Digital Filter Design Using Non-Uniform Sampling

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Abstract: In this paper, a non-uniform sampling algorithm is proposed for one-dimensional bandlimited time varying signals. The main feature of the proposed sampling scheme is that the sampling steps are inversely proportional to signal gradient or slope of the signal. As a result, a smaller sampling step is obtained whenever the gradient is high and vice-versa. Thus, a better representation of the signal is obtained when it is linearly interpolated between the samples. The proposed sampling algorithm is then applied to the design of digital filters using the well-known impulse invariance method. It is shown that the nonuniform sampling allows to design a FIR filter whose length is 14 times shorter than what is obtained with uniform sampling; with both filters having similar characteristiscs. Simulations are provided to show the performance and characteristics of the sampling sequence.

1 Introduction

Digital filters are widely employed in many engineering applications especially for the purpose of noise attenuation and suppression from signals measured by sensors. Most common digital filters manipulate input signals that are sampled regularly or uniformly by means of an analogue-to-digital converter. However, one of the main shortcoming of uniform sampling is that it is not well suited for nonstationary signals [1]; that is signals that have different frequency components at different interval of time. For such signals, if regular sampling is employed, valuable data can be lost if the sampling theorem is not obeyed. Conversely, large amount of unnecessary data will be stored if the signal is constant or slowly varying within a given interval of time. Consequently, it would be more judicious to adopt an irregular sampling strategy to deal with such signals.

In this paper, a non-uniform sampling algorithm is proposed for one-dimensional bandlimited time varying signals, $x(t)$. It is important to note that uniform sampling consists in selecting or measuring instantaneous values of the amplitude of $x(t)$ according to the following arithmetic sequence of time $t = \tau(n) = nT$ where $T > 0$ (sampling period). The sequence $\tau(n)$ is called a sampling sequence. When the sampling sequence is monotone increasing but not arithmetic the sampling is said to be non-uniform or irregular. A number of research works has been conducted on irregular sampling, for both bandlimited and nonbandlimited signals, using various types of nonuniform sampling sequences (see eg. [2]-[8]). However, in this paper, we propose an alternative sampling sequence such that the steps are inversely proportional to signal gradient or slope. As a result, a smaller sampling step is obtained whenever the gradient is high, and conversely, when the gradient is low a larger step is employed.

The nonuniform sampling algorithm is applied for the design of digital filters using the well-known impulse invariance method. It is shown that the proposed sampling algorithm provide a digital filter which has a frequency response that is closer to the frequency response of the analogue filter compared to what is obtained using a uniform sampling scheme. Simulations are provided to show the performance and characteristics of the sampling sequence. Finally, some remarks are made regarding the improvement and further extension of the algorithm.

2 A special nonuniform sampling sequence

Consider a continuous real valued bandlimited causal signal $x(t)$. Recall that $x(t)$ is causal if $x(t) = 0$ for $t < 0$. Here, we shall consider only causal signals since in practice we generally deal with such signals. We assume that the time derivative of $x(t)$ exists and is continuous and bounded. More precisely, there exists a positive constant M such that $\left|\frac{d}{dt}x(t)\right| \leq M$ for all $t \geq 0$.

We define the following nonuniform sampling sequence:

$$\tau(n+1) = \tau(n) + (1-\beta(n))\, T_{\min} + \beta(n) T_{\max} \qquad (1)$$

where $n = 0, 1, 2, ...$ and where $0 < T_{\min} < T_{\max}$ and

$\beta(n)$ is defined as

$$\beta(n) = \frac{1}{\gamma \left| \dfrac{x(\tau(n)) - x(\tau(n-1))}{\tau(n) - \tau(n-1)} \right| + 1} \qquad (2)$$

with $\gamma > 0$. We assume the following initial conditions $\tau(0) = 0$ and $\beta(0) = 0$ so that $\tau(1) = T_{\min}$. The values of $T_{\min}$ and $T_{\max}$ are chosen such that if $x(t)$ is sampled uniformly with the sampling intervals $T_{\min}$ and $T_{\max}$, then the Shannon sampling theorem is obeyed. More precisely, $T_{\min}, T_{\max} \leq \dfrac{\pi}{\omega_{\max}}$ where $\omega_{\max}$ is the highest frequency component of $x(t)$.

Before applying the sequence to a particular signal, we shall first of all discuss the characteristics of the sampling sequence (1):

i) First, note that the derivative of $x(t)$ evaluated at the time instant $t = \tau(n)$ can be approximated by the forward Euler approximation as:

$$\frac{d}{dt}x(t)\bigg|_{t=\tau(n)} \approx \frac{x(\tau(n)) - x(\tau(n-1))}{\tau(n) - \tau(n-1)}.$$

Consequently, the term $\beta(n)$ is inversely proportional to the derivative of the signal at time instant $t = \tau(n)$. This means that if the signal is fast varying then the derivative (or slope) will be high and as a result the value of $\beta(n)$ will be small and vice-versa. Also, note that $0 \leq \beta(n) \leq 1$. The largest value of $\beta(n) = 1$ occurs in the limiting case where the signal is constant in which case the slope of the signal will be zero.

ii) The sampling step, $s(n) = \tau(n+1) - \tau(n) = (1 - \beta(n))\,T_{\min} + \beta(n)T_{\max}$ is nothing more than the formula describing a particular point lying on the segment $[T_{\min,}\, T_{\max}]$. In effect, if $\beta(n) = 0$ then the step size $s(n) = T_{\min}$. On the other hand when $\beta(n) = 1$, then $s(n) = T_{\max}$. More precisely, $T_{\min} \leq s(n) \leq T_{\max}$.

iii) It is clear that the sequence $\{\tau(n)\}$ is a monotone increasing sequence since $\tau(n+1) - \tau(n) > 0$. This confirms that $\tau(n)$ is indeed a sampling sequence.

iv) Finally, the positive number γ is a weight that is attached to the slope of the signal. If $\gamma > 1$, then more weight will be attached to the slope and the sampling step will be smaller. Otherwise, if $0 \leq \gamma < 1$ then the sampling step will be larger.

2.1 Simulation results:

The above sampling sequence was programmed (using the Maple software) to test its performance. The following analogue signal:

$$x(t) = e^{-t} \qquad (3)$$

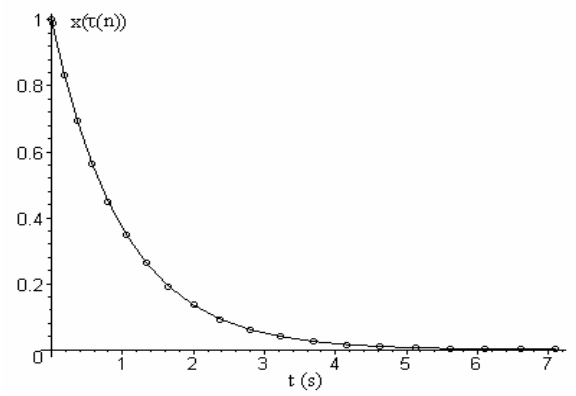

Figure 1: Nonuniformly sampled signal $x(\tau(n))$ with $T_{\max} = 0.01, T_{\max} = 0.5, \gamma = 2$

was sampled using the above nonuniform sampling sequence. The numerical values employed were: $T_{\max} = 0.5s$, $T_{\min} = 0.01s$, $\gamma = 2$. The choice of these particular numerical values of will be detailed in the next section. For comparison purposes we have also sampled $x(t)$ regularly using the sampling interval $T_{\min}$ and $T_{\max}$. Figure 1 shows the samples of $x(t)$ when it is nonuniformly sampled using (1); that is $x(\tau(n)) = e^{-\tau(n)}$. Figure 2 shows the samples of $x(t)$ when it is uniformly sampled using the sampling interval $T = T_{\min}$ that is $x(nT_{\min}) = e^{-nT_{\min}}$. Figure 3 shows the samples of $x(t)$ when it is uniformly sampled using the sampling interval $T = T_{\max}$ that is $x(nT_{\max}) = e^{-nT_{\max}}$. In each case the simulation was run for 7 seconds. It can be observed that more samples were taken with the nonuniform sampling sequence (20 samples) than with the uniform sampling sequence with sampling interval $T_{\max}$ (14 samples). On the other hand, 700 samples were taken when $x(t)$ is uniformly sampled with the sampling interval $T_{\min}$. The effect of a change in weight were also investigated. Figure 4 shows the samples of $x(t)$ when it is nonuniformly sampled using (1); that is $x(\tau(n)) = e^{-\tau(n)}$ when the weight was increased 5 times, to $\gamma = 10$ while $T_{\max}$ and $T_{\min}$ were kept at the same value; $T_{\max} = 0.5$, $T_{\min} = 0.01$. In this case, it can be seen that more samples are taken in the same interval of time. More precisely, 35 samples were obtained. Another way to obtained more samples, by using the nonuniform sampling sequence within a given interval of time, while keeping the weight fixed would be obviously to decrease the maximum allowable step size $T_{\max}$. The above results are summarised in the Table below:

Sampling method	No. of samples
Uniform sampling, $T_{\max} = 0.5s$	14
Uniform sampling, $T_{\min} = 0.01s$	700
Nonuniform sampling, $\gamma = 2$	20
Nonuniform sampling, $\gamma = 10$	35

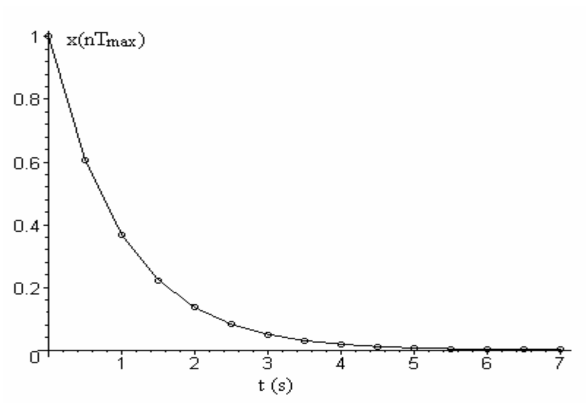

Figure 2: Uniformly sampled signal $x(nT_{\max})$ with $T_{\max} = 0.5s$

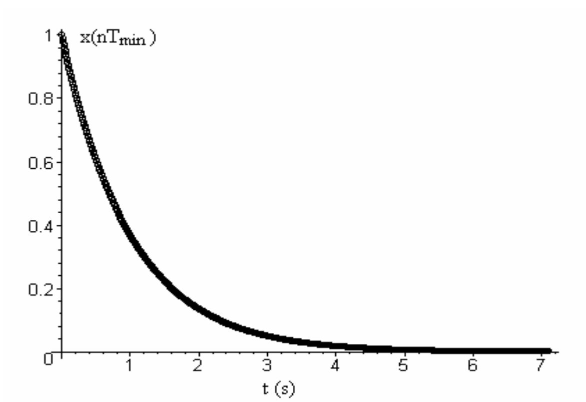

Figure 3: Uniformly sampled signal $x(nT_{\min})$ with $T_{\min} = 0.01s$

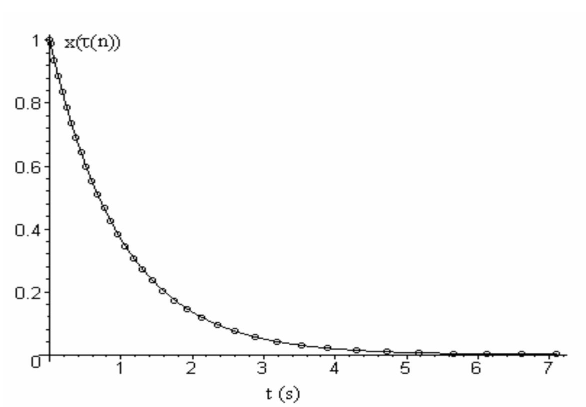

Figure 4: Nonuniformly sampled signal $x(\tau(n))$ with $T_{\min} = 0.01s$, $T_{\max} = 0.5s$, $\gamma = 10$

3 Application to digital filter design

In this section, we shall apply the nonuniform sampling algorithm for digital filter design, We shall concentrate mainly in the impulse invariance method [9]. The impulse invariance method is mainly used for the design of IIR (Infinite Impulse Response) filters. However, we shall here adapt it for FIR (Finite Impulse Response) filters design. The traditional design methodology is as follows:

Given the transfer function, $H(s)$, of an analogue filter, which has the characteristics of the desired digital filter, we first find the impulse response by using the inverse Laplace transform of the transfer function; that is, $h(t) = \mathcal{L}^{-1}\{H(s)\}$. Next, we sample the impulse response regularly so that $h(nT) = h(t)|_{t=nT}$ where T is the sampling period. Finally, we calculate the transfer function of the digital filter by applying the z-transform on the sampled impulse response; that is $H(z) = \mathcal{Z}\{h(nT)\}$.

It is clear that the impulse invariance method is only valid if the sampling interval is very small. In order to adapt this method for FIR filter design in a nonuniform sampling context, we employ the previous sampling sequence for the sampling process of the impulse response so that we now have

$$h(\tau(n)) = h(t)|_{t=\tau(n)}$$

where $\tau(n)$ is given by (1). To design an FIR filter, the sampling is done only over a finite interval of time $[0, \tau(N)]$ for some index $N > 0$. This also determines the length of the filter.

Then, the discrete-time Fourier transform (DTFT) of the sequence is performed and compared with that of the original analogue signal. At this stage it is important to clarify the definition of the nonuniform DTFT employed. Recall that the Fourier transform of the function $h(t)$ over the interval $[0, \tau(N)]$ is given by [10]:

$$\begin{aligned} H_{nu}(\omega) &= F\{h(t)\} \\ &= \int_{\tau(0)=0}^{\tau(N)} h(t)e^{-j\omega t}dt \\ &= \int_{\tau(0)}^{\tau(1)} h(t)e^{-j\omega t}dt + \int_{\tau(1)}^{\tau(2)} h(t)e^{-j\omega t}dt + \\ &\quad \cdots + \int_{\tau(N-1)}^{\tau(N)} h(t)e^{-j\omega t}dt \end{aligned}$$

If we adopt a rectangular approximation of the integral over each of the interval $[\tau(k-1), \tau(k)]$, then we will have

$$H_{nu}(\omega) \approx \sum_{k=0}^{\tau(N)} [\tau(k) - \tau(k-1)]\, h(\tau(k-1))e^{-j\omega(\tau(k-1))}. \tag{4}$$

Note that is not obvious to derive a z-transfer function from the samples $h(\tau(k))$; $k = 0, ..., N$, unless a new definition of the z-transform is established in the nonuniform sampling context.

3.1 Simulation result:

We have applied this method to the design of a low-pass digital filter of order 1 with cut-off frequency 1rad/s. That is we started the procedure with an analogue filter with the following transfer function:

$$H(s) = \frac{1}{s+1}$$

so that

$$h(t) = e^{-t}. \quad (5)$$

This is exactly the signal $x(t)$ described in (3). This was done on purpose as a follow up of the previous simulation results.

The impulse response $h(t)$ was sampled using the above nonuniform sampling sequence. The same mimimum sampling interval, $T_{\min} = 0.01s$ was employed as before so as to ensure that the digital filter and the analogue filter have matching magnitude responses. Note that the above filter has a cut-off frequency of 1rad/s. The value of the maximum sampling frequency was chosen 6 times greater than the Nyquist frequency, thus yielding $T_{\max} = 0.5s$. In practice, it is customary to use a sampling frequency 5 to 8 times greater than the Nyquist frequency [9]. The sampling was done over a fixed interval of time $[0, 7]$. As a result, 20 samples was obtained with the nonuniform sampling algorithm. For comparison purposes, the impulse response $h(t)$ was also sampled regularly using the sampling interval $T_{\min}$ and $T_{\max}$ over the same interval of time $[0, 7]$. Consequently, 14 samples was obtained when the sampling interval is $T_{\max}$ and 700 samples was obtained when the sampling interval is $T_{\min}$. The frequency responses of the three set of samples was calculated and simulated. Note the frequency response of the sampled signal $h(nT_{\min})$ is given by

$$H_{\min}(\omega) \approx T_{\min} \sum_{k=0}^{\tau(N)=7} h(kT_{\min}) e^{-j\omega k T_{\min}}. \quad (6)$$

and the frequency response of the sampled signal $h(nT_{\max})$ is given by

$$H_{\max}(\omega) \approx T_{\max} \sum_{k=0}^{\tau(N)=7} h(kT_{\max}) e^{-j\omega k T_{\max}}. \quad (7)$$

The three magnitude responses $|H_{nu}(\omega)|$, $|H_{\min}(\omega)|$ and $|H_{\max}(\omega)|$ was compared with that of the analogue signal:

$$|H(\omega)| = \frac{1}{\sqrt{\omega^2 + 1}}. \quad (8)$$

Figure 5 shows this comparison.. It can be seen that $|H(\omega)|$ and $|H_{\min}(\omega)|$ are very similar and coincident. This is to be expected since a huge number of samples were taken (700!) with the sampling interval $T_{\min}$. As for $|H_{nu}(\omega)|$ and $|H_{\max}(\omega)|$, it can be seen that $|H_{nu}(\omega)|$ has a much

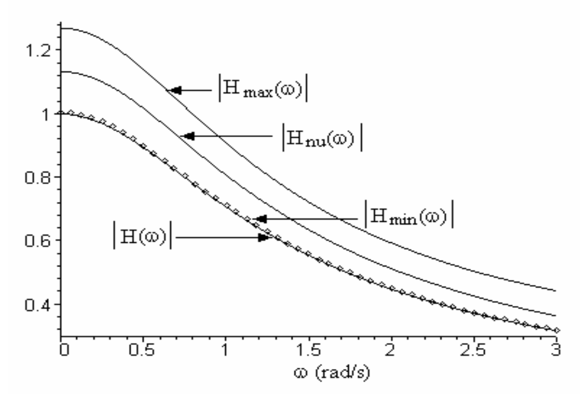

Figure 5: Comparison of magnitude responses, $\gamma = 2$

closer match to $|H(\omega)|$ than $|H_{\max}(\omega)|$. This is because the nonuniform sampling has allowed to take more 'relevant' information when the signal was fast varying than with the regular sampling (with sampling interval $T_{\max}$). In fact, a closer match will be obtained with the nonuniform sampling if either the weight γ is increased or the maximum allowable step size is decreased. Note that at $\omega = 0$ the magnitude responses $|H_{\max}(\omega)|$ and $|H_{nu}(\omega)|$ are not equal to 1. This is due to the effect of the truncation of the number of samples. For larger time intervals and larger number of samples, we would obviously have $|H_{\max}(0)| = |H_{nu}(0)| \approx 1$. This is shown in Figure 6 where the weight γ is increased ($\gamma = 15$) while the other numerical values were kept constant. In this case 35 samples were taken by the nonuniform sampling algorithm and it can be seen that $|H_{nu}(\omega)|$ gets closer to $|H(\omega)|$. In fact, even though not shown here, if ($\gamma = 20$) then 50 samples would have been taken and $|H_{nu}(\omega)|$ will match almost exactly to $|H(\omega)|$.

Consequently, the main advantage of the nonuniform sampling is that a comparable frequency response to that with $|H_{\min}(\omega)|$ or $|H(\omega)|$ is obtained but with less samples (here only 50 compared to 700!). Hence, the nonuniform sampling allows to design an FIR filter whose length is 14 times shorter than what is obtained with uniform sampling; with both filters having similar characteristiscs. In terms of memory requirement, one should store both the sampling instants and the sampling value in the nonuniform sampling case. Thus, for the present applications 100 terms need to be stored compared to the 700 samples in the uniform sampling case. Therefore, the nonuniform sampling reduces the memory space by 8 times.

4 Conclusions

In this paper, we have proposed a nonuniform sampling algorithm for one-dimensional bandlimited time varying signals. The main feature of the proposed sampling scheme is that the sampling steps are inversely proportional to signal

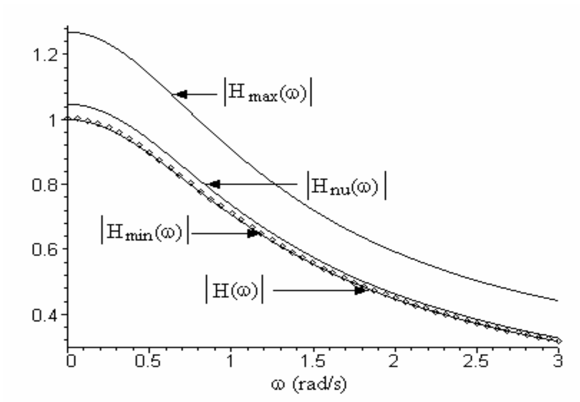

Figure 6: Comparison of magnitude responses, $\gamma = 15$

gradient or slope. As a result, a smaller sampling step is obtained whenever the gradient is high, and vice-versa. Thus, a better representation of the signal is obtained when it is linearly interpolated between the samples. The sampling algorithm was applied to the design of a digital FIR filter using the impulse invariance method. The filter thus obtained has a length that is 14 times shorter than a FIR filter obtained using uniform sampling and with similar characteristiscs. In addition, for that particular filter, a reduction of memory space by 8 times was also achieved. Finally, even though not demonstrated here, the algorithm can also be employed to design FIR filters using the frequency sampling method.

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