Combined Techniques for 3D Stresses Smoothing

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Abstract: In this work it is presented the combined application of improved SPR-R and REP-R techniques, individually implemented in [1], for smoothing the tensions on hexahedrons for three-dimensional domain problems. This allows obtaining more exact results when using the finite elements method. In order to analyze the combination behavior of these techniques, an h_adaptive procedure is made, using the Zienkiewicz-Zhu estimator (Z^2) [2] for estimating the error that involves the smoothing stress range. In this way, the effectiveness indices and the error convergence for problems with smooth solution are analyzed and a case where the solution is singular is presented. The convergence results and effectiveness index show that both techniques separately are computationally analogous when the meshes have elevated degrees of freedom as appears in [3]. The most suitable composition for the sphere global solution is: SPR-R to REP-R, whereas for the crack is the REP-R to SPR-R composition. In both cases, these combinations display that greater convergence speed and effectiveness index threshold can be obtained with smaller amount of refinements.

Key-Words: Superconvergent Patch Recovery, Tensions Smoothing, Error estimator, Convergence in Finite Elements.

1 Introduction

The error is intrinsic in the simulations for finite elements, since the discretization carried out when transforming a continuous model into a discreet one cannot capture all the information required to obtain an exact solution. A very important progress toward the efficient techniques of postprocess that improve the solution for finite elements (EF) has been presented by Zienkiewicz and Zhu [4] with the procedure of tensions recovery "superconvergent patch recovery" (SPR), which is based on the tensions smoothing for areas; in this a polynomial expansion describes the tensions, using a denominated group of contiguous elements "patch", around the nodes where it is wanted to carry out the tensions smoothing. This expansion is made using Minimum Square and the tensions are calculated starting from the values of tension evaluated by means of EF in the points of numeric integration. After, many authors have presented improvements like those carried out by: Lee and Him [5], Blacker ET to the [6], Wiberg ET to the [7], [8] and Ródenas [9]. Another technique that looks for to balance the tensions improved in the patch in the same way that in the EF method and with a similar procedure to the one used in the SPR, is the called "Recovery by Equilibrium Patches (REP) proposed by Booromand and Zienkiewcz [10], which has also been improved (REP-R) considering the contour conditions and it is implemented individually the same as the SPR-R in [1]. Additionally, in [11] another interesting technique is presented (LP), which is a combination of SPR and REP techniques, where according to its authors this presents more accurate tension values that the SPR overalls if it considered problems with singular solution in the two-dimensional domain.

This work presents the combined implementation from the REP-R to SPR-R techniques and viceversa, similar to that suggested in the technical LP, with the difference that REP-R and SPR-R are implemented for separate and is considered for three-dimensional domains. For defining the kindness of the composition, the effectiveness and convergence indexes are presented in a model with smoothed solution and another with singular solution. This article is organized in five sections. In section 2 the formulation of the technical SPR-R and REP-R in function of the improvements developed by Ródenas [9] is presented. In section 3 the parameters to carry out the numeric verification of the combination of technical and the used models are depicted. Later on, in the section 4 the analysis of simulation obtained results are shown and finally, in the section 5 the appropriate conclusions are presented.

2 SPR-R and REP-R Techniques

These techniques are recent; in them the restrictions of tension imposed in the nodes of the contour with a small cost extra computational regarding the technical SPR and original REP are completed exactly, and they are implemented with hexahedrons in [1]. These are based on their original formulation, like can be detailed in [4] and [10], and their improvement can be summarized as: in the tensions interpolation polynomial in patch ($\boldsymbol{\sigma}^*_p$) each one of their components (σ_x, σ_y, σ_z, σ_{xy}, σ_{xz}, σ_{yz}) are expressed according to:

$$\begin{aligned}\sigma^*_{pi} &= \alpha_{i1} + \alpha_{i2}x + \alpha_{i3}y + \alpha_{i4}z + \alpha_{i5}x^2 + \alpha_{i6}y^2 + \\ &+\alpha_{i7}z^2 + \ldots \qquad i = x,\ y,\ z,\ xy,\ xz,\ yz\end{aligned} \tag{1}$$

The exact execution of the restrictions of tension imposed in the nodes of patch assembling can be forced incorporating, in the equations system to be solved, the known values of this restrictions. For this, in the nodes patch's located in the contour, the tensions interpolation polynomials of ($\boldsymbol{\sigma}^*_p$) will be outlined in a system of local reference of axes ξ, η and ζ with the origin in the node of patch assembling and with the axes guided according to the normal and tangential addresses to the contour. To outline the equations of the technical SPR in the new coordinated system it is carried out an adjournment of the coordinated system and two turns, one in y and then in x. Therefore, the local coordinates (ξ, η, ζ) corresponding to a point of global coordinated (x, y, z) will be evaluated, by means of the expression:

$$\begin{Bmatrix}\xi\\ \eta\\ \zeta\end{Bmatrix} = \mathbf{Rot}\begin{Bmatrix}x-x_n\\ y-y_n\\ z-z_n\end{Bmatrix} \tag{2}$$

Where x_n. y_n, z_n are the global coordinates of the patch assembling node and the rotation matrix comes given by:

$$\mathbf{Rot} = \begin{bmatrix}\cos\phi & 0 & -sen\phi\\ -sen\theta\, sen\phi & \cos\theta & -sen\theta\cos\phi\\ \cos\theta\, sen\phi & sen\theta & \cos\theta\cos\phi\end{bmatrix} \tag{3}$$

The angles ϕ, θ define the contour normal direction regarding the global axes (x, y, z). The tensions in any point of the patch expressed in the new coordinated system will be evaluated by means of the expression:

$$\begin{bmatrix}\sigma_\xi & \sigma_{\xi\eta} & \sigma_{\xi\zeta}\\ \sigma_{\xi\eta} & \sigma_\eta & \sigma_{\eta\zeta}\\ \sigma_{\xi\zeta} & \sigma_{\eta\zeta} & \sigma_\zeta\end{bmatrix} = \mathbf{Rot}\begin{bmatrix}\sigma_x & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_y & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_z\end{bmatrix}\mathbf{Rot}^T \tag{4}$$

The inverse transformation for obtaining σ_x, σ_y, σ_z, σ_{xy}, σ_{xz}, σ_{yz} starting from σ_ξ, σ_η, σ_ζ, $\sigma_{\xi\eta}$, $\sigma_{\xi\zeta}$, $\sigma_{\eta\zeta}$ is calculated carrying out the inverse outlined in (4).

$$\begin{bmatrix}\sigma_x & \sigma_{xy} & \sigma_{xz}\\ \sigma_{xy} & \sigma_y & \sigma_{yz}\\ \sigma_{xz} & \sigma_{yz} & \sigma_z\end{bmatrix} = \mathbf{Rot}^T\begin{bmatrix}\sigma_\xi & \sigma_{\xi\eta} & \sigma_{\xi\zeta}\\ \sigma_{\xi\eta} & \sigma_\eta & \sigma_{\eta\zeta}\\ \sigma_{\xi\zeta} & \sigma_{\eta\zeta} & \sigma_\zeta\end{bmatrix}\mathbf{Rot} \tag{5}$$

The new reference system is appropriate in order to force the exact execution of tension restrictions imposed in the patch assembling nodes, since the contour normal tensions are known.

In the polynomial of interpolation of tensions in the patch given by equation (1), each one of their components in the new system of coordinated σ_ξ, σ_η, σ_ζ, $\sigma_{\xi\eta}$, $\sigma_{\xi\zeta}$, $\sigma_{\eta\zeta}$ and their corresponding derivatives with respect to this coordinates can be expressed, for example as a second order polynomial, as:

$$\begin{aligned}\sigma^*_{pi} &= \alpha^*_{i1} + \alpha^*_{i2}\xi + \alpha^*_{i3}\eta + \alpha^*_{i4}\zeta + \alpha^*_{i5}\xi^2 + \alpha^*_{i6}\eta^2 + \\ &+\alpha^*_{i7}\zeta^2 + \alpha^*_{i8}\xi\eta + \alpha^*_{i9}\eta\zeta + \alpha^*_{i10}\xi\zeta\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\partial\sigma^*_{pi}}{\partial\xi} &= \alpha^*_{i2} + 2\alpha^*_{i5}\xi + \alpha^*_{i8}\eta + \alpha^*_{i10}\zeta\\ \frac{\partial\sigma^*_{pi}}{\partial\eta} &= \alpha^*_{i3} + 2\alpha^*_{i6}\eta + \alpha^*_{i8}\xi + \alpha^*_{i9}\zeta\\ \frac{\partial\sigma^*_{pi}}{\partial\zeta} &= \alpha^*_{i4} + 2\alpha^*_{i7}\zeta + \alpha^*_{i9}\eta + \alpha^*_{i10}\xi\\ i &= \xi,\ \eta,\ \zeta,\ \xi\eta,\ \xi\zeta,\ \eta\zeta\end{aligned} \tag{7}$$

Now it can be forced the exact execution of the tension restrictions imposed in the assembling nodes considering the expressions (6) and (7) particularized in this node. The previous expressions evaluated in the patch assembling node of coordinates $\xi = 0$, $\eta = 0$, $\zeta = 0$ are:

$$\begin{aligned}&\sigma^*_{pi}\big|_0 = \alpha^*_{i1}\\ &\frac{\partial\sigma^*_{pi}}{\partial\xi}\Big|_0 = \alpha^*_{i2},\quad \frac{\partial\sigma^*_{pi}}{\partial\eta}\Big|_0 = \alpha^*_{i3},\quad \frac{\partial\sigma^*_{pi}}{\partial\zeta}\Big|_0 = \alpha^*_{i4}\end{aligned} \tag{8}$$

If the value of some of the tensions or of their derivatives is known in the patch assembling node, then the corresponding value of the coefficient α^* of the outlined equations system is eliminated, condensing this system.

3 Implementation of the Combination

To carry out the study of the behavior of the combined techniques SPR-R to REP-R and vice versa in a particular problem, it is necesary to have the exact solution of this problem. Because if it is known the exact solution of the problem from the thick wall sphere exposed to inner pressure, this will be analyzed initially with each composition. This way, it can be obtained data of effectiveness that allow, together with the convergence values, to determine which presents the best results of the combinations. Later on, the problem of plate with crack will be analyzed, using the combinatory technique REP-R to SPR-R, since in [1] it is demonstrated that for this type of problems the technical REP-R presents better results of effectiveness (θ) and convergence that the SPR-R. To study the combinatory behavior, it should be determined the relative error and the index of effectiveness, which will be detailed next.

The estimated relative error in the energy norm at global level is expressed according to:

$$\eta\% = \frac{\|e_{es}\|}{\left(\|u_{ef}\|^2 + \|e_{es}\|^2\right)^{1/2}} \cdot 100 \qquad (9)$$

Where $\|e_{es}\|$ it represents the estimated error, Z^2 and $\|u_{ef}\|$ is the energy norm of the finite elements solution that corresponds to the square root of double the deformation energy:

$$\| u_{ef} \| = \sqrt{\int_V \sigma_{ef}{}^T \, \mathbf{D}^{-1} \, \sigma_{ef} \, dV} \qquad (10)$$

Where $\mathbf{D}$ it is the material properties matrix. So, the convergence speed is obtained as the slope of the generated curve according to the logarithm of the relative error ($\eta\%$ = logη %) against the logarithm of the freedom degrees (*GDL* = log *GDL*).

The index of effectiveness θ represents the relationship among the values of the estimated error (Z^2) and the exact error and is obtained according to:

$$\theta = \frac{\|e_{es}\|}{\|e_{ex}\|} \qquad (11)$$

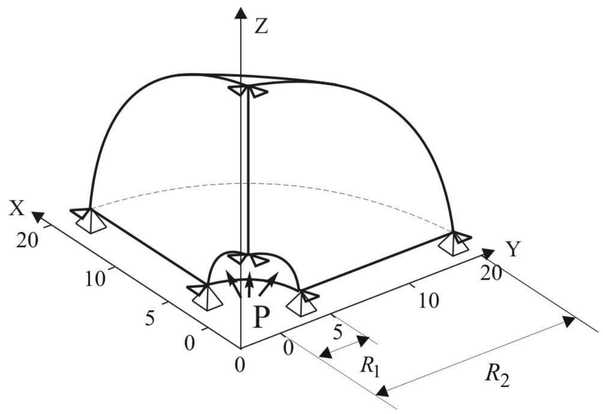
Where $\|e_{ex}\|$ represents the exact error and it is calculated according to:

$$\| e_{ex} \| \approx \sqrt{\| u_{ex} \|^2 - \| u_{ef} \|^2} \qquad (12)$$

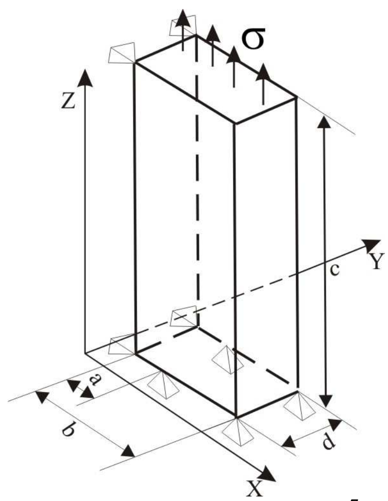
$\|u_{ex}\|$ represents the energy norm of the exact solution.

3.1 Models Used for Verification

To evaluate each one of the necessary parameters in the technical SPR-R to REP-R combinatory and vice versa, two examples are used: thick wall sphere exposed to inner pressure and a plate with crack and plate loaded to tension with lateral crack, which have the dimensions and characteristic of the material that are shown in Fig 1. The value of the square of the energy of exact deformation that is necessary for the computation of (9) and (10) has been obtained analytically for the sphere, while in the crack it has been the result of a very refined mesh. For the numeric analyses a model will be used that corresponds to 1/8 of the sphere. The contour conditions are imposed as is shown in the Fig. 1 for both problems.



a. R_1=5, R_2=20, P=1, E=1000, υ=0.3,
$\|\mathbf{u}_{ex}\|^2 = 0.1308996939$



b. σ=1000, a=0.6, b=2, c=6, d=1, E=10^7, υ=0.333,
$\|\mathbf{u}_{ex}\|^2 = 0.112007$

Fig. 1- a. Sphere exposed to inner pressure
b. Plate with lateral crack

4. Results Analysis

4.1 Results for the Problem With Smoothed Solution

Next the results of the different numeric studies are presented carried out in the sphere exposed to inner pressure, where it is presented the index of effectiveness and convergence in the combination in sphere SPR-R to REP-R (see Fig. 2) with linear elements. In Fig. 3, the obtained meshes are presented in the h-adaptative procedure. Later on, similar results with the combinatory REP-R to SPR-R is presented (see Fig. 4 and Fig. 5); in these results there are only shown the last three meshes since the first ones are identical to the initial combination.

When carrying out an analysis of the evolution of the reliability according to the Fig. 2, it is observed that when applying the technical REP-R immediately after the SPR-R the obtained results makes that the reliability index is closer to the unit. The magnitude of the convergence speed is similar in the first refinements, however when increasing the GDL it diminishes until reaching a value of 0.28.

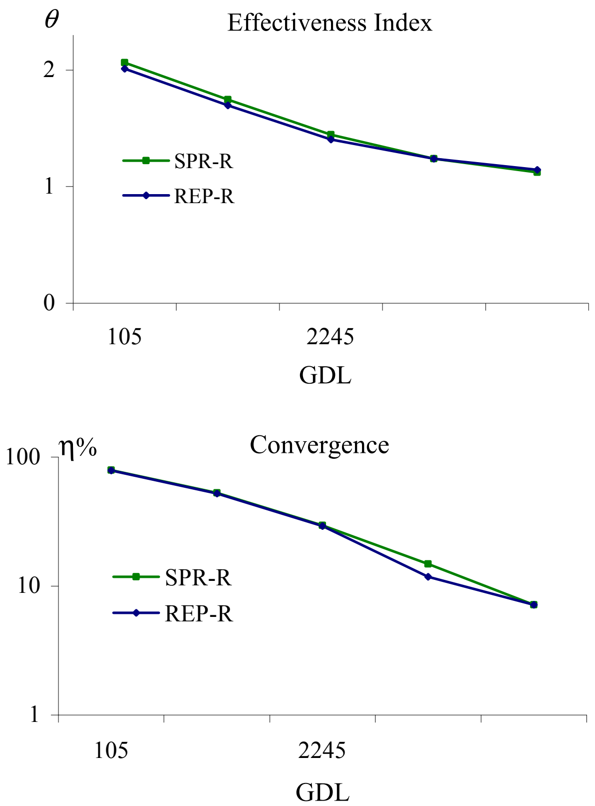


Fig. 2 - Combination in sphere SPR-R to REP-R

Concerning the reliability evolution in Fig. 4, it is observed that when applying the technical SPR-R after the REP-R it doesn't improve the reliability index, since this is superior with SPR-R in all the mesh refinements. The magnitude of the convergence speed is almost identical; the average is of 0.35 for all the refinements. Regarding the number of elements for mesh, the first combination uses smaller quantity to reach the similar magnitudes of θ and convergence speed, like it can be observed in the Fig. 3.

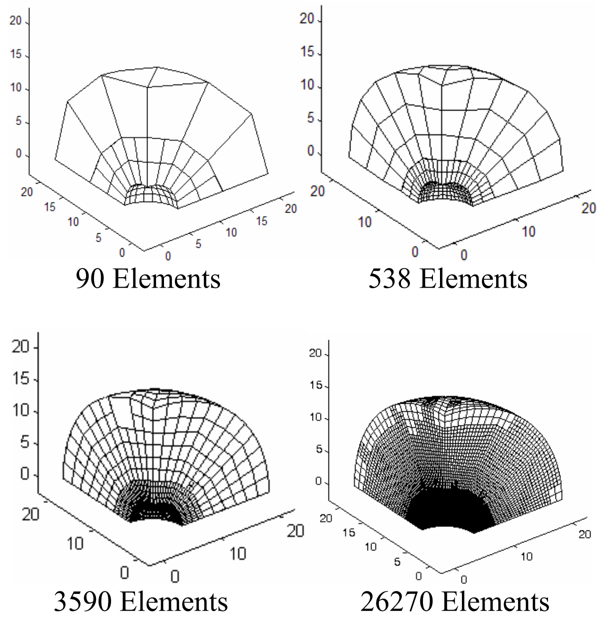


Fig. 3 - Meshes for sphere SPR-R to REP-R

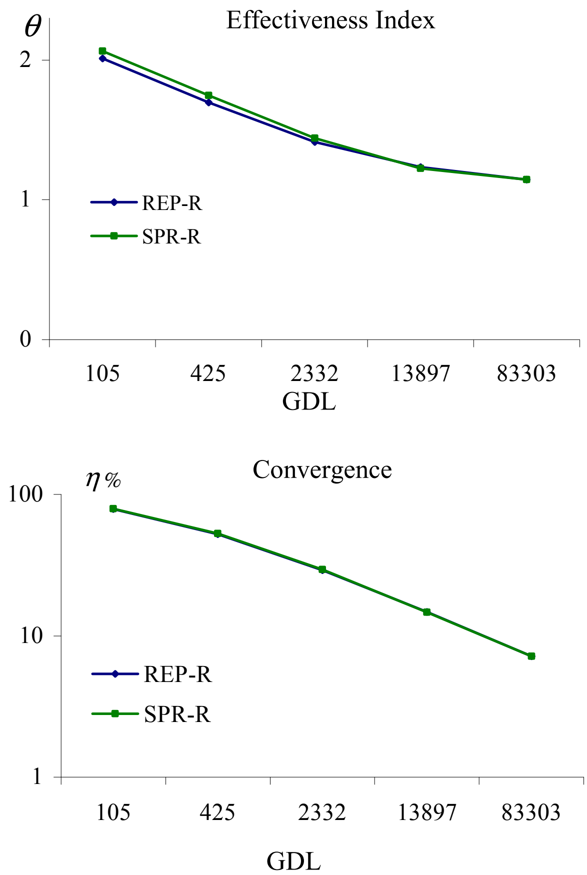


Fig. 4 - Combination in sphere REP-R to SPR-R

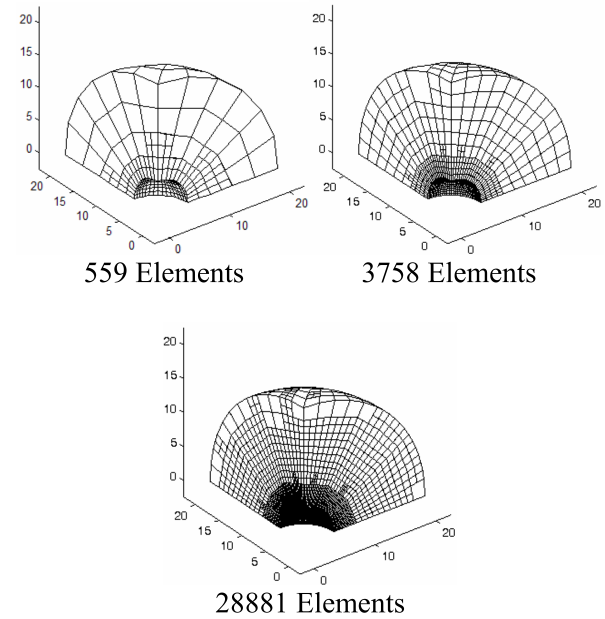


Fig. 5- Last three meshes REP-R to SPR-R

4.2 Results for the Problem with Singularities

In this problem, the effectiveness index converges to the desired value, in a accurate way when SPR-R is applied after REP-R. An equivalent behavior exhibits it the convergence, with an average value of 0.19 for the last three meshes.

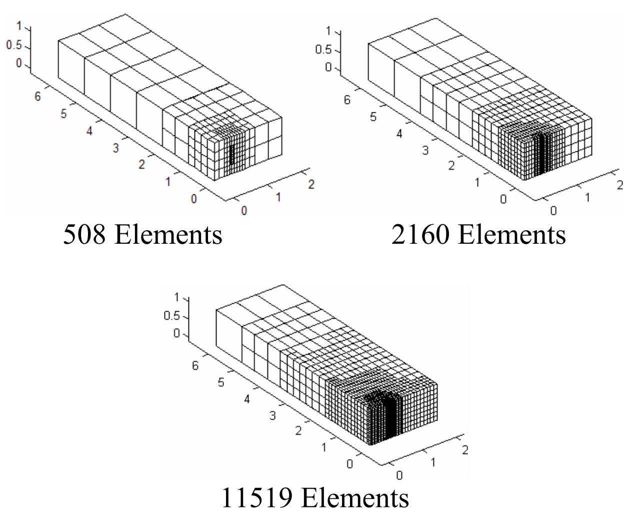


Fig. 6- Last three meshes for crack REP-R to SPR-R

5 Conclusions

The combined procedures implemented for the reconstruction of the tensional field present a similar behavior, since for each case, like it was expected, the magnitudes of the effectiveness indexes converge to the unit when the GDL increases as it can be observed in Figs. 4 and 7 similar behavior presents it the convergence speeds, which is near to the theoretical value for each studied case.

For the combined techniques implemented in this work, it can be concluded that for the sphere case, the combined technique REP-R to SPR-R doesn't improve the effectiveness index neither the convergence increases significantly. While when applying the technical REP-R to SPR-R it is improved the effectiveness index without affecting the convergence speed significantly. In the problem with singular solution, when applying the technical SPR-R to the procedure REP-R, it is improved the effectiveness index, while the convergence speed stays almost constant.

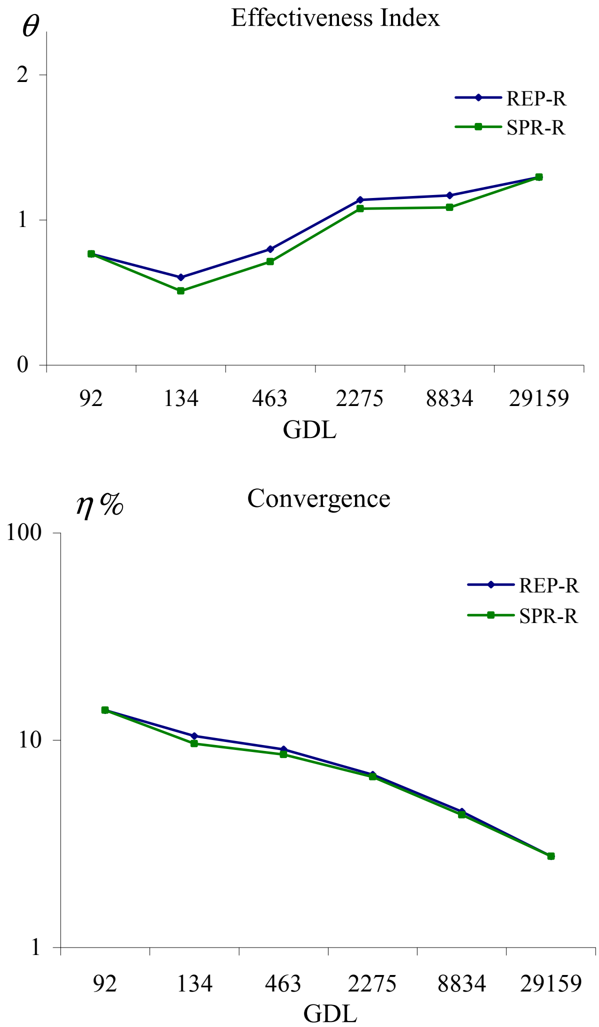


Fig. 7 - Combination in crack REP-R to SPR-R

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