Robust Quadratic Stabilization for a Class of Discrete-time Nonlinear Uncertain Systems: A Genetic Algorithm Approach.

SAMIR NEJIM[1,2], NAWEL TEJ[1] & NACEUR BENHADJ BRAIEK[1].
[1] Laboratoire d'Etude et Commande Automatique de Processus (L.E.C.A.P.)
Ecole Polytechnique de Tunisie
BP 51, 2070 La Marsa
TUNISIA
[2] Académie Navale,
5070 Menzel Bourguiba.
TUNISIA

Abstract : - We present in this paper a new approach, to check the stability robustness of discrete-time systems under nonlinear perturbations. The systems are composed of a linear constant part perturbed by an additive nonlinearity. The unique information about the nonlinearity is that it satisfies a quadratic constraint.

Our objective is to design a state feedback control law that maximizes the bound on the nonlinearity which the system can tolerate without destroying its stability. All results are obtained within the framework of a Genetic Algorithm (GA). The effectiveness of the method is illustrated through an example.

Key- words: - genetic algorithms, stability, field of stability, robustness measure, uncertain systems, LMI.

1 Introduction

The analysis of stability robustness of linear invariant systems subject to nonlinear perturbations has been of considerable interest to researchers for quite some time. Recently, Siljak and Stipanovic [1] and [2], presented a method of robust stability measure and stabilization for linear continuous time and discrete-time systems under nonlinear perturbations using the linear Matrix inequalities (LMI) approach, respectively.

When applied to discrete-time systems, Stipanovic and Siljak's method can be used only for single input systems and presents some structural restrictions on the positive definite matrix and the corresponding variables. Daniel.W.C and Guoping Lu [3], and Zhiqiang Zuo, Jinzhi Wang and Huang [4] removed these drawbacks, and then derived new results in the same area of interest for the linear system with nonlinear perturbations. In comparison with existing methods, less conservative results have been obtained.

Within the framework of this study, we deal with the same problem of robust quadratic stabilization of discrete-time linear system subjected to nonlinear uncertainties that satisfies a quadratic constraint. Our objective is to find an optimal feedback controller such that the closed-loop system is stable for all admissible parameter perturbations.

A Genetic Algorithm Approach is then proposed to cast this problem into an optimization one involving nonlinear matrix inequalities, in fact, the main advantage of the use of genetic algorithm in optimization lies in improved possibilities of finding the global optimum even if we have to resolve nonlinear matrix inequalities.

The paper is organized as follows. In section 2 we present the formulation problem. Section 3 briefly recalls existing LMI methods for stabilizing nonlinear systems by state feedback control. We present in section 4, the proposed approach based on genetic Algorithm and observe the performances of such a method. The effectiveness of the proposed tuning method is illustrated in section 5 through a numerical example.

2 Problem formulation

We consider the case of a discrete-time linear system with nonlinear perturbations, the state representation is given by:

$$X(k+1) = AX(k) + BU(k) + g(k, X(k)) \qquad (1)$$

where $X(k) \in \Re^n$ represents the state vector, $U(k) \in \Re^m$ represents the control input, $A \in \Re^{nxn}$ is a constant matrix which may be instable, $B \in \Re^{nxm}$ is a constant matrix and the function $g(k, X(k))$ represents an unstructured uncertainty which can be a linear or nonlinear function. We assume that, $g(k, X(k))$ satisfies the following quadratic constraint condition :

$$g^T(k, X(k)).g(k, X(k)) < \alpha^2 (X^T F^T F X) \qquad (2)$$

with $\alpha \in \Re_*^+$, and F a constant matrix of suitable size.

Our objective is to determine a state feedback control law:

$$U(k) = -KX(k) \qquad (3)$$

such that the system described by (1) remains stable and at the same time maximizes the domain of uncertainties measured by α, for which the system preserves its stability.

The constraint (2) is equivalent to :

$$\begin{pmatrix} X \\ g \end{pmatrix}^T \begin{pmatrix} -\alpha^2 F^T F & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} X \\ g \end{pmatrix} \leq 0 \qquad (4)$$

When we apply the feedback (3) to the system (1), we obtain the closed-loop system as :

$$\begin{aligned} X(k+1) &= (A - BK)X(k) + g(k, X(k)) \\ &= \overline{A}X(k) + g(k, X(k)) \end{aligned} \qquad (5)$$

where $\overline{A} = A - BK$. (6)

is the closed-loop system matrix.

Motivated by Stipanovic and Siljak [2] we introduce the following definition:

Definition 1: system (1) is robustly stabilized by the control law (3) with degree α, if the equilibrium X=0 of the closed-loop system (5) is globally asymptotically stable for all $g(k, X(k))$ satisfying constraint (2).

To study the stability of the closed-loop system, we consider the quadratic lyapunov function given for discrete-time systems by:

$$V(X(k)) = X^T(k) P X(k) \qquad (7)$$

with P a positive definite matrix such that the difference $\Delta V(X(k)) = V(k+1) - V(k)$ is negative definite for all $k \in Z_+^*$

$$\begin{aligned} \Delta V(X(k)) &= X^T\left[\left(\overline{A}^T P \overline{A} - P\right)X + \left(\overline{A}^T P\right)g\right] \\ &\quad + g^T\left[\left(P\overline{A}\right)X + (P)g\right] < 0 \qquad (8) \\ &= \begin{pmatrix} X \\ g \end{pmatrix}^T \begin{pmatrix} \overline{A}^T P \overline{A} - P & \overline{A}^T P \\ P\overline{A} & P \end{pmatrix} \begin{pmatrix} X \\ g \end{pmatrix} < 0 \end{aligned}$$

Using the S-procedure of Yakubovich [7] [appendix-1], the inequality (8) with constraint (4) is equivalent to the existence of a matrix $P > 0$ and a scalar $\varepsilon \geq 0$ such that :

$$\begin{pmatrix} \overline{A}^T P \overline{A} - P + \varepsilon \alpha^2 F^T F & \overline{A}^T P \\ P\overline{A} & P - \varepsilon I \end{pmatrix} < 0 \qquad (9)$$

In LMI, the inequality (9) is a non strict LMI, since $\varepsilon \geq 0$. For minimization problem, it is well-known [Boyd et al.] [6] that the result of minimization under non strict LMI constraints is equivalent to that under strict LMI constraints. Thus $\varepsilon \geq 0$ is substituted by $\varepsilon > 0$.

The relation (9) can be written in the following form:

$$\begin{pmatrix} \overline{A}^T \tilde{P} \overline{A} - \tilde{P} + \alpha^2 F^T F & \overline{A}^T \tilde{P} \\ \tilde{P}\overline{A} & \tilde{P} - I \end{pmatrix} < 0 \qquad (10)$$

with: $\tilde{P} = \dfrac{P}{\varepsilon}, \qquad P > 0, \qquad \varepsilon > 0$

Obviously, inequality (10) is nonlinear for variables $\tilde{P}$ and K

3 LMI approaches

Stipanovic and Siljak [2], Daniel and Goubing [3], Zuo, Wang and Huang [4] presented several approaches of robust stability and stabilization of discrete-time systems under nonlinear perturbations using the linear matrix inequalities (LMIs).

3.1 Stipanovic and Siljak's approach

Stipanovic and Siljak [2] formulated this problem of robust stabilization with maximization of the robustness measure, for a class of single input systems, in a convex optimization problem with constraints LMI where several tools associated with LMIs were used: the S-procedure, the Schur complement…

In [2], the robust stability problem of a discrete-time system under nonlinear perturbation is formulated into a constrained convex optimization problem involving linear matrix inequalities (LMIs) to design a linear state feedback control law which stabilizes the closed loop system and maximizes the robustness measure given by α.

To resolve this problem, and to contouring problems of non linearities Stipanovic and Siljak assume that the pair (A,B) is given in the controllable form and consider that for $L = \tilde{P}BK$, they impose many restrictions on the matrices $\tilde{P}, L$ and B.

$$\tilde{P} = \begin{pmatrix} \tilde{P}_1 & 0_{(n-1)} \\ 0^T & \tilde{P}_2 \end{pmatrix} \qquad (11)$$

$$L = \begin{pmatrix} 0_{(n-1)xn} \\ l \end{pmatrix} \quad (12)$$

$$B = \begin{pmatrix} 0_{(n-1)} \\ l \end{pmatrix} \quad (13)$$

where:

$\widetilde{P}_1$ is a $(n-1)$ x $(n-1)$ matrix ;

$\widetilde{P}_2 \in \Re$ and $l = (l_1, l_2, ..., l_n); l_i \in \Re$.

with :

$$K = \widetilde{P}_2^{-1} l \quad (14)$$

Other constraints were imposed on L and $\widetilde{P}_2^{-1}$ which are represented in LMI by :

$$\begin{pmatrix} -K_l & l \\ l & -I \end{pmatrix} < 0 \quad (15)$$

$$\begin{pmatrix} K_p & I \\ I & \widetilde{P}_2 \end{pmatrix} > 0 \quad (16)$$

These two constraints allowed restricting the gain K as follows:

$$K^T K = \widetilde{P}_2^{-2} .l.l^T < K_l K_p^2 . \quad (17)$$

Stipanovic et Siljak formulated the optimization problem in the following form where $\beta = \frac{1}{\alpha^2}$;

minimize $\beta + K_l + K_p$

Subject to : $\widetilde{P} > 0$

$$\begin{pmatrix} -\widetilde{P} & A^T\widetilde{P} - L^T & A^T\widetilde{P} - L^T & F^T \\ \widetilde{P}A - L & \widetilde{P} - I & 0 & 0 \\ \widetilde{P}A - L & 0 & -\widetilde{P} & 0 \\ F & 0 & 0 & -\beta I \end{pmatrix} < 0$$

$$\begin{pmatrix} -K_l & l \\ l^T & -I \end{pmatrix} < 0 \quad (18)$$

$$\begin{pmatrix} K_p & I \\ I & \widetilde{P}_2 \end{pmatrix} > 0$$

Stipanovic et Siljak stated the following theorem:

Theorem 1[2]:

The system (1) is robustly stabilized by control law (3) if problem (18) is feasible.

3.2 Daniel and Guoping's approach

The robust stabilization problem with maximization of the robustness measure using the LMIs, was extended in [3], by Daniel and Guoping for a class of multi-input and multi-output (MIMO) discrete-time nonlinear systems represented by this equation

$$X(k+1) = AX(k) + BU(k) + g(k, X(k), U(k)) \quad (19)$$

where $A \in \Re^{nxn}$, $B \in \Re^{nxm}$ represent constant matrices, it is assumed here that the pair (A,B) is given in the controllable form. $X(k) \in \Re^n$ represents the state vector, $U(k) \in \Re^m$ the control input and the function $g(X,U,k)$ an unstructured uncertainty which can be linear or nonlinear function of both state and control inputs. It is assumed that $g(X,U,k)$, satisfies the following quadratic constraint condition :

$$g^T(k, X(k), U(k).g(k, X(k), U(k)) \le \alpha^2 (X^T F^T FX + U^T H^T HU) \quad (20)$$

with $\alpha \in \Re_*^+$, H and F constant matrices of suitable sizes.

The authors propose a linear feedback $U = -KX$ which stabilizes the closed loop system described by :

$$X(k+1) = \overline{A}X(k) + g(k, X(k), U(k)) \quad (21)$$

The quadratic constraint is given by the relation (20), this constraint can be written in an equivalent form :

$$g^T(k, X(k)).g(k, X(k)) \le \alpha^2 (X^T G^T GX) \quad (22)$$

with $G = \begin{pmatrix} F \\ HK \end{pmatrix}$.

Daniel and Guoping [3] stated the following theorem:

Theorem 2[3]:

The closed-loop system (21) is globally asymptotically stable with a maximum limit of nonlinearities α if there exist matrices $Q \in \Re^{nxn}$ and $L \in \Re^{mxn}$ such that the convex optimization problem is feasible:

minimize β with :

- $\beta \in \Re_+^*$
- There exist $Q \in \Re^{nxn}$ and $L \in \Re^{mxn}$, verifying the following constraints:

$Q > 0$

$$\begin{pmatrix} -Q & QA^T - L^T B^T & \begin{pmatrix} FQ \\ HL \end{pmatrix}^T \\ AQ - BL & I - Q & 0 \\ \begin{pmatrix} FQ \\ HL \end{pmatrix} & 0 & -\beta I \end{pmatrix} < 0 \quad (23)$$

with: $L = kQ$.

The gain is then given by: $K = LQ^{-1}$ and robustness measure is given by $\alpha = \frac{1}{\sqrt{\beta}}$.

4 Genetic Algorithm resolution Approach

In this paragraph we will propose a new resolution approach of the Problem of robust stabilization with maximization of the robustness measure by solving nonlinear matrix inequality (NLMI) (10) using Genetic Algorithm.

Genetic Algorithms have been shown to solve linear and nonlinear problems by exploring all regions of the state space and exponentially exploiting promising areas through mutation, crossover and selection operations applied to individuals in the populations (Michalewicz) [9].

GAs maintain and manipulate a family or population of solutions and implements a “survival of the fittest” strategy in their search for better solutions.

We consider the discrete type quadratic lyapunov function (7).

In order to guarantee the stability of the closed-loop system given by the relation (5), we just require that the difference of $V(x(k))$ along the trajectory of the system is negative, i.e :

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k)) < 0$$

We also consider the nonlinear matrix inequality given by the relation (10).

$$\begin{pmatrix} \overline{A}^T \tilde{P} \overline{A} - \tilde{P} + \alpha^2 F^T F & \overline{A}^T \tilde{P} \\ \tilde{P} \overline{A} & \tilde{P} - I \end{pmatrix} < 0$$

As known, inequality (10) is nonlinear for variables $\tilde{P}$ and K.

In this paper, the use of Genetic Algorithm is considered as being an optimization problem seeking for the maximum of the robustness measure α.

For this intention, we have to define the variables of the problem and the fitness function to be maximized.

Variables of the problem are elements of the feedback vector K and elements of the matrix $\tilde{P}$.

For each optimization variable x_i, we correspond a gene. A chromosome is composed of several genes. A chromosome representation is needed to describe each individual in the population. In this problem K is an nm elements vector, $\tilde{P}$ is a vector with $\frac{n(n+1)}{2}$ elements, all these elements constitute the variables of the problem.

Therefore, the chromosome is a vector with $\frac{n(n+1)}{2} + nm$ elements, composed of coefficients of the feedback matrix and those of the symmetric positive definite matrix $\tilde{P}$.

GA must be provided with an initial population and randomly generates solutions for the entire population.

GA must respect constraints and chooses the feedback vector such that the relation given by the inequality (10) is verified; this relation is transformed as follows:

$$M(\alpha, K) = M_0(K) + \alpha^2(K) M_\alpha < 0 \qquad (24)$$

with :

$$M_0(K) = \begin{pmatrix} \overline{A}^T \tilde{P} \overline{A} - \tilde{P} & \overline{A}^T \tilde{P} \\ \tilde{P} \overline{A} & \tilde{P} - I \end{pmatrix} \qquad (25)$$

$$M_\alpha = \begin{pmatrix} F^T F & 0 \\ 0 & 0 \end{pmatrix} \qquad (26)$$

GA seeks for a maximal value of $\alpha(K)$ verifying the relation (24).

The maximum value of $\alpha(K)$ is then given by:

$$\alpha_{\max}(K) = \max\left(\left\{\alpha \in \Re^+ / \max_{1 \le i \le N}(\lambda_i(M(\alpha, K))) < 1\right\}, 0\right) \qquad (27)$$

with λ_i, $1 \le i \le N$, are the eigenvalues of the matrix $M(\alpha, K)$, N is the order of the matrix $M(\alpha, K)$.

Fittest individuals that lead to a stable closed-loop systems have a robustness measure $\alpha > 0$, we apply to them the following scaled cost function:

$$fitness = a + b.\alpha \qquad (28)$$

where a and b are positive scaling coefficients permitting to reduce or to increase variations between individuals.

For the case where $\alpha = 0$, there is no robustness margin. In this case, inferior individuals lead to an unstable closed-loop system thus if we apply to them a $fitness = 0$, they will be systematically eliminated and will not survive at next generations, so we apply to them the following fitness:

$$fitness = a - \max\left|\lambda_i\left(M\left(\alpha, K\right)\right)\right| \qquad (29)$$

with $1 \le i \le n$

Reporting to the fact that best individuals have an increased chance of being selected, this second form of cost function lets the chance to inferior individuals to survive and also reproduce at next generations in order to preserve the gene diversity on a generation.

The proposed GA must respect another constraint, it searches for a symmetric positive definite matrix $\widetilde{P}$, verifying the relation (24).

The application of the genetic algorithm is given by the following steps:

1- Define the variables of the problem and the cost function.

The variables of the problem are the $\frac{n(n+1)}{2}+nm$ elements of the chromosome composed of coefficients of the feedback matrix and those of the symmetric positive definite matrix $\widetilde{P}$.
The cost function is given by relations (28) and (29).

2 - Genesis of the population:

To test GA, one defines for each simulation:
- the number of parameters,
- the number of bits used to code various parameters,
- lower and upper bounds (vlb, vub respectively) of each parameter,
- the stop condition,
- the maximum number of generations,
- the size of the initial population.

3- Evaluation

From the initial population, binary GA calculates for each chromosome: the state feedback K , the matrix $\widetilde{P}$ and the cost function to maximize.

4- Selection-elimination.

GA classifies the chromosomes according to their fitness; the GA uses the Roulette wheel method to assign a probability of selection P_j to each individual j proportionally to its fitness value.

5- Reproduction (crossover operator)

The most common recombination operator is the one-point crossover method. A crossover point is selected along the chromosome and the genes up to that point are swapped between the two parents. A probability of crossover P_c will be fixed for each simulation, more it increases, more the population undergoes significant changes.

6- Mutation (mutation operator)

Mutation alters one individual to produce a single new solution. Binary mutation flips each bit in every individual in the population with probability P_m to be fixed for each simulation. This rate is generally weak since a high rate can lead to a sub-optimal solution. Generally allowed values are between 0,001 and 0,03.

7- Simulation Results

GA finds after N generations, optimal values of the elements of the state feedback vector K , and the elements of the matrix $\widetilde{P}$ as well as the maximal cost α .

5 An example Study

Let us consider a single-input discrete-time system represented by the following state equation

$$X(k+1)=\begin{pmatrix}0 & 1\\ -2 & -3\end{pmatrix}X(k)+\begin{pmatrix}0\\ 1\end{pmatrix}U(k)+g(k,X(k)) \quad (30)$$

With nonlinear perturbation function that satisfies the quadratic constraint given by the relation (2):

$$g^T(k,X(k)).g(k,X(k))\le \alpha^2(X^TF^TFX)$$

where : $F=I_2$.

The optimization Problem (18) using LMIs, proposed by Stipanovic and Siljak [1] and stated by the theorem 1, give a stabilizing feedback gain:

$$K=\begin{bmatrix}-1.9252 & -2.8878\end{bmatrix}$$

and a maximal value of the degree of stability :

$$\alpha_{\max}=0.6015.$$

with the eigenvalues of closed-loop system matrix $\overline{A}$ of (8) located at $-0.0561\pm i0.2676$.

These values were improved by solving the convex optimization problem (23) proposed by Daniel and Guoping [3] and given by the theorem 2. This method leads to a stabilizing feedback gain:

$$K=\begin{bmatrix}-2 & -3\end{bmatrix}$$

and a maximal value of the degree of stability $\alpha_{\max}=0.6179$ corresponding to :

$$Q=\begin{bmatrix}2.6184 & 0\\ 0 & 1.003\end{bmatrix}.$$

We use the proposed method described in section 4 and we represent each variable by a 16 bits length chromosome, the variables are searched in an interval [vlb,vub]. A one-point crossover and a one-bit mutation are applied with rates of 0.7 and 0.03 respectively. The reproduction method used is the Roulette wheel where each chromosome is reproduced in the next generation proportionally to its fitness. The considered size of population in this study is of 500 individuals. The genetic algorithm converges after 158 generations.

This leads to the following parameters of the feedback matrix gain:

$$K = [-2.001 \quad -2.9996]$$

and with the following value of the fitness function :

$$\alpha_{\max} = 0.6174 \ .$$

where :

$$\tilde{P} = \begin{bmatrix} 0.3812 & 0.0001 \\ 0.0001 & 0.9994 \end{bmatrix}$$

As we can see the state feedback gain stabilises the system (30) and at the same time maximizes the field of uncertainties given by α , for which system preserves its stability. The obtained result shows a clear improvement of the maximal bound given by the proposed approach using Genetic Algorithm compared to the result given in [1] where several restrictions were imposed (improvement of 2.64 %). This result is nearly identical to the result given by the optimization approach proposed by Daniel and Guoping [2] (the difference is at the level of the fourth decimal).

6 Conclusion

In this work we proposed a new numerical method for the synthesis of a state feedback law that stabilizes the closed loop system and maximizes the robustness measure by solving nonlinear matrix inequality (NLMI).

This method uses the optimization abilities of Genetic Algorithm and uses for its development the Lyapunov quadratic function.

In the application developed in this paper, GA converged toward the optimal parameters values of the state feedback vector that stabilizes the closed loop system and maximizes the robustness measure.

The approach developed by Stipanovic and Siljak in 2001 [1] using LMIs is limited for a class of single-input and single-output (SISO) discrete-time nonlinear systems, more recent results using the LMIs, developed by Daniel et Guoping 2003 [2] and by Zhiqiang,, Jinzhi and Lin in 2004 [3] were applied for a class of multi-input and multi-output (MIMO) discrete-time nonlinear systems. These approaches present the disadvantage of imposing restrictions on the particular structure of the matrices (Stipanovic and Siljak approach case) or necessitate much theoretical developments), whereas in the case of the proposed Genetic Algorithm approach, there isn't any restriction imposed so results are more powerful. More over the proposed GA method is not limited for SISO discrete-time nonlinear systems, it can also be applied for a class of MIMO discrete-time nonlinear systems.

7 Appendix

A-1 :Lemme: The S-procedure of Yakubovich [6]

$\Omega_0(x)$ et $\Omega_1(x)$ two arbitrary quadratic forms of $\Re^n$, then $\Omega_0(x) < 0$

for all $x \in \Re^n / \{0\}$ checking $\Omega_1(x) \leq 0$, if and only if: There exists $\tau \geq 0$ such that: $\Omega_0(x) - \tau.\Omega_1(x) < 0$, $\forall\ x \in \Re^n / \{0\}$.

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