Switching properties of an arbitrarily excited nonlinear electron-wave directional coupler

ANDREAS F TERZIS
Department of Physics
University of Patras
Patras, Rion, GR-26504
GREECE
http://www.physics.upatras.gr/

EMMANUEL PASPALAKIS
Materials Science Department
University of Patras
Patras, Rion, GR-26504
GREECE
http://www.matersci.upatras.gr/

Abstract: We theoretically investigate the switching properties of nonlinear electron-wave directional couplers by applying the coupled mode theory. Analytical results for the probability of finding the electron wave in each waveguide, for arbitrary initial occupation of each waveguide, are presented. Our findings reveal that several technologically important cases can be achieved, such as significant or complete electron transfer between the waveguides or trapping of the electron wave in the initial state, depending on the structural parameters of the device. In addition, for specific values of the initial state of the system the important case of symmetry breaking is realized.
Key-Words: - Electron-wave directional coupler, switching, arbitrary initial conditions, electronic oscillations, self-trapping, symmetry breaking

1 Introduction

Several studies have proposed devices that could realize coherent electron transport. An important device in this area of quantum electronics is the electron-wave directional coupler [1-10]. This device has also been recently proposed for applications in quantum computation [11-15]. In a particular study, Tsukada *et al.* [7] proposed and analyzed numerically with the use of coupled mode theory a nonlinear electron-wave directional coupler. This device consists of two closely spaced, parallel electron waveguides with extremely small capacitances. They showed that this device exhibits either complete electron transfer from one waveguide to the other or electron self-trapping to the initially excited waveguide for specific values of the system parameters. A symmetry breaking effect is also found in the case that both waveguides are initially excited in a coherent superposition state.

In a later work [16] we revisited the nonlinear electron-wave directional coupler of Tsukada *et al.* [7] and presented analytical results for the probability of finding the electron wave in each waveguide for the case that only one waveguide is initially excited. We also calculated the critical length of the device. In addition, we proposed a new type of device, a bent nonlinear electron-wave directional coupler and studied its switching characteristics.

In this article we study further the switching characteristic of the nonlinear electron-wave directionall coupler of Tsukada *et al.* [7] and obtain analytical results for the probability of finding the electron wave in each waveguide for the case that the system is initially in an arbitrary superposition state. These results show the general behavior of the system. With the analytical solution we also manage to explain the effect of symmetry breaking that can occur in this system [7] and also show the crucial dependence of this effect to the form of the initial state of the system.

This article is organized as follows: in the next chapter we present the basic equations based on coupled mode theory. In chapter 3 we present analytical results for the case of arbitrary initial conditions. Then, in the following three chapters the behavior of the system for several cases of initial conditions is presented. Finally, we summarize our findings in chapter 7.

2 Theoretical Model

For the theoretical analysis of the nonlinear electron-wave directional coupler we use coupled mode theory [1-3,7]. According to this approach the wavefunction $\psi(z)$ of the electron propagating along the z direction can be written as a linear combination of the eigenfunctions of the individual waveguides ψ_a, ψ_b,

$$\psi(z) = a(z)\psi_a + b(z)\psi_b. \quad (1)$$

If this wavefuction is substituted into the Schrödinger equation we obtain the following set of nonlinear coupled differential equations (we use $\hbar = 1$ in this paper)

$$i\frac{da(z)}{dz} = -\Omega\left(1-2|a(z)|^2\right)a(z) + k_0 b(z), \quad (2)$$

$$i\frac{db(z)}{dz} = \Omega\left(1-2|a(z)|^2\right)b(z) + k_0 a(z), \quad (3)$$

where k_0 is the waveguide coupling constant per unit length. In the derivation of Eqs. (2) and (3) a symmetric waveguide structure is assumed. The coefficient k_0 arises due to electron tunnelling between the waveguides.

The first terms in the right hand side of Eqs. (2) and (3) are due to the Coulomb charging effect that exists due to the extremely small capacitance of the structure. Also, $\Omega = Q/(2C)$ where Q is the total charge on the waveguides per unit length and C is the effective capacitance of the waveguide [7]. We note that k_0, Ω are taken positive in our work. The quantity of interest is the probability of finding the electron wave in each waveguide at a specific distance, given by $P_a(z) = |a(z)|^2$, $P_b(z) = |b(z)|^2$.

3 Analytical Results for Arbitrary Initial Conditions

A convenient quantity for obtaining analytical results is the difference of these probabilities $p(z) = P_a(z) - P_b(z)$. As $P_a(z) + P_b(z) = 1$, we find $P_a(z) = [1+p(z)]/2, P_b(z) = [1-p(z)]/2$.

Our goal is to obtain analytical results for arbitrary initial conditions at the input of the waveguide coupler $a(0) = a_0$, $b(0) = b_0$ with a_0, b_0 being complex numbers with $|a_0|^2 + |b_0|^2 = 1$.

We combine Eqs. (2)-(3) and obtain

$$\frac{du}{dz} = -2\Omega p(z)v(z) - 2k_0 p(z), \quad (4)$$

$$\frac{dv}{dz} = 2\Omega p(z)u(z), \quad (5)$$

$$\frac{dp}{dz} = 2k_0 u(z), \quad (6)$$

where $v(z) = -2\,\mathrm{Re}\left(a(z)b^*(z)\right)$ and $u(z) = -2\,\mathrm{Im}\left(a(z)b^*(z)\right)$. Here, $p^2 + u^2 + v^2 = 1$.

We are interested for analytical results in the case of arbitrary initial conditions $p(0) = p_0$, $u(0) = u_0$ and $v(0) = v_0$ with the constraint $p_0{}^2 + u_0{}^2 + v_0{}^2 = 1$.

By combining Eqs. (5) and (6), we obtain $\frac{dv(z)}{dz} = \frac{\Omega}{k_0}p(z)\frac{dp(z)}{dz}$, which by integration gives the following equation

$$v(z) = v_0 + \frac{\Omega}{2k_0}\left[p^2(z) - p_0^2\right]. \quad (7)$$

Then, from Eq. (4) we obtain

$$\dot{u}(z) = \left[\frac{\Omega^2}{k_0}p_0^2 - 2\Omega v_0 - 2k_0\right]p(z) - \frac{\Omega^2}{k_0}p^3(z). \quad (8)$$

If now we differentiate Eq. (6) and use Eq. (8) we obtain a second order differential equation for $p(z)$

$$\frac{d^2p}{dz^2} = \left[2\Omega^2 p_0^2 - 4\Omega k_0 v_0 - 4k_0^2\right]p(z) \quad (9)$$
$$-2\Omega^2 p^3(z).$$

For arbitrary initial conditions, the solution of Eq. (9) is [17,18]

$$p(z) = D\ cn\left[\frac{D\Omega}{\sqrt{k}}(z - z_0)\middle| k\right], \quad (10)$$

where

$$k = \frac{1}{2}\left(1 + \frac{p_0^2 - \zeta}{\sqrt{\zeta^2 + 4\xi^2 u_0^2}}\right), \quad (11a)$$

$$D = sign(p_0)\sqrt{p_0^2 - \zeta + \sqrt{\zeta^2 + 4\xi^2 u_0^2}} \quad (11b)$$

$$\xi = \frac{k_0}{\Omega}, \quad (11c)$$

$$\zeta = 2\xi(\xi + v_0), \quad (11d)$$

$$z_0 = \frac{\sqrt{k}}{D\Omega}\int_0^{\arccos(p_0/D)} \frac{dx}{\sqrt{1 - k\sin^2 x}}, \text{ if } u_0 = 0,$$

and

$$z_0 = sign(u_0)\frac{\sqrt{k}}{D\Omega}\int_0^{\arccos(p_0/D)}\frac{dx}{\sqrt{1-k\sin^2 x}}, \qquad (11e)$$

if $u_0 \neq 0$.

Obviously, for the case of $p_0 = 1$, i.e. the case that only the waveguide a is initially excited, then $u_0 = v_0 = 0$ and Eq. (10) reduces to

$$p(z) = cn\left[2k_0 z \middle| k = \frac{\Omega^2}{4k_0^2}\right], \qquad (12)$$

that is identical to Eq. (10) of Ref. 16.

The behaviour of Eq. (10) is governed by the value of parameter k and in turn parameter ξ. In the case that $k = 1$, $p(z) = D\sec h\left[D\Omega(z - z_0)\right]$ and the probabilities in either waveguides reach a steady state as the electron-wave propagates in the waveguide. In the cases that $k < 1$ or $k > 1$ then the evolution of the system is periodic.

4 Results for the $u_0 = 0$ case

We start with the case that $u_0 = 0$. This can occur, for example, in the case that both a_0, b_0 are real or imaginary. As $p_0^2 + u_0^2 + v_0^2 = 1$, v_0 is not an independent parameter as $v_0 = \pm\sqrt{1 - p_0^2}$.

We can easily find the critical value of the nonlinearity parameter ξ, ξ^c. The value ξ^c specifies the value of ξ at which $k = 1$. Therefore, ξ^c is given by $\xi^c = (-v_0 \pm 1)/2$. In our study we consider only positive values for the nonlinearity parameter, so we only adopt the positive root and conclude that

$$\xi^c = \frac{1 - v_0}{2}. \qquad (13)$$

The critical value of nonlinearity parameter ξ^c corresponds to the value above which the system undergoes extended periodic electron transfer with the value of $p(z)$ changing from p_0 to $-p_0$. This is practically the behavior of the $cn(x\,|\,k)$ elliptic function for $k < 1$. Actually, for vanishing k-parameter the *cn* function becomes a cosine function. For values of ξ below ξ^c suppression of electronic transfer is found and self-trapping to the initial state of the system is obtained. This again basically depicts the behavior of the $cn(x\,|\,k)$ elliptic function for $k > 1$. Finally, in the case that $k = 1$, as we have noted above the *cn* function becomes the *sech* function and no oscillations are found. The behavior analyzed above can be seen in Fig. 1, where we plot the spatial evolution of the probabilities for positive and negative values of v_0.

As can be seen from Figs. 1(e) and 1(f) the behavior of the self-trapping region can be different. For certain values of Ω one obtains self-trapped oscillations between the initial values [as in Fig. 1(e)] and for other values of Ω there are oscillations above and below the initial values [as in Fig. 1(f)].

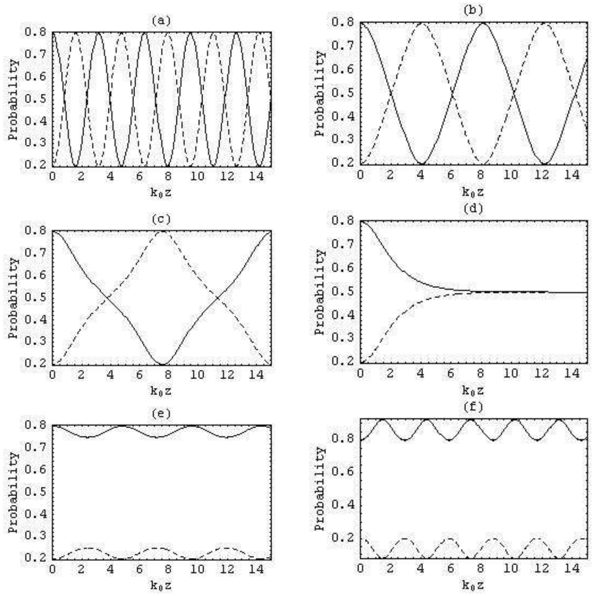

Fig.1. We present $P_a(z)$ (solid curve) and $P_b(z)$ (dashed curve) as a function of the normalized propagation distance for initial conditions $a_0 = \sqrt{0.8}, b_0 = \sqrt{0.2}$. This leads to $p_0 = 0.6, u_0 = 0, v_0 = -0.8$. The parameters of this figure are (a) $\Omega = 0.01k_0$, (b) $\Omega = k_0$, (c) $\Omega = 1.1k_0$, (d) $\Omega = 10k_0/9$, (e) $\Omega = 1.2k_0$, and (f) $\Omega = 1.5k_0$.

We note that for $\xi < -v_0$, the *cn* elliptic function becomes a *dn* elliptic function. Also, in the case that $\xi > -v_0$ the oscillations are between the initial values [as in Fig. 1(e) where $\xi = 0.833$] while for $\xi < -v_0$ the oscillations are above and below the initial values [as in Fig. 1(f) where $\xi = 0.667$]. Of special interest is the case where $\xi = -v_0$, then k becomes infinity and we have a stable stationary (time-independent) solution.

We now pay some more attention to the case of almost equal initial population in both waveguides, i.e. the case that $p_0 \approx 0$. We first note that in the case that $p_0 = 0$ then $v_0 = \pm 1$ and

$$\xi^c = \left(1 - v_0\right)/2 = \begin{cases} 0, \text{ for } v_0 = +1 \\ 1, \text{ for } v_0 = -1 \end{cases}. \qquad (14)$$

As we can see from Eq. (10) in this case p_0 stays zero at all distances. However, in the case where the system is in a superposition state such that p_0 is very close to zero but not exactly zero with $v_0 = -\sqrt{1 - p_0^2} \approx -1$ we find that for values of the non-linearity parameter below its critical value, ξ^c, a symmetry breaking situation is achieved. This can be seen in Figs. 2 and 3.

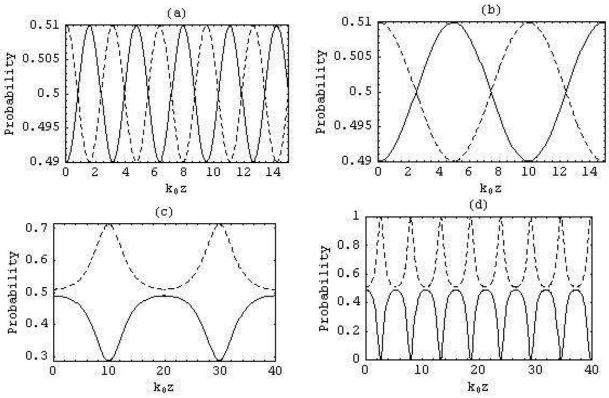

Fig.2. The same as in Fig. 1 for initial conditions $a_0 = \sqrt{0.49}, b_0 = \sqrt{0.51}$. This leads to $p_0 = -0.02, u_0 = 0, v_0 = -0.9998$. The parameters of this figure are (a) $\Omega = 0.01k_0$, (b) $\Omega = 0.9k_0$, (c) $\Omega = 1.05k_0$, (d) $\Omega = 2k_0$.

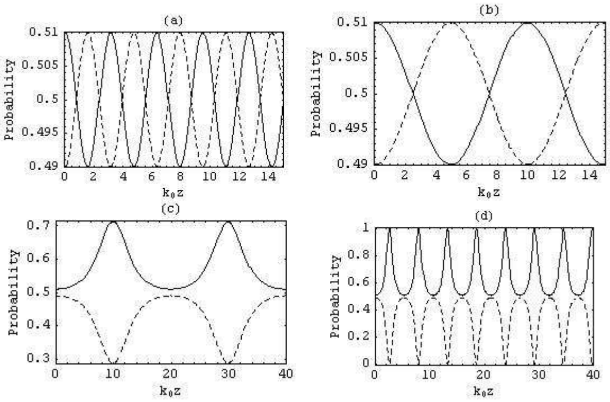

Fig.3. The same as in Fig. 2 for initial conditions $a_0 = \sqrt{0.51}, b_0 = \sqrt{0.49}$. This leads to $p_0 = 0.02, u_0 = 0, v_0 = -0.9998$.

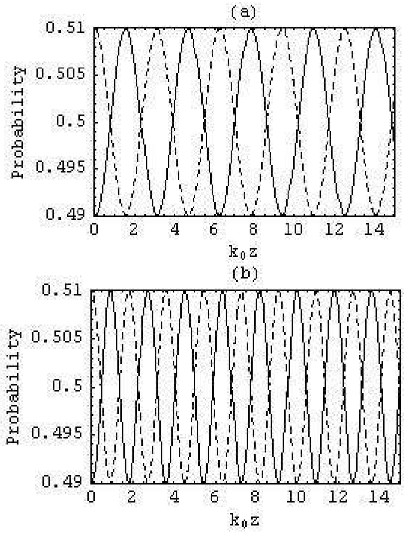

Fig.4. The same as in Fig. 2 for initial conditions $a_0 = \sqrt{0.49}, b_0 = -\sqrt{0.51}$. This leads to $p_0 = -0.02, u_0 = 0, v_0 = 0.9998$. The parameters of this figure are (a) $\Omega = 0.01k_0$, (b) $\Omega = 2k_0$.

By symmetry breaking we mean that for $\xi > \xi_c$ we obtain small amplitude oscillations with the probability changing between the initial probabilities of the two waveguides, while for $\xi < \xi_c$ oscillations with large amplitude occurs and even complete transfer to one of the waveguides is achieved. In the latter case the waveguide that will be mainly excited during propagation depends on the sign of p_0.

This is explained as follows: for negative v_0 we obtain a negative ζ, which occurs for $0 < \xi < \xi^c \left[= (1 - v_0)/2 \sim 1\right]$, and the coefficient in front of the elliptic Jacobi function becomes $D = sign(p_0)\sqrt{p_0^2 - 2\zeta} \approx sign(p_0)\sqrt{-2\zeta}$. Also, $k = 1 - \dfrac{p_0^2}{\zeta}$ which is almost unity, but always larger than unity. As $\zeta \approx 2\xi\left(\xi - 1\right)$ the parameter D obtains a maximum value approximately at $D \approx sign(p_0)$ for $\xi \approx 0.5$. So, in this case for $\xi = 0.5$, depending on the sign of the small initial value of p_0, we obtain complete transfer to waveguide a (for positive p_0) or waveguide b (for negative p_0). This is shown especially in Figs. 2(d) and 3(d).

The spontaneous breaking effect is also dependent on the sign of v_0. This can be seen in Fig. 4. For positive

v_0, with $v_0 \approx 1$, we obtain small amplitude electronic oscillations between the two waveguides, as in this case $\xi_c \approx 0$ and we are in the regime that $\xi > \xi_c$ even for very small values of ξ. The only thing that changes with the change of ξ is the period of oscillations.

5 Results for the $v_0 = 0$ case

We continue with the case that $v_0 = 0$. An example of this case is the situation that either a_0 or b_0 is imaginary and the other is real. In this case we get a much simpler expression for the k parameter as now ζ does not depend of the initial conditions ($\zeta = 2\xi^2$). From this expression we can find the critical value of the nonlinearity parameter as a function of p_0 (this that corresponds to $k=1$). Easily we obtain that the critical value is

$$\xi^c = 0.5 p_0^2 . \tag{15}$$

Therefore, we have the same critical value ξ_c for both positive ($u_0 = +\sqrt{1-p_0^2}$) or negative value ($u_0 = -\sqrt{1-p_0^2}$) of u_0.

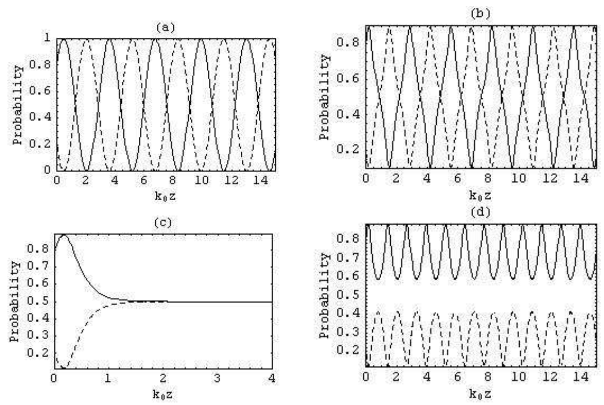

Fig.5. The same as in Fig. 1 for initial conditions $a_0 = \sqrt{0.8}, b_0 = i\sqrt{0.2}$. This leads to $p_0 = 0.6, u_0 = 0.8, v_0 = 0$. The parameters of this figure are (a) $\Omega = 0.01k_0$, (b) $\Omega = 5k_0$, (c) $\Omega = 50k_0/9$, (d) $\Omega = 6k_0$.

In Figs. 5 and 6 we observe that for values of the nonlinearity parameter above the critical value oscillations occurs with $p(z)$ changing between the maximum values $-D$ and D. We stress that, in this case, $|D|$ can be larger than $|p_0|$, so the oscillations can occur between values of the probability that are larger than the initial probabilities in the waveguides. Also, for $\xi < \xi_c$ suppression of oscillations and transfer is found and self-trapping of the system close to its initial state occurs. Results for this are shown in Figs. 5 and 6. As it is expected, by comparing the results for negative and positive initial u_0, we obtain similar results that only differ by a phase factor. The symmetry breaking situation along the lines discussed in the case of section 4 does not occur here, as in this case for very small values of p_0 the critical value ξ_c becomes also very small and we are always practically in the regime that $\xi > \xi_c$.

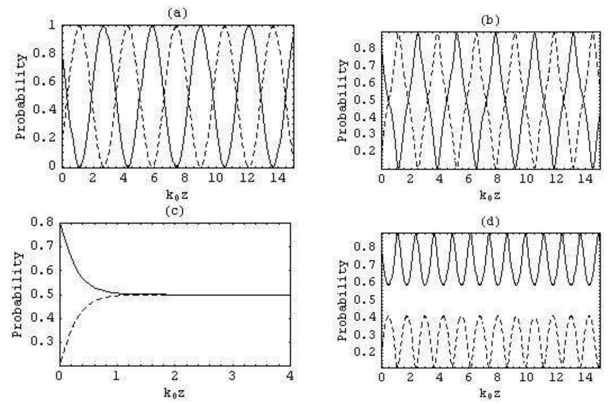

Fig.6. The same as in Fig. 6 for initial conditions $a_0 = \sqrt{0.8}, b_0 = -i\sqrt{0.2}$. This leads to $p_0 = 0.6, u_0 = -0.8, v_0 = 0$.

6 Results for the $u_0, v_0 \neq 0$ case

This is the case that both a_0, b_0 are complex in general. From the general expression for the parameter k [Eq. (11a)] we obtain the critical value of the nonlinearity parameter as a function of p_0. After some algebra we find that the critical value is given by

$$\xi^c = \frac{p_0^2 (v_0 \pm 1)}{2(v_0^2 - 1)} . \tag{16}$$

We concentrate on the positive critical value and get

$$\xi^c = \frac{p_0^2}{2(1+v_0)} = \frac{1-u_0^2-v_0^2}{2(1+v_0)} . \tag{17}$$

Even in this most general case that all values of u_0, v_0, p_0 are non-vanishing the behavior of the system is rather similar to that discussed in the two previous cases. The system executes electronic oscillations between the two waveguides with $p(z)$ changing between the maximum values $-D$ and D once the nonlinearity parameter is above the critical value ξ^c. In this case, too, $|D|$ can be larger than

$|p_0|$, so the electronic oscillations can occur between values of probability that are larger than the initial probability values. In the case that ξ is smaller than ξ_c the system is trapped in its initial state and performs oscillations of small amplitude between the two waveguides.
In addition, as in the case of Section 5 the symmetry breaking situation does not occur in this general case. Only in the case that u_0 takes very small values (such that it can be assumed that it practically goes to zero) we can recover the symmetry breaking result of Section 4.

7 Summary

In this work we have studied the switching characteristic of a nonlinear electron-wave directional coupler in the case that the device at the entrance is prepared in a general superposition state. We first obtained analytical results for the probability of finding the electron wave in each waveguide. Then, we analyzed the switching behavior of the system for specific initial conditions. For different parameters of the system extended electronic oscillations between the two waveguides, self-trapping in the initial state and symmetry breaking can be obtained.

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