Mutual Inductance Between Coaxial Circular Coils of Rectangular Cross Section and Thin Coaxial Circular Coils with Constant Current Density in Air (Filament Method)

CEVDET AKYEL and SLOBODAN I. BABIC
Département de Génie Électrique
École Polytechnique de Montréal
Montréal C.P. 6079, Succ. Centre Ville
H3C 3A7 QUÉBEC, CANADA

Abstract: - This paper deals with an efficient and fast approach for determining the mutual inductance between coaxial circular coils of rectangular cross section in combination with thin coaxial circular coils as the thin wall solenoid, the thin disk coil and the filamentary coil. This approach is based on the filament method where conductors are approximated by the set of Maxwell's coils. The obtained expressions are expressed over the complete elliptical integrals of the first and second kind and permit fast calculation of the mutual induction for mentioned systems. These new expressions are accurate and simple for useful applications. The paper also points out the computational cost and the accuracy.

Key-Words: - Computational electromagnetics, mutual inductance, filament circular coils, disk coils, thin wall solenoids.

1 Introduction

Circular coils are widely used in various electromagnetic applications such as coil guns, tubular linear motors, single-layer coils, coils with rectangular cross section, current reactors, transmission lines and modern VLSI design systems. The mutual inductance as a fundamental electrical engineering parameter for a coil can be computed by applying the Biot - Savart law directly or using other alternate methods [1-23]. Before the advent of digital computers, calculations relied almost entirely on various power series [4-6]. Exact methods based on elliptic integral solutions for current loops, thin current cylinders, thin disks, massive coils have existed since at least the time of Maxwell but were laborious without computers. Today FEM and BEM methods are routinely used for magnetostatic problems, but these methods have accuracy problems near sharp surface singularities unless a high density of elements is used, [21]. The purpose of this article is to present an elliptic integral-based solution for coaxial different current loops involving coils of rectangular cross section for witch the mutual inductance will be calculated. The Maxwell's circular coils are used to replace the current loops that will be treated in this paper. This method is well known as the filament method, which leads to accurate expressions of the mutual inductance. The rapidity of this calculation is very important regarding to the computational cost and the accuracy.

2 Problem Formulation

The main idea of this method, called the Filament Method, is the using of Maxwell's coils where coils are divided into coaxial circular coils, [3-5], Fig. 1 for which the mutual inductance is given by the expression,

$$M_{Maxwell} = \frac{2\mu_0\sqrt{R_I R_{II}}}{k}[(1-\frac{k^2}{2})K(k) - E(k)] \qquad (1)$$

$$k^2 = \frac{4R_I R_{II}}{(R_I + R_{II})^2 + c^2}$$

In order to account for the finite dimensions of the coils, massive solenoids of rectangular cross section are considered to be subdivided into meshes of filamentary coils [1] as shown at Fig. 2.

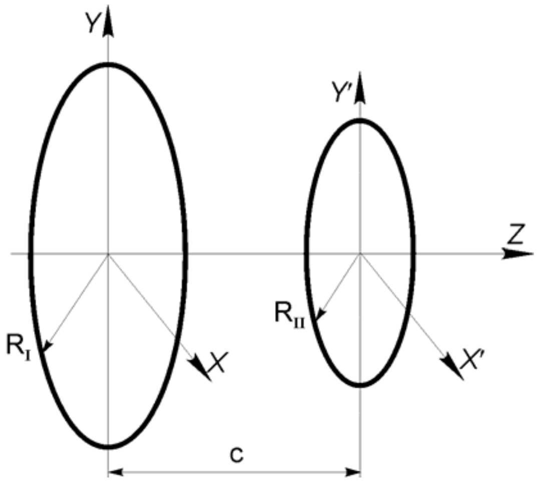

Fig 2. Maxwell's coils

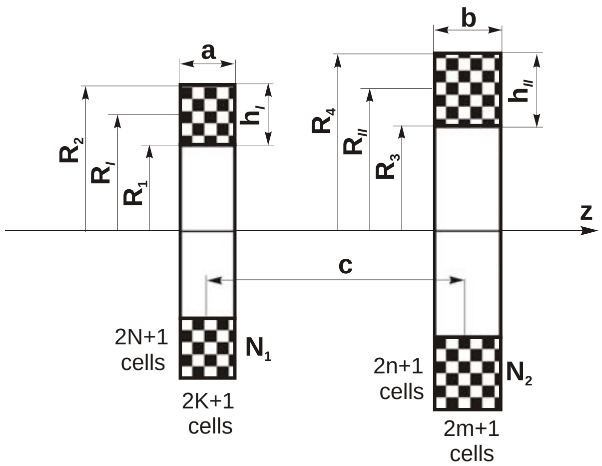

Fig.2. Configuration of mesh coils: Two circular coils of rectangular cross section

In [1] we showed the mutual inductance can be expressed in the following form,

$$M = \frac{N_1 N_2 \sum_{g=-K}^{g=K} \sum_{h=-N}^{h=N} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} M(g,h,p,l)}{(2K+1)(2N+1)(2m+1)(2n+1)} \qquad (2)$$

where $M(g,h,p,l)$ is the mutual inductance between two filamentary coils (Maxwell's coils), [4-6] that form the coil of rectangular cross section,

$$M(g,h,p,l) = \frac{\mu_0 \sqrt{R_{11}(h) R_{22}(l)}}{k} [(2-k^2)K(k) - 2E(k)]$$

where

$$R_{11}(h) = R_I + \frac{h_I}{(2N+1)} h\,, \; h = -N,...,0,...,N$$

$$R_{22}(l) = R_{II} + \frac{h_{II}}{(2n+1)} l\,, \; l = -n,...,0,...,n$$

$$R_I = \frac{R_1 + R_2}{2}, \; R_{II} = \frac{R_3 + R_4}{2}$$

$$h_I = R_2 - R_1, \; h_{II} = R_4 - R_3$$

$$z(g,p) = c - \frac{a}{(2K+1)} g + \frac{b}{(2m+1)} p$$

$$g = -K,...,0,...,K \; ; \; g = -m,...,0,...,m$$

$$k^2(g,h,p,l) = \frac{4R_{11}(h) R_{22}(l)}{(R_{11}(h) + R_{22}(l))^2 + z^2(g,p)}$$

$E(k)$ and $K(k)$ are complete elliptic integrals of the first and second kinds, [24-25].
The formula (2) is the basic formula to obtain all possible combinations of coaxial circular coils either finite or infinite thickness.

3 Problem Solution

3.1 Circular coil of rectangular cross section and thin wall solenoid

Let us consider the system: the wall solenoid and the circular coil of rectangular cross section with N_1, N_2 number of turns, respectively. Dimensions of this system are shown at Fig. 3.
This system can by obtained from (2) replacing $h_I = 0$, $R_{11} = R$ for which the corresponding mutual inductance is,

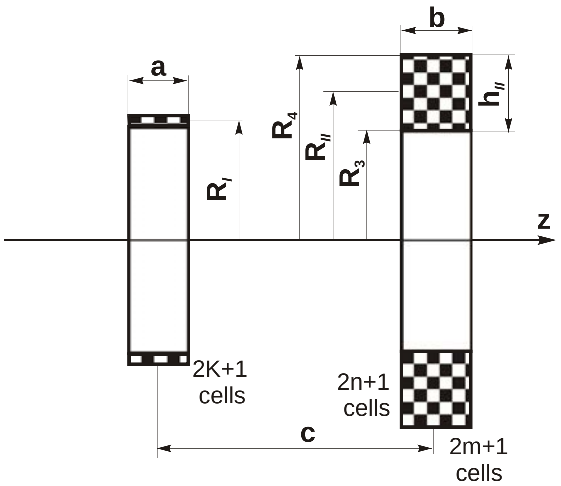

Fig 3. Configuration of mesh matrix: Thin wall solenoid-circular coil of rectangular cross section.

$$M = \frac{N_1 N_2 \sum_{g=-K}^{g=K} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} M(g,p,l)}{(2K+1)(2m+1)(2n+1)} \qquad (3)$$

where

$$M(g,p,l) = \frac{\mu_0 \sqrt{R_{11} R_{22}(l)}}{k(g,p,l)} [(2-k^2)K(k) - 2E(k)]$$

$$R_{11} = R_I = R, \quad R_{22}(l) = R_{II} + \frac{h_{II}}{2n+1} l$$

$$R_{II} = \frac{R_4 + R_3}{2}, h_{II} = R_4 - R_3$$

$$z(g,p) = c - \frac{a}{2K+1} g + \frac{b}{2m+1} p$$

$$k^2(g,p,l) = \frac{4 R_{11} R_{22}(l)}{(R_{11} + R_{22}(l))^2 + z(g,p)^2}$$

3.2 Circular coil of rectangular cross section and thin disk coil

Let us consider the system: the thin disk coil and the circular coil of rectangular cross section with N_1, N_2 number of turns, respectively. Dimensions of this system are shown at Fig. 4.

This system can by obtained from (2) replacing $a = 0$ for which the corresponding mutual inductance is,

$$M = \frac{N_1 N_2 \sum_{n=-N}^{h=N} \sum_{p=-m}^{p=m} \sum_{l=-n}^{l=n} M(h,p,l)}{(2N+1)(2m+1)(2n+1)} \qquad (4)$$

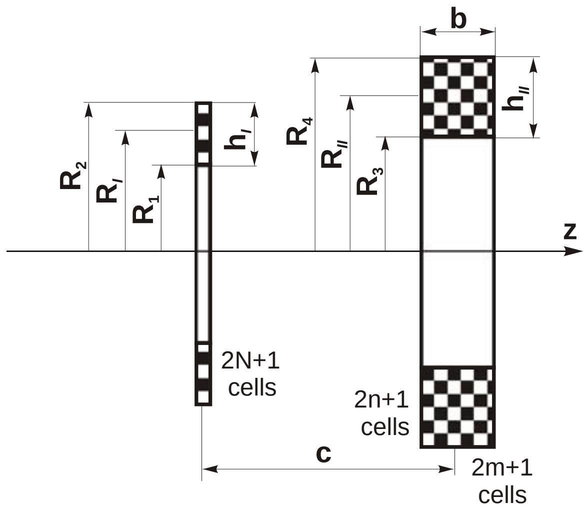

Fig 4. Configuration of mesh matrix: Thin disk coil-circular coil of rectangular cross section.

where

$$M(h, p.l) = \frac{\mu_0 \sqrt{R_{11}(h) R_{22}(l)}}{k(h,p,l)} [(2-k^2)K(k) - 2E(k)]$$

$$R_{11}(h) = R_I + \frac{h_I}{2N+1} h, \quad h_I = R_2 - R_1$$

$$R_I = \frac{R_2 + R_1}{2}, \quad R_{22}(l) = R_{II} + \frac{h_{II}}{2n+1} l$$

$$R_{II} = \frac{R_4 + R_3}{2}, \quad h_{II} = R_4 - R_3$$

$$z(p) = c + \frac{b}{2m+1} p$$

$$k^2(h,p,l) = \frac{4 R_{11}(h) R_{22}(l)}{(R_{11}(h) + R_{22}(l))^2 + z(p)^2}$$

3.3 Circular coil of rectangular cross section and filamentary circular coil

Let us examine the system: the filamentary coil and the circular coil of rectangular cross section with N_2 being number of turns. Dimensions of this system are shown in Fig. 5.

This system can by obtained from (2) replacing replacing $N_1 = 1$, $h_I = 0$, $R_{11} = R$, $a = 0$ for which the corresponding mutual inductance is,

$$M = \frac{N_2}{(2m+1)(2n+1)} \sum_{l=-n}^{l=n} \sum_{p=-m}^{p=m} M(l,p) \qquad (5)$$

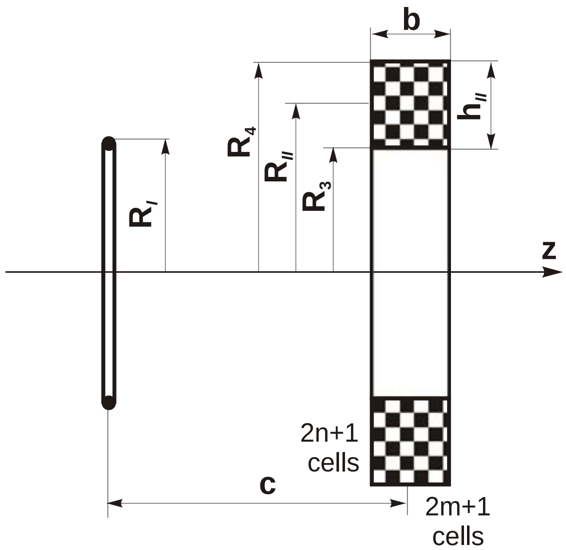

Fig.5. Configuration of mesh matrix: Filamentary circular coil-circular coil of rectangular cross section.

where

$$M(l,p)=\frac{\mu_0\sqrt{R_{11}R_{22}(l)}}{k(l,p)}[(2-k^2)K(k)-2E(k)]$$

$$R_{11}=R_I=R, \quad R_{22}(l)=R_{II}+\frac{h_{II}}{2n+1}l$$

$$h_{II}=R_4-R_3, \quad z(p)=c+\frac{b}{2m+1}p$$

$$k^2(l,p)=\frac{4R_{11}R_{22}(l)}{(R_{11}+R_{22}(l))^2+z(p)^2}$$

4 Examples

4.1 Example 1 (Two circular coaxial coils of rectangular cross section)

Consider two coils of 200 turns each (Brooks coils), mean radii R_{mean} = 10 cm, separation between their median planes d = 6 cm, and side of the square cross section r = 6 cm, [4]. Calculate their mutual inductance.

From [4] the mutual inductance is

$$M = 3.7384\ mH$$

The mutual inductance of these coils can be calculated analytically [23]

$$M = 3.73728054\ mH$$

Execution time was 0.61 s.

In Table 1 we show values of the mutual inductance using the proposed filament method, expression (2). Also the corresponding computational time and the absolute error of calculation regarding the exact value are given. It is obvious that the number of subdivisions has a big influence on the computational cost but one does not need to make enormous subdivisions to have the satisfactory accuracy.

Table1. Comparison of computational efficiency

K/N/m/n Subdiv.	$M_{Filament}$ (mH)	Computational Time (Seconds)	Error (%)
All 2	3.73647532	0.330	0.0355
All 4	3.73703152	3.399	0.0202
All 8	3.73721070	42.730	0.0152
All 15	3.73725953	503.660	0.0139
All 20	3.73726853	2733.810	0.0135

With only K = N = m = n = 8 we obtain satisfactory accuracy and computational time. The further increasing of subdivisions will considerably increase the computational time but not the accuracy.

4.2 Example 2 (Circular coil of rectangular cross section and thin wall solenoid)

Given a thin solenoid of radius R = 10 cm, having azimuthal length a = 40 cm with N_1 = 1000 turns and a circular coil of rectangular cross section of radius R_1 = 30 cm, R_2 = 50 cm, having azimuthal length b = 80 cm, N_2 = 1000 turns, calculate the mutual inductance of this system.

The exact method [26] gives the mutual inductance,

$$M = 0.21586877\ H$$

The execution time was 0.05 s.

Table V shows values for mutual inductance using the filament method, expressions (3). Also the corresponding computational time and the absolute error of calculation regarding to the exact value [26] are given. From Table 2 one can see that all results are in good agreement but the best accuracy by the filament method can be achieved for the same number of subdivisions. For $K = m = n = 30$ the computational time was 127.79 s and absolute error 0.00397 %. The further increasing of subdivisions will considerably increase the computational time but not the accuracy.

Table2. Comparison of computational efficiency

K/m/n Subdivisions	$M_{Filament}$ *(H)*	*Computational Time (Seconds)*	*Error (%)*
5/10/10	0.21597876	2.72	0.05095
10/15/15	0.21590970	11.26	0.01896
15/30/30	0.21588225	64.93	0.00624
30/30/30	0.21587734	127.79	0.00397
30/50/50	0.21587299	370.80	0.00195
40/70/70	0.21587103	980.80	0.00104
60/80/80	0.21587019	1914.70	0.00066
100/100/100	0.21586956	5484.05	0.000037

4.3 Example 3 (Circular coil of rectangular cross section and thin disk coil)

Given a thin disk of radius R_1 = 1 cm, R_2 = 5 cm, with N_1 = 100 turns and a circular coil of rectangular cross section of radius R_3 = 6 *cm*, R_4 = 8 cm, having azimuthal length a = 5 *cm* and N_2 = 100 turns, calculate the mutual inductance of this system.

The exact method proposed in [27] gives the mutual inductance,

$$M = 249.92384781\ \mu H$$

The execution time was 0.3 s.

Table3. Comparison of computational efficiency

N/m/n Subdivisions	$M_{Filament}$ *(μH)*	*Computational Time (Seconds)*	*Error (%)*
10/10/10	249.8991787	6.24	0.08244
20/20/20	249.9166741	55.61	0.00287
30/30/30	249.9200823	151.92	0.00151
50/50/50	249.9218660	688.41	0.00079
70/70/70	249.9223646	1901.52	0.00059
100/100/100	249.9226315	7478.63	0.00049

Table 3 shows values for mutual inductance using the filament method, expressions (4). Also the corresponding computational time and the absolute error of calculation regarding to the exact value [27] are given. All results are in good agreement regarding accuracy and computational time. It is enough to take $N = m = n = 20$ subdivisions to have fast and accuracy results.

4.4 Example 4 (Circular coil of rectangular cross section and filamentary circular coil)

Consider the filamentary coil of radii R = 2 cm that is at plane z_Q = 0 cm. The circular coil of rectangular cross section of radius R_1 = 4 cm, R_2 = 6 cm and length (z_2 - z_2) where z_1 = - 0.5 cm, z_2 = 0.5 cm with $N_2 = N$ = 100 turns, [11]. Calculate their mutual inductance.

The exact method [11] gives the mutual inductance,

$$M = 1.70465909\ \mu H$$

The computational time was 0.13 s.

Applying the filament method (5), we obtain the mutual inductance values for different number of subdivisions, Table 4. The corresponding computational time and absolute error of the calculation with respect to exact value [11] are also given. From Table 4, one can conclude that all results obtained using either expression (5) are in excellent agreement with the given results in [14].

Table 4. Comparison of computational efficiency

m Subd	*n* Subd	$M_{Filament}$ *(μH)*	*Computational Time (Seconds)*	*Error (%)*
2	2	1.70378587	0.01999	0.05125
4	4	1.70438877	0.04999	0.01586
10	10	1.70460938	0.29	0.00292
15	15	1.70463627	0.61	0.00134
25	25	1.70465065	1.69	0.00050
50	50	1.70465693	6.55	0.00013
100	100	1.70465853	25.849	0.00003
200	200	1.70465894	104.37	0.0000086

4 Conclusion

The efficient and fast procedures for the calculation of the mutual inductance of coaxial current loops are presented. These procedures are based on an elliptic integral-based approach so that one can use them to calculate practically the mutual inductance of all possible coaxial conductors of rectangular cross section with coaxial circular coils of infinity thickness or finite thickness. This method known as the filament method permits to calculate precisely and quickly the mutual inductance of previously mentioned conductors. From obtained results one can conclude that in many cases the mutual inductance has been calculated precisely with few subdivisions of conductors that directly has an influence on the computational cost. All results obtained by proposed approach have been compared to well-known methods. In many presented examples we show a good agreement with already published data. For practical engineering applications this approach can be considered as easy, lucid and rapid. All software is coded in MATLAB. With a personal computer, one can obtain nice and rapid results in calculating the mutual inductance of all presented coaxial current loops that are interesting in engineering applications.

Acknowledgment

We would like to thank the Natural Science and Engineering Research Council of Canada (NSERC) that supported this work under Grant RGPIN 4476-05 NSERC NIP 11963.

References:

[1] C.Akyel, S. I. Babic, Ke Wu and F. J. Mojca,"Improvement in the Mutual Inductance Calculation of Thin Coaxial Circular Coils with Constant Current Density inAir",***WSEAS Transactions on Communications***, Issue 6, Vol.5,June2006, ISSN:1109-2742, pp.970-977.

[2] S.I. Babic and C. Akyel, "An Improvement in the Calculation of the Self-Inductance of Thin Disk Coils with Air-Core", ***WSEAS Transaction On Circuits and Systems***, Issue 8, Vol. 3, October 2004, ISSN: 1109-2734, pp. 1621-1626.

[3] S.I. Babic and C. Akyel, "Magnetic Force between Circular Filament Coil and Massive Circular Coil with Rectangular Cross Section", ***WSEAS Transaction On Circuits and Systems***, Issue 6, Vol. 4, June 2005, ISSN: 1109-2734, pp. 610-617.

[4] F. W. Grover, "*Inductance Calculations*", New York: Dover, 1964, chs.2 and 13.

[5] H.B.Dwight, "*Electrical Coils and Conductors*", McGraw-Hill Book Company, New York, 1945.

[6] Chester Snow, "*Formulas for Computing Capacitance and Inductance*", National Bureau of Standards Circular 544, Washington DC, December 1954.

[7] G. Zhong and C. Koh, "Exact Closed Form Formula for Partial Mutual Inductance of On-Chip Interconnects", Proceedings of the 2002 *IEEE* International Conference on Computer Design: VLSI in Computers and Processors, ICCD 2002.

[8] Ki-Bong Kim, E. Levi, Z. Zabar and L. Birenbaum, "Mutual Inductance of Noncoaxial Circular Coils with Constant Current Density", ***IEEE Transactions On Magnetics***, Vol. 33, No. 5, pp. 3916-3921, September 1997.

[9] Z. Zabar, X.N.Lu., E. Levi, L. Birenbaum, and J. Creedon, "Experimental Results and Performance Analysis of a 500 m/sec Linear Induction Launcher (LIL)",***IEEE Transactions On Magnetics***, Vol. 31, No. 1, pp. 522-527, January 1995.

[10] M. Liao, Z. Zabar, E. Levy, and L. Birenbaum, "Numerical Calculation of the Inductance of Circular Coils", 5^{th} Symposium on Electromagnetic Launch Technology, Touluse, France, April, 10-13, 1995.

[11] S.Babic, C.Akyel and S. J. Salon, "New Procedures for Calculating the Mutual Inductance of the system: Filamentary Circular Coil-Massive Circular Solenoid", ***IEEE Transactions On Magnetics***, Vol. 38, No. 5, pp. 1131-1134, May, 2003.

[12] S.Babic, S.Salon and C.Akyel, "The Mutual Inductance of Two Thin Coaxial Disk Coils in Air", ***IEEE Transactions on Magnetics***, Vol. 40, No. 2, pp. 822-825, March 2004.

[13] C.Akyel, S.Babic and S. Kincic, "New and Fast Procedures for Calculating the Mutual Inductance of Coaxial Circular Coils (Disk Coil-Circular Coil)", ***IEEE Transactions On Magnetics***, Vol. 38, No. 5, Part 1, pp. 1367-1369, September, 2002.

[14] S. Babic and C. Akyel, "An Improvement in Calculation of the Self-and Mutual Inductance of Thin-Wall Solenoids and Disk Coils", ***IEEE Transactions On Magnetics***, Vol. 36, No.4, pp. 678-684, July 2000.

[15]Dingan Yu and K. S. Han,"Self-Inductance of Air-Core Circular Coils with Rectangular Cross - Section", ***IEEE Transactions On Magnetics***, Vol. 23, No. 6, pp. 3916-3921, November 1987.

[16]A. V. Kildishev,"Application of Spheroidal Functions in Magnetostatics", ***IEEE Transaction on Mag*****netics**, Vol. 40, No. 2, pp.-846-849, March 2004.

[17] C.A. Borghi, U. Reggiani, G. Zama, "Calculation of Mutual Inductance by Means of the Toroidal Multipole Expansion Method", ***IEEE Transactions On Magnetics***, Vol. 25, No. 4, pp. 2992-2994, July 1989.

[18] Garet M. W. Calculation of Fields, "Forces and Mutual Inductances of Current Systems by Elliptic Integrals", ***Journal of Applied Physics***, Vol. 34. pp. 2567-2573, 1963.

[19] Kazuhiro Kajikawa and Katsuyuki Kaiho, "Usable Ranges of Some Expressions for Calculation of the Self-Inductance of a Circular Coil of Rectangular Cross Section, Cryogenic Engineering", Vol. 30, No. 7 (1995), pp. 324-332, (In Japanese).

[20] L. Bottura, "Inductance Calculation for Conductors of Arbitrary Shape", CRYO /02 /028, April, 5, 2002.

[21] J. T.Conway, "Exact Solutions for the Magnetic Fields of Axisymmetric Solenoids and Current Distributions",***IEEE Transactions On Magnetics***, Vol. 37, No.4, pp. 2977-2988, July 2001.

[22] S. I. Babic and C. Akyel" New mutual inductance calculation of the magnetically coupled coils: Thin disk coil- Thin wall solenoid", ***Journal of Electromagnetic Waves and Applications***, Vol. 20, No.10, 1666-1669, 2006.

[23] S. I. Babic and C.Akyel, "New Analytic-Numerical Solutions for the Mutual Inductance of Two Coaxial Circular Coils with Rectangular Cross Section in Air", ***IEEE Trans. on Magnetics***, Vol. 42, No. 6, pp. 1661-1669, June 2006.

[24] M. Abramowitz and I. A. Stegun, "*Handbook of Mathematical Functions*", National Bureau of Standards Applied Mathematics, Washington DC, December 1972, Series 55, p. 595.

[25] I.S.Gradshteyn and I.M.Ryzhik, "*Table of Integrals, Series and Products*", New York and London, Academic Press Inc., 1965.

[26] S.I.Babic and C. Akyel, " New Mutual Inductance Calculation of the Magnetically Coupled Coils: Thin wall Solenoid – Circular Coil with Rectangular Cross Section", (ready to be submitted).

[27] S.I.Babic and C. Akyel," Novel Mutual Inductance Calculation of the Magnetically Coupled Coils: Circular Coil of Rectangular Cross Section and Thin Disk Coil", (submitted to Electrical Engineering-Springer, 2006).

This paper is deduced to genius Nikola Tesla.