Introduction of a Mathematical Storage Function Model Based on Lumping Process of Infiltration Theory

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Abstract: Estimation and water prediction of the infiltration of rainfall has been always an important task for hydrologists. Richard equation is adopted as a fundamental equation to develop a storage function model. At first stage, a new boundary condition has been introduced and its accuracy has been cross checked with experiments. In the second stage, on the lumping process of non-dimensional form of Richard equation, a compensation factor has been proposed to equalize the semi-lumped equations. Finally the relation between storage and discharge has been achieved by fully lumped equation.

Key-words: Richard equation, lumping of unsaturated flow equation, semi lumped equations, storage function model

1 Introduction:

A huge volume of yearly precipitation infiltrates in to the ground surface of various catchment's areas in the world and produces extensive water resources under the ground surface. Suitable estimation and prediction of ground water can be useful to manage the domestic, agricultural and industrial demands in different areas.
Mathematical model is one of the important tolls to provide valuable information about ground water at minimum time and cost. Basically in the nature, infiltration of water into the soil depends on the properties of soil such as hydraulic conductivity, water retention, porosity, soil layer thickness, slope length, slope depth and also rainfall density. Of course, there are many other parameters in the field which are difficult to obtain. On the other hand, the computation based on Richards equation takes much time. Therefore, to decrease the computation time, a storage function model needs to be derived. Fujita,M.(1981) and Takasao,T.(1985) derived a storage function model using kinematics wave theory and Matsubayashi,U.(1994) didn't explain the lumping process in details. One of the aims of this paper is a detail elaboration of lumping process using Richard's equation.

2 Research Pattern:

Infiltration of rain into the soil can be expressed by Richard's equation in its two dimensional unsaturated flow form as the following:

$$\frac{\partial \theta}{\partial t} = -\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y} \qquad (1)$$

Where (θ) is the moisture content in the soil,(t) is time, and (V_x) and (V_y) are fluxes in (x) and (y) directions. Equation (1) applied to a soil column and its schematic condition is shown in figure (1). Where (l) is length, (d) is depth and (α) is slope length of soil column. A rainfall density as (r) is applied to the surface of soil column. The fluxes in (x) and (y) directions can be derived by Darcy's equation.

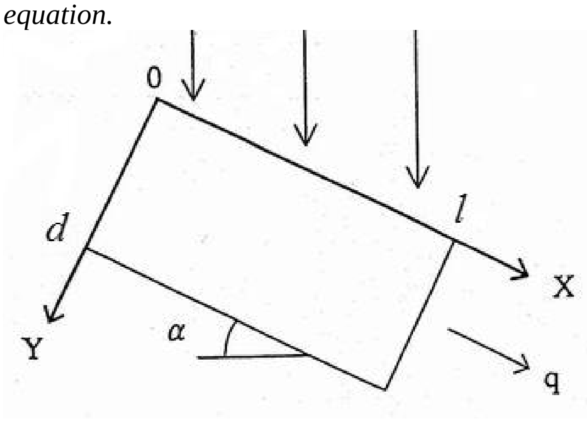

Figure 1: profile of soil column

$$V_x = -k\frac{\partial \Phi}{\partial x} \quad (2)$$

$$V_y = -k\frac{\partial \Phi}{\partial y} \quad (3)$$

(Φ) is water potential in the soil and (k) is hydraulic conductivity of soil which can be expressed as:

$$k = k_s\left(\frac{\theta-\theta_r}{\theta_s-\theta_r}\right)^{\beta} \quad (4)$$

Where (k_s) is saturated conductivity of soil, (β) depends on soil property, (θ_s) and (θ_r) are maximum (saturated) and minimum water content of the soil. The equation of water content can be derived by Haverkamp equation.

$$\theta = \frac{a^2}{a^2+\Psi^2}(\theta_s-\theta_r)+\theta_r \quad (5)$$

(Ψ) is suction in the soil and (a) is depends on soil property. Equations (2) and (3) can be rewritten as follows:

$$V_x = k_s\left(\frac{a^2}{a^2+\Psi^2}\right)^{\beta}\left(Sin\alpha-\frac{\partial \Psi}{\partial x}\right) \quad (6)$$

$$Vy = k_s\left(\frac{a^2}{a^2+\Psi^2}\right)^{\beta}\left(Cos\alpha-\frac{\partial \Psi}{\partial y}\right) \quad (7)$$

The solution of Richards equation has been considered by the following boundary conditions:

$$y = 0 \quad \text{and} \quad V_y = r.Cos\alpha \quad (8)$$

$$y = d \quad \text{and} \quad V_y = 0 \quad (9)$$

$$x = 0 \quad \text{and} \quad V_x = 0 \quad (10)$$

At ($X = l$) where discharge of stored water in soil porous media appears due to the slope length of soil column, different boundary conditions have been proposed by researchers. For example:

$$\frac{\partial \Psi}{\partial x} = 0 \quad (11)$$

$$\frac{\partial^2 \Psi}{\partial x^2} = 0 \quad (12)$$

These equations are physically explained that the suction Ψ, doesn't change abruptly near the boundary and therefore they have an effect to depress the discharge. Matsubayashi,u.(1994) increased the horizontal hydraulic conductivity to avoid such a phenomena. In the present study, we introduce a boundary condition (equation (13)) and its validity has been cross checked with experiments as shown in figure (3), (4) and (5).

$$x = l \quad \text{and} \quad \frac{\partial V_x}{\partial x} = 0 \quad (13)$$

An experimental rectangular soil column (500×30×40 cm) with artificial rain on the top installed in sand with 21 tensiometers as it is shown in figure (2). Water flow potential through soil porous media and soil suction are measured by tensiometers at different depth and distance in soil column. As changing of water flow potential is more abruptly at outlet of soil column ($x = l$) therefore most of the tensiometers concentrated in such a place. Two cases of slope length (0.1 and 0.2 rdn) are considered for soil column which are almost similar to slope length of catchments areas in nature. By separate soil mechanic experiments on sand following properties are achieved ($\theta_r = 0.05$, $\theta_s = 0.4$, $\beta = 2$, $a = 65cm$, $k_s = 0.009\frac{Cm}{Sec}$).

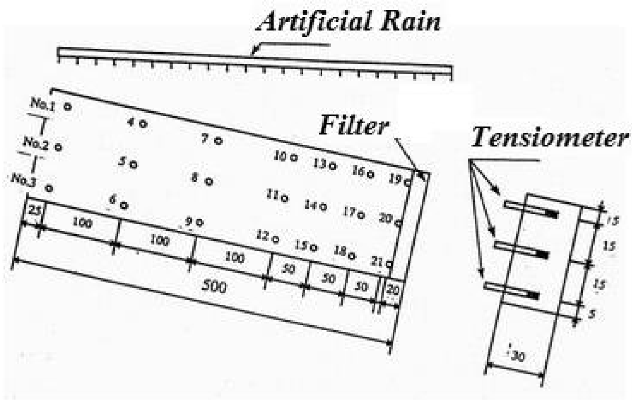

Figure 2: profile of experimental soil

Using experimental conditions and boundary condition in equation (13), the Richards equation has been solved and the results are shown in figure (3). Further the ($\bar{s}-\bar{q}$) relation and storage is computed in two ways. The relation ($\bar{s}_1-\bar{q}$) is derived according to the equation (14) and the results is shown in figure (4). Besides, the relation ($\bar{s}_2-\bar{q}$) is derived by using equation (15) and the results is shown in figure (5). Both relations show good accuracy with experimental results ($\bar{S}_0$ is the initial storage)

$$\bar{S}_1 = \int_0^t r dt - \int_0^t \bar{q} dt \quad (14)$$

$$\bar{S}_2 = \int_0^l \int_0^D \theta(x,y,t) dy dx - S_0 \quad (15)$$

Non dimensional form of the basic equation. Physical parameters of basic equations can be reduced by converting it into the non dimensional form. For this purpose the following relationships have been considered:

$V_x = V_0 \nabla_x$, $y = y_0 \nabla Y$, $x = x_0 x$, $\Psi = \Psi_0 \Phi$, $t = t_0 T$

$\bar{S} = S_0 S$, $\tilde{q} = q_0 Q$, $\theta = \theta_0 \theta$, $r = r_0 R$, $V_y = V_0 \nabla_y$ (16)

Following assumptions have been made for non dimensionalizing process.

$\theta_0 = \theta_s - \theta_r$, $r_0 = k_s$, $V_0 = k_s$, $y_0 = d$, $x_0 = l$

$$S_0 = \theta_0 dl \ , q_0 = k_s d \ , \Psi_0 = d \ , t_0 = \frac{d\theta_0}{k_s} \quad (17)$$

The non dimensional form of equations (1),(6),(7),(4) and (5) can be written as the following:

$$\frac{\partial \theta}{\partial t} = -(\frac{d}{l})\frac{\partial \nabla_x}{\partial x} - \frac{\partial \nabla_y}{\partial y} \quad (18)$$

$$\frac{a}{d} = A \quad (19)$$

$$\nabla_x = \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} (Sin\alpha - \frac{d}{l}\frac{\partial \Phi}{\partial x}) \quad (20)$$

$$\nabla_y = \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} (Cos\alpha - \frac{\partial \Phi}{\partial y}) \quad (21)$$

$$k = \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} \quad (22)$$

$$\theta = \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} + \frac{\theta_r}{\theta_s - \theta_r} \quad (23)$$

The non dimensional initial and boundary conditions can be written as:

$$\Phi = \frac{l}{d}(x-1)Sin\alpha + (y-1)Cos\alpha + \frac{c}{d} \quad (24)$$

$$\nabla_x = 0 , \ x = 0 \quad (25)$$

$$\frac{\partial \nabla_x}{\partial x} = 0 , \ x = 1 \quad (26)$$

$$\nabla_y = RCos\alpha , \ y = 0 \quad (27)$$

$$\nabla_y = 0 , \ y = 1 \quad (28)$$

The initial condition in equation (24) means initially, water is available in the soil but there is no discharge. This initial condition has been satisfied the experiments initial condition.

3 Semi Lumped Unsaturated Flow Equation:

By integration of the equations (18), (20), (25) and (26) along y we can obtain semi lumped equations.

$$\frac{\partial \bar{\theta}}{\partial t} + (\frac{d}{l})\frac{\partial Q_x}{\partial x} = RCos\alpha \qquad 0 \le x \le 1 \quad (29)$$

$$Q_x = \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} (Sin\alpha - \frac{d}{l}\frac{\partial \bar{\Phi}}{\partial x})^2 \quad (30)$$

$$Q = 0 \qquad x = 0 \quad (31)$$

$$\frac{\partial Q}{\partial x} = 0 \qquad x = 1 \quad (32)$$

($\bar{\theta}$) in equation (29) and ($\bar{\Phi}$) in equation (30) are averaged water content and suction along depth. However, when the relation in equation (32) replaced in equation (29),it has been created disturbance in calculation and it gave defaulted result. To avoid such a condition, following relation has been considered:

$$x = 1 \text{ and } \frac{\partial \bar{\theta}}{\partial T} = 0 \text{ or } \bar{\theta} = \bar{\theta}_0 = const \quad (33)$$

to equalize the approximation of integration, following relation has been considered.

$$f_1 = \frac{\int_0^1 \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} dy}{\left(\frac{A^2}{A^2 + \bar{\Phi}^2}\right)^{\beta}} \quad (34)$$

When relation of equation (34) replaced in equation (30), it has been realized that it destroys the situation of boundary condition at ($x = 0$). Therefore to realize the relation in equation (30), equation (35) has been introduced.

$$Q_x = \gamma \left(\frac{A^2}{A^2 + \Phi^2}\right)^{\beta} \left(Sin\alpha - \frac{d}{l}\frac{\partial \bar{\Phi}}{\partial x}\right) \quad (35)$$

Where (γ) is obtained through various numerical calculations.

$$\gamma = Exp[0.1\beta - 0.07 + (0.09 - 0.17\beta)ln(A)] \quad (36)$$

For different ranges of parameters the validity of (γ) is shown in figure (6). (γ) is obtained under uniform rain, therefore to check its accuracy, we applied different types of rain as it is shown in figure (7).

4 Mathematical Storage Function Model:

The relation between storage and discharge can be derived under the steady state condition.

$$Q_x = \frac{l}{d} RxCos\alpha \quad (37)$$

Through further integration of equation (29) and (35) under the condition ($Sin\alpha >> \frac{d}{l}\frac{\partial \bar{\Phi}}{\partial x}$), equations (38) and (39) are obtained.

$$\frac{dS}{dT} + \left(\frac{d}{l}\right) Q_{x=1} = RCos\alpha \quad (38)$$

$$S=\left(\frac{\beta}{\beta+1}\right)\left(\frac{Q}{\gamma Sin\alpha}\right)^{\frac{1}{\beta}}+\left(\frac{\theta_r}{\theta_s-\theta_r}\right) \quad (39)$$

(S) is storage and figure (8) shows the result obtained by semi lumped equations. In principle, the equations (29) and (35) have to be solved theoretically to find (S ~ Q) relation at unsteady state. However it is difficult to solve them theoretically. Therefore, a convenient method has been presented by authors. Let's assume:

$$\frac{A^2}{A^2+\Phi^2}=e^{\varepsilon\overline{\Phi}} \quad (40)$$

By the substituting the relation in equation (40) into equation (35) the following relation can be achieved.

$$Q=\gamma Sin\ \alpha e^{\beta\varepsilon\overline{\Phi}}-\frac{\gamma d}{l\beta\varepsilon}\frac{\partial e^{\beta\varepsilon\overline{\Phi}}}{\partial x} \quad (41)$$

Considering the relationship between x and t. With the respect to different given time, means the steady state condition of discharge along (x) where the relation in equation (43) means the unsteady state condition.

$$Q(x)=Q_{x=1}x \quad (42)$$

$$Q(x)=-x(x-2)Q_{x=1} \quad (43)$$

Let's assume a new considerate system where according to boundary condition at $x'=0$:

$$e^{\beta\varepsilon\overline{\Phi}}=e^{\beta\varepsilon\overline{\Phi}_0} \quad (44)$$

By considering the new considerate system and substituting the relation in equation (42) into equation (35) the following relation can be achieved.

$$Z_1=e^{\beta\varepsilon\overline{\Phi}}=\left(\frac{Q_{x=0}}{\gamma Sin\alpha}\right)*$$

$$\left(1-e^{-c_1x'}-\frac{1}{c_1}(c_1x'-1)-\frac{1}{c_1}e^{-c_1x'}\right) \quad (45)$$

Where:

$$c_1=\frac{l\beta\varepsilon Sin\alpha}{d} \quad (46)$$

Similarly, by replacing the equation (43) in equation (35) we can derive.

$$Z_2=e^{\beta\varepsilon\overline{\Phi}}=\left(\frac{Q_{x'=0}}{c_1}\right)\left\{\left(1-e^{-c_1x'}\right)-\left(x'^2-\frac{2x'}{c_1}+\frac{2}{c_1^2}\right)+\frac{2}{c_1^2}e^{-c_1x'}\right\}$$

(47)

Finally, the (S ~ Q) relation for steady and unsteady state can be obtained by following equations:

$$S_1=\int_0^1 Z_1^{\frac{1}{\beta}}dx=\left(\frac{Q_{x'=0}}{\gamma Sin\alpha}\right)^{\frac{1}{\beta}}G_1(c_1-\beta) \quad (48)$$

$$S_2=\int_0^1 Z_2^{\frac{1}{\beta}}dx=\left(\frac{Q_{x'=0}}{\gamma Sin\alpha}\right)^{\frac{1}{\beta}}G_2(c_1-\beta) \quad (49)$$

Where $G_1(c_1,\beta)$ and $G_2(c_1,\beta)$ are obtained by numerical calculations.

$$G_1(c_1,\beta)=0.151\log c_1+0.331\log\beta+0.241 \quad (50)$$

$$G_2(c_2,\beta)=(-0.064\log\beta+0.21)\log c_1+0.31\log\beta+0.31$$

(51)

5 Conclusions:

At the first stage, we introduced a new boundary condition at the outlet of soil column and its accuracy is examined with experimental results. In the second stage, we proposed a compensating factor (γ) to equalize the semi lumped equation to the Richards equation. Fully lumped storage function model in two cases of steady and unsteady states are introduced and examined in different physical conditions. In steady state, with increasing and slope length, the result of storage function model shows higher accuracy in comparison with experimental results. Besides in case of unsteady state, when $\beta=1$ and $0.05\le D\le 0.08$ the most suitable result can be achieved.

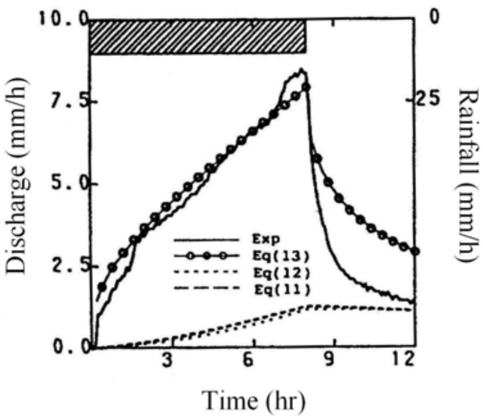

Figure 3: Profile of discharge α=0.1

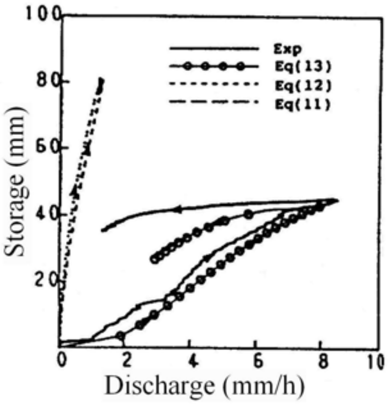

Figure 4: Relationship of storage and discharge($\bar{S}_1 \sim \bar{q}$),α=0.1

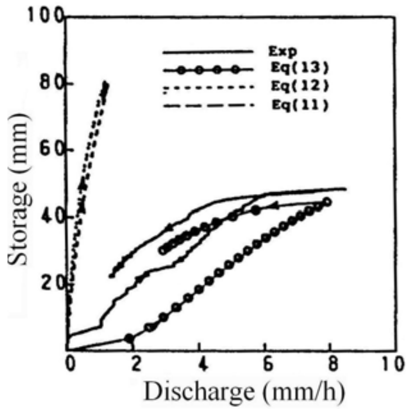

Figure 5: Relationship of storage and discharge ($\bar{S}_2 \sim \bar{q}$), α=0.1

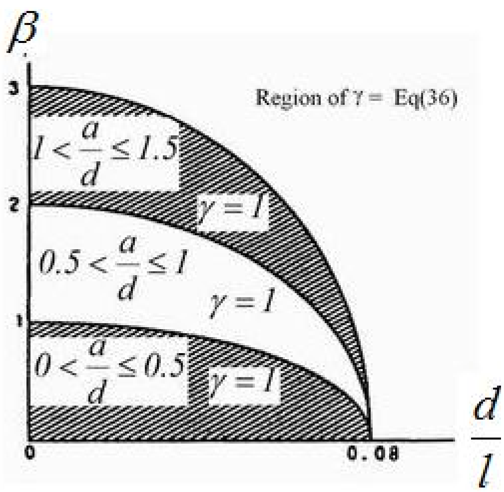

Figure 6: range of parameters for γ

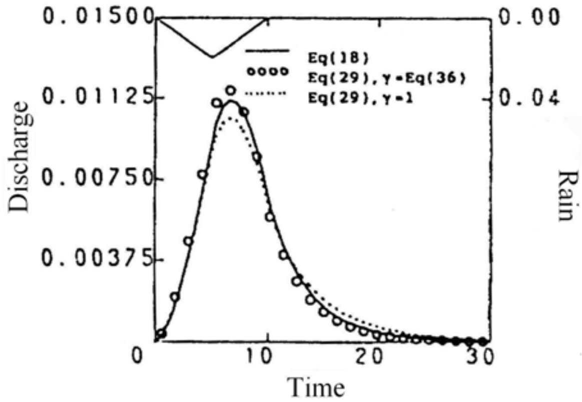

Figure 7: profile of discharge

$$\frac{d}{l} = 0.5, \frac{a}{d} = 0.05, \alpha = 0, \beta = 2$$

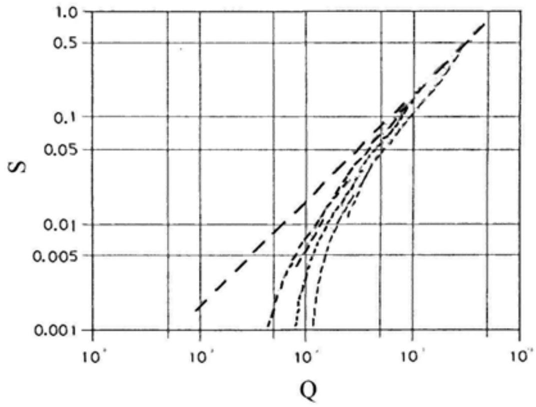

Figure 8: relationship of storage and discharge

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