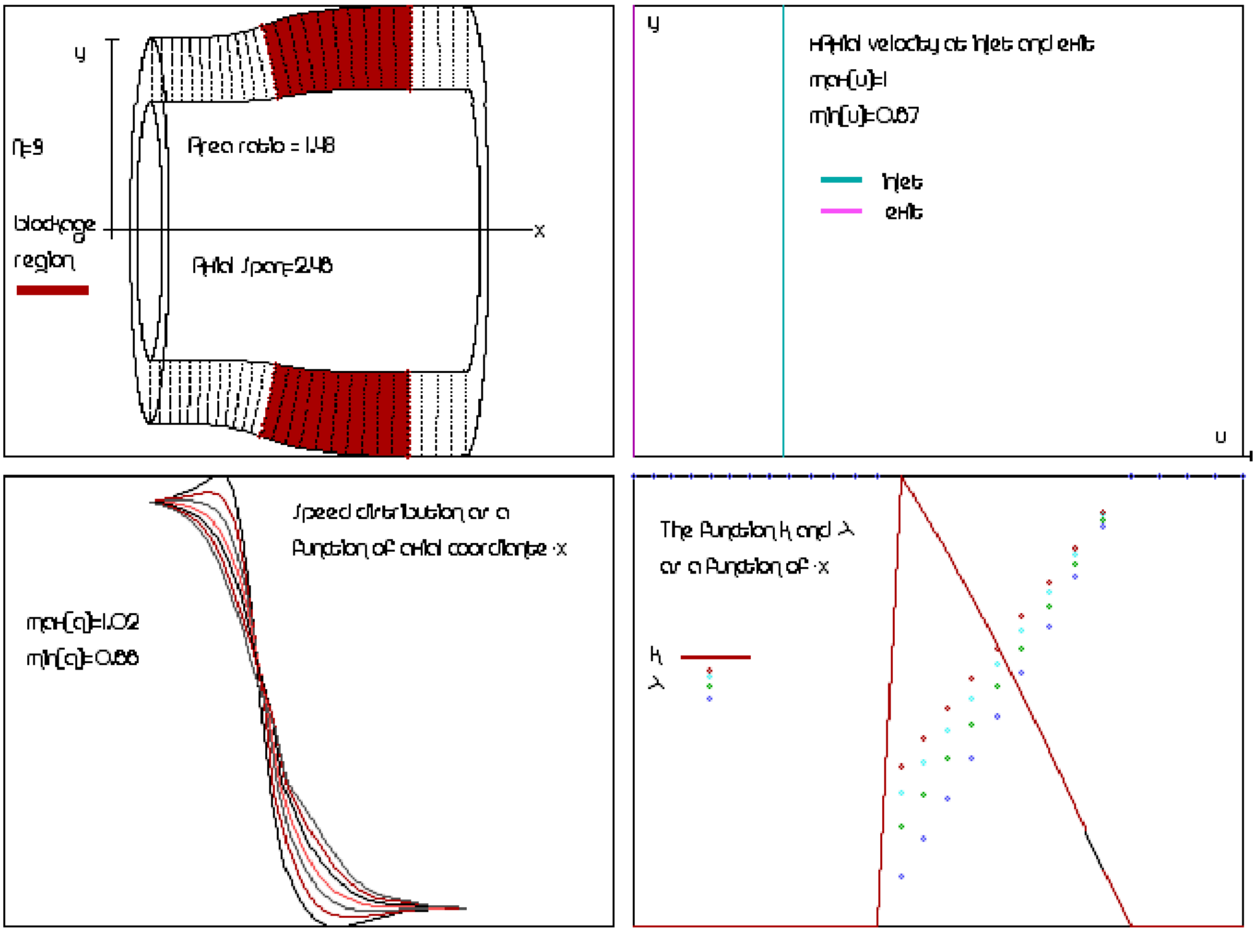

Fig 21.1. The geometry and speed distribution produced with $N=3$.

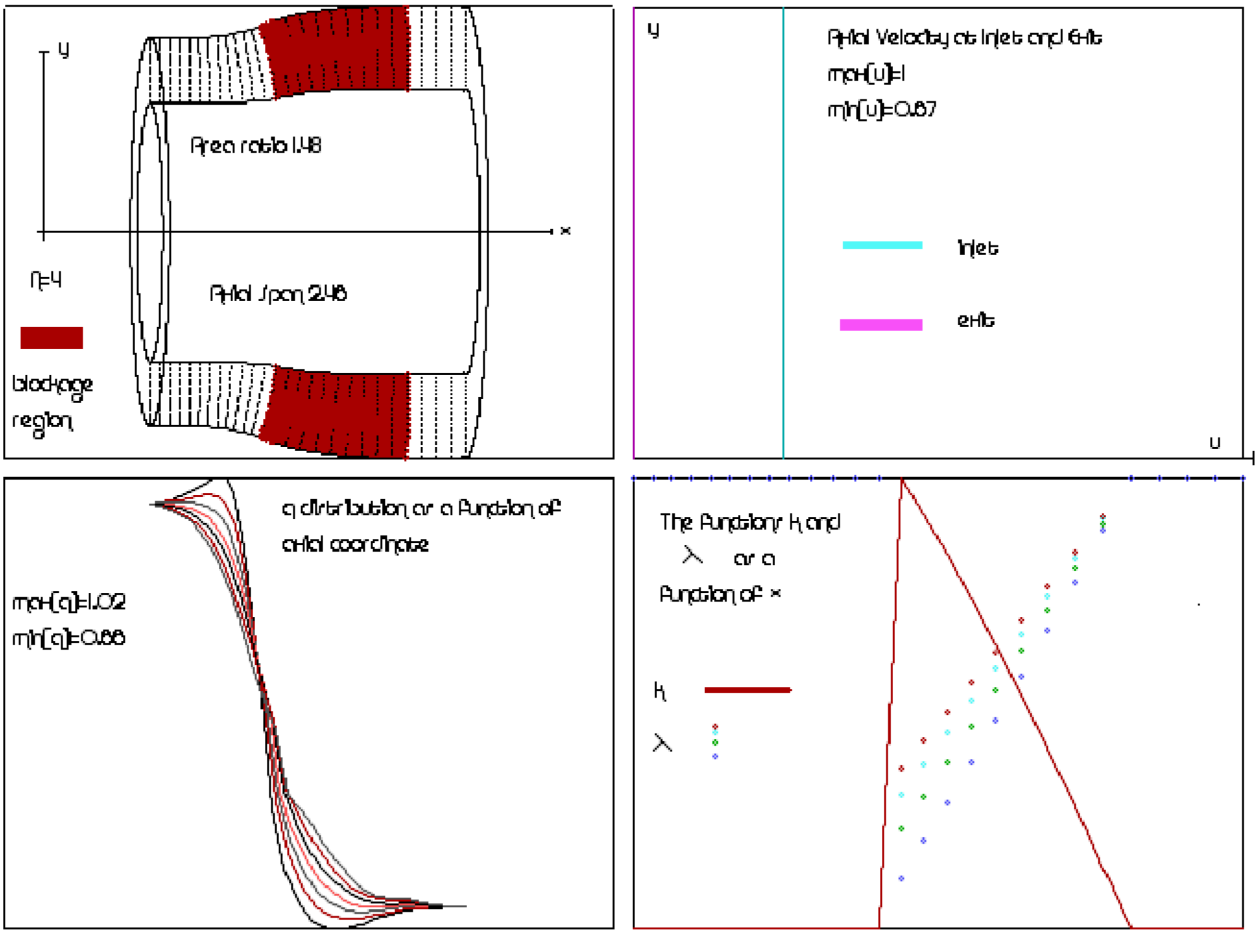

Fig 21.2. The geometry and speed distribution produced with $N=4$.

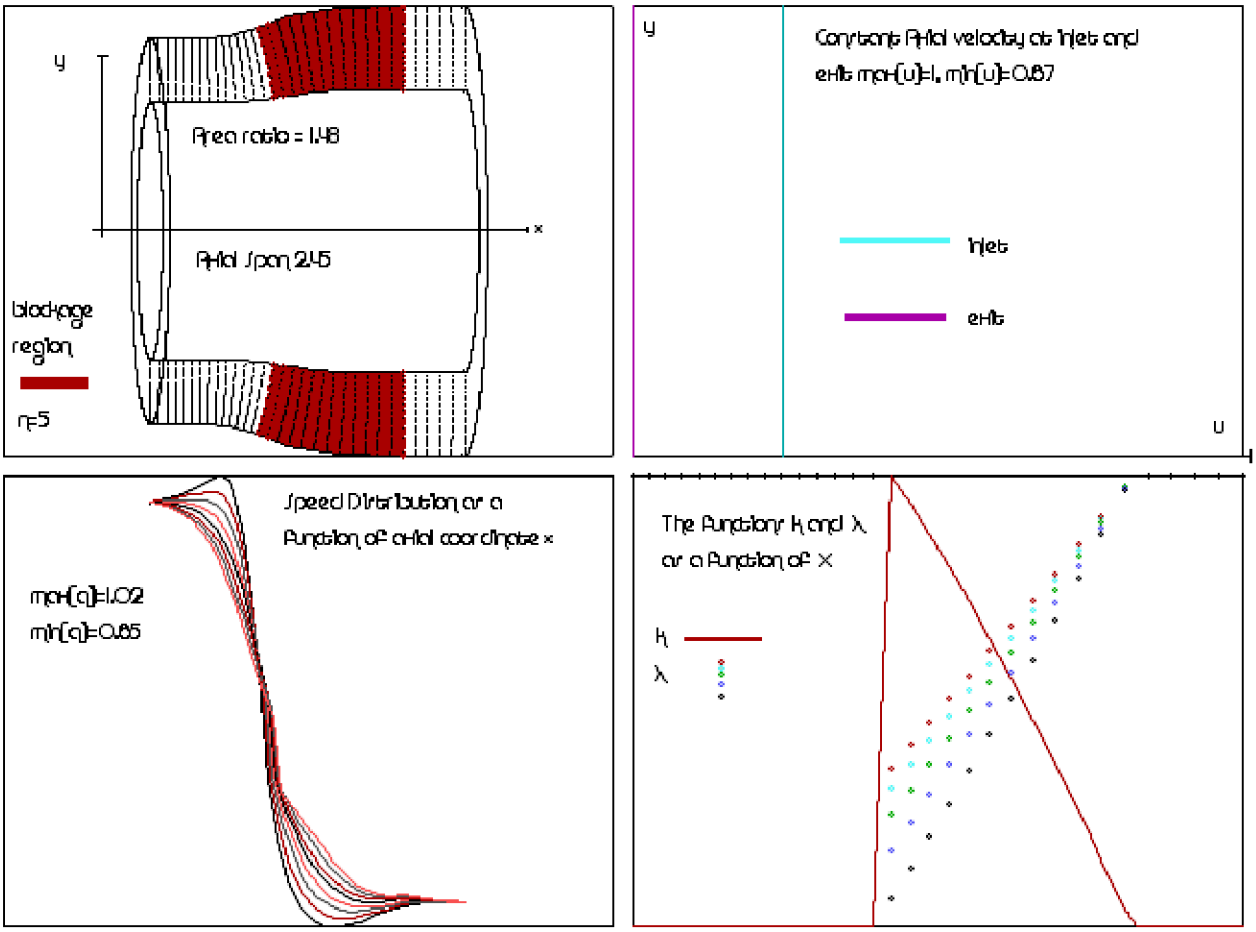

Fig 21.3. The geometry and speed distribution produced with $N=5$.