Calculation of the Longitudinal Stability Derivatives of a Transport Aircraft and Analysis of Longitudinal Modes

UĞUR ÖZDEMİR[1] ELBROUS M. JAFAROV[2] MEHMET Ş. KAVSAOĞLU[3]
Department of Aeronautics and Astronautics
Istanbul Technical University
Maslak, 34469 Istanbul
TURKEY

Abstract: -In this study, the longitudinal stability derivatives of a transport aircraft have been calculated by modeling the aircraft configuration and flight condition. The applying method is able to use for all sizes of civil and military aircrafts. As a second step, the longitudinal motion of aircraft is investigated and the transfer functions of linearized equations of motion have been obtained. Phugoid and short period modes have been investigated. Frequency-response and transient-response analysis have been performed respectively by using Matlab Simulink.

Key-Words: - Longitudinal stability derivatives, phugoid and short modes, time-response.

1 Introduction

During the dynamic modeling of aircrafts, stability derivatives are generally obtained from the experimental or previous studies about the related aircraft. Therefore, obtaining new derivatives of an aircraft whose derivatives have not calculated yet is difficult for researchers. The Advanced Aircraft Analysis (AAA) program applies to most fixed wing configurations (civil or military) and permit engineers to fast calculate stability derivatives straightforwardly [1-4]. In this study, a longitudinal motion of a transport aircraft has been studied. This aircraft model approximately represents the characteristic of Boeing 737-400. Firstly, configuration modeling and result have been given. Secondly, the transfer functions of longitudinal linearized equations of motion have been obtained. Phugoid and Short Period modes have been investigated. Finally, frequency-response and transient-response Analyses have been performed respectively.

2 AAA Analyses

2.1 Configuration Modeling in AAA

Configuration of the aircraft has been set in Geometry Module of AAA. The aim of the Geometry module assist the user to characterize the geometry of the fuselage, wing, horizontal tail, vertical tail and work out related parameters [1-4]. After the parameters of the aircraft are set, corresponding plots have been obtained.

Configuration modeling parameters of the aircraft: fuselage, wing, horizontal and vertical tails, aileron, elevator and rudder control surfaces are determined and shown in Fig. 1-7 respectively.

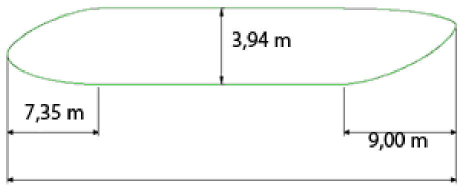

Fig.1 Fuselage Geometry

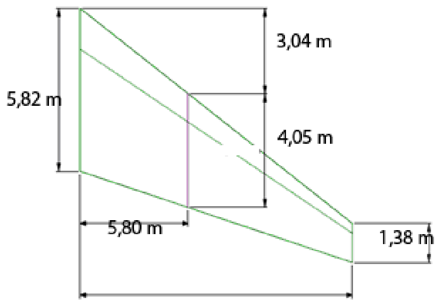

Fig.2 Wing Geometry

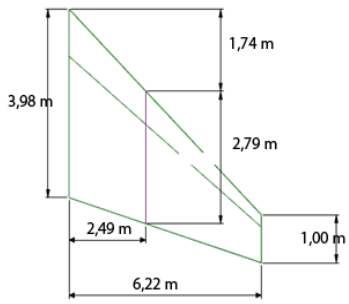

Fig.3 Horizontal Tail Geometry

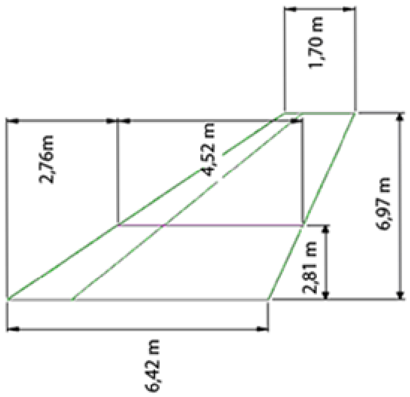

Fig.4 Vertical Tail Geometry

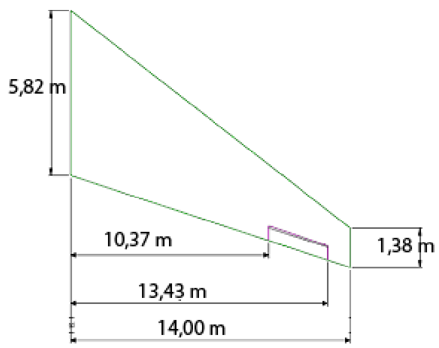

Fig.5 Aileron Geometry

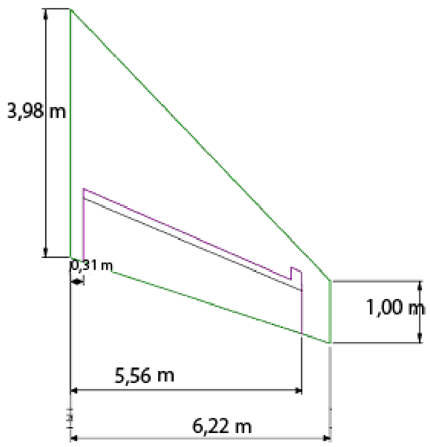

Fig.6 Elevator Geometry

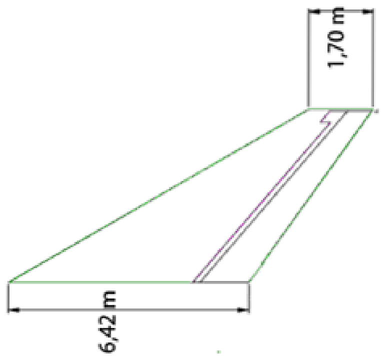

Fig.7 Rudder Geometry

These parameters are used in determining of stability derivatives and transfer functions.

2.1 Results of the AAA

The steady-state flight conditions and longitudinal stability derivatives are obtained by using AAA programming the result of which are presented in Table 1.

Table 1 The calculation results

Steady State Flight Speed: $U_1 = 537{,}0$ km/hr
Airplane Current Weight: $W_{current} = 500000{,}0$ N
Wing Area: $S_w = 105{,}00\ \text{m}^2$
Steady State Pitch Attitude: $\theta_1 = 3{,}0$ deg
Wing Mean Geometric Chord: $\bar{c}_w = 4{,}05$ m
Airplane Moment of Inertia about the Y-body Axis: $I_{yy_B} = 2552412{,}6\ 2$ kg-m^2
Steady State Flight Mach Number: $M_1 = 0{,}459$
Dynamic Pressure in Steady State: $\bar{q}_1 = 9151{,}60$ N/m^2
Wing Loading at Current Flight Condition $W/S = 4761{,}90$ N/m^2
Forward Acceleration Imparted to the Airplane as a Result of a Unit Change in Speed: $X_u = -0{,}0082$ s^{-1}
Forward Acceleration Imparted to the Airplane due to Thrust as a Result of a Unit Change in Speed: $X_{Tu} = -0{,}0000$ s^{-1}
Forward Acceleration Imparted to the Airplane as a Result of a Unit Change in Angle-of-attack: $X_\alpha = 5{,}2770$ m/ s^2
Vertical Acceleration Imparted to the Airplane as a Result of a Unit Change in Speed: $Z_u = -0{,}1476$ s^{-1}
Vertical Acceleration Imparted to the Airplane as a Result of a Unit Change in Angle-of-attack: $Z_\alpha = -102{,}0394$ m/s^2
Vertical Acceleration Imparted to the Airplane as a Result of a Unit Change in Rate of Change of Angle-of-attack: $Z_{\dot{\alpha}} = -0{,}6563$ m/s
Vertical Acceleration Imparted to the Airplane as a Result of a Unit Change in Pitch Rate: $Z_q = -2{,}0160$ m/s
Pitch Angular Acceleration Imparted to the Airplane as a Result of a Unit Change in Speed: $M_u = 0{,}0023$

Pitch Angular Acceleration Imparted to the Airplane due to Thrust as a Result of a Unit Change in Speed: M_{T_u} =0,0000 1/m.s
Pitch Angular Acceleration Imparted to the Airplane as a Result of a Unit Change of Angle-of-attack: M_α = -2,7367 s^{-2}
Pitch Angular Acceleration Imparted to the Airplane due to Thrust as a Result of a Unit Change in Angle-of-attack: $M_{T\alpha}$ = 0,0000 s^{-2}
Pitch Angular Acceleration Imparted to the Airplane as a Result of a Unit Change in Rate of Change of Angle-of-attack: $M_{\dot{\alpha}}$ = -0,1915 s^{-1}
Pitch Angular Acceleration Imparted to the Airplane as a Result of a Unit Change in Pitch Rate: M_q = -0,6106 s^{-1}
Forward Acceleration Imparted to the Airplane as a Result of a Unit Change in Elevator Deflection Angle: $X_{\delta e}$ = -0,1921 m/ s^2
Vertical Acceleration Imparted to the Airplane as a Result of a Unit Change in Elevator Deflection Angle: $Z_{\delta e}$ =-9,3490 m/ s^2
Pitch Angular Acceleration Imparted to the Airplane as a Result of a Unit Change in Elevator Deflection Angle: $M_{\delta e}$ = -2,7284 s-^{2}

3 Longitudinal Dynamics

In this section the longitudinal equations of motion are derived. The parameters of phugoid and short modes are obtained.

3.1 Longitudinal Linearized Equations of Motions

The general longitudinal linear equations of aircraft with the assumption of initial trimmed flight and the wings level condition can be presented as follows [5];

$$\begin{aligned}
\dot{u} &= -g\theta\cos\cos\theta_1 + X_u u + X_{T_u} u + X_\alpha \alpha + X_{\delta e}\delta_e \\
\dot{w} - U_1 q &= -g\theta\sin\theta_1 + Z_u u + Z_\alpha \alpha + Z_{\dot{\alpha}}\dot{\alpha} + Z_q q + Z_{\delta e}\delta_e \\
\dot{q} &= M_u u + M_{T_u} u + M_\alpha \alpha + M_{T_\alpha}\alpha + M_\alpha \alpha + M_{\dot{\alpha}}\dot{\alpha} + M_q q + M_{\delta e}\delta_e
\end{aligned} \tag{1}$$

where;
u = change of velocity in longitudinal flight;
α = change of angle of attack in longitudinal flight;
w = change of velocity in Z direction;
δ_e = elevator deflection;

Defining
$q = \dot{\theta}, \quad \dot{q} = \ddot{\theta}$

where

q = change of angular velocity about Y axis.
θ = change of pitching angle

and taken into account the following, for small perturbations

$\alpha \approx \dfrac{w}{U_1} \Rightarrow w = \alpha U_1$ and $\dot{w} = \dot{\alpha} U_1$,

equation (1) with zero initial conditions can be represented in Laplace transform as follows:

$$\begin{bmatrix} (s - X_u - X_{T_u}) & -X_\alpha & g\cos\theta_1 \\ -Z_u & [s(U_1 - Z_{\dot{\alpha}}) - Z_\alpha] & [-(Z_q + U_1)s - g\sin\theta_1] \\ -(M_u + M_{T_u}) & [M_{\dot{\alpha}}s + M_\alpha + M_{T_\alpha}] & (s^2 - M_q S) \end{bmatrix} \begin{bmatrix} \frac{u(s)}{\delta_e(s)} \\ \frac{\alpha(s)}{\delta_e(s)} \\ \frac{\theta(s)}{\delta_e(s)} \end{bmatrix} = \begin{bmatrix} X_{\delta e} \\ Z_{\delta e} \\ M_{\delta e} \end{bmatrix} \tag{2}$$

Equation (2), the characteristic equation for longitudinal motion can be obtained as:

$$CE = \begin{vmatrix} (s - X_u - X_{T_u}) & -X_\alpha & g\cos\theta_1 \\ -Z_u & [s(U_1 - Z_{\dot{\alpha}}) - Z_\alpha] & [-(Z_q + U_1)s - g\sin\theta_1] \\ -(M_u + M_{T_u}) & [M_{\dot{\alpha}}s + M_\alpha + M_{T_\alpha}] & (s^2 - M_q S) \end{vmatrix} = 0 \tag{3}$$

Substitutiong the parameters from Table 1, we can calculate CE as:

$$491.5451S^4 + 731.3739S^3 + 1533.7677S^2$$
$$+15.5175S + 20.5709 = 0$$

or factoring,

$$491.5451(S^2 + 1.4842S + 3.1013)$$
$$(S^2 + 0.0037S + 0.0135) = 0 \tag{4}$$

Then from CE we can obtain the phugoid and short periods modes parameters: the roots, natural frequencies, damping ratios and the time required for the oscillation to damp to one-half amplitude as follows:

$$T_{1/2_{pm}} = \frac{0.693}{\xi\omega_n}$$

Phugoid	Short Period
$\xi_{pm} = 0.016$	$\xi_{sp} = 0.421$
$\omega_{n_{pm}} = 0.1162 \dfrac{rad}{s}$	$\omega_{n_{sp}} = 1.1761 \dfrac{rad}{s}$
$T_{1/2_{pm}} = 372$ sec $= 6.21$ min	$T_{1/2_{sp}} = 1.026$ sec

These values are is quite typical.

3.2 **Longitudinal Transfer Functions for Elevator** Displacement

Each of the three longitudinal transfer functions can be determined by using Cramer's rule [5].

3.2.1 $\dfrac{u(s)}{\delta_e(s)} = \dfrac{N_u}{\overline{D}_1}$

$$\frac{N_u}{\overline{D}_1} = \frac{\begin{vmatrix} X_{\delta e} & -X_\alpha & g\cos\theta_1 \\ Z_{\delta e} & \{s(U_1 - Z_{\dot\alpha}) - Z_\alpha\} & \{-(Z_q + U_1)s + g\sin\theta_1\} \\ M_{\delta e} & -\{M_{\dot\alpha}s + M_\alpha + M_{T_\alpha}\} & (s^2 - M_q s) \end{vmatrix}}{\begin{vmatrix} (s - X_u - X_{T_u}) & -X_\alpha & g\cos\theta_1 \\ -Z_u & \{s(U_1 - Z_{\dot\alpha}) - Z_\alpha\} & \{-(Z_q + U_1)s + g\sin\theta_1\} \\ -(M_u + M_{T_u}) & -\{M_{\dot\alpha}s + M_\alpha + M_{T_\alpha}\} & (s^2 - M_q s) \end{vmatrix}} \tag{5}$$

$$\overline{D}_1 = Es^4 + Fs^3 + Gs^2 + Hs + I$$

where;

$$E = U_1 - Z_{\dot\alpha}$$

$$F = -(U_1 - Z_{\dot\alpha})(X_u + X_{T_u} + M_q) - Z_\alpha - M_{\dot\alpha}(U_1 + Z_q)$$

$$\begin{aligned} G = {} & (X_u + X_{T_u})\{M_q(U_1 - Z_{\dot\alpha}) + Z_\alpha + M_{\dot\alpha}(U_1 + Z_q)\} \\ & + M_q Z_\alpha - Z_u X_\alpha + M_{\dot\alpha} g\sin\theta_1 - (M_\alpha + M_{T_\alpha})(U_1 Z_q) \end{aligned}$$

$$\begin{aligned} H = {} & g\sin\theta_1\{M_\alpha + M_{T_u} - M_{\dot\alpha}(X_u + X_{T_u})\} \\ & + g\cos\theta_1\{Z_u M_{\dot\alpha} + (M_u + M_{T_u})(U_1 - Z_\alpha)\} \\ & + (M_u + M_{T_u})\{-X_\alpha(U_1 + Z_q)\} + Z_u X_\alpha M_q \\ & + (X_u + X_{T_u})\{(M_\alpha + M_{T_u})(U_1 + Z_q) - M_q Z_\alpha\} \end{aligned}$$

$$\begin{aligned} I = {} & g\cos\theta_1\{(M_\alpha + M_{T_u})Z_u - Z_\alpha(M_u + M_{T_u})\} \\ & + g\sin\theta_1\{(M_u + M_{T_u})X_\alpha - (X_u + X_{T_u})(M_\alpha + M_{T_u})\} \end{aligned}$$

$$N_u = A_u s^3 + B_u s^2 + C_u s + D_u$$

where;

$$A_u = X_{\delta e}(U_1 - Z_{\dot\alpha})$$

$$B_u = -X_{\delta e}\{(U_1 - Z_{\dot\alpha})M_q + Z_\alpha - M_{\dot\alpha}(U_1 + Z_q)\} + Z_{\delta e}X_\alpha$$

$$\begin{aligned} C_u = {} & X_{\delta e}\{M_q Z_\alpha + M_{\dot\alpha} g\sin\theta_1 - (M_\alpha M_{T_\alpha})(U_1 + Z_q)\} \\ & + Z_{\delta e}\{M_{\dot\alpha} g\cos\theta_1 - X_\alpha M_q\} + M_{\delta e}\{X_\alpha(U_1 + Z_q) \\ & - (U_1 - Z_{\dot\alpha})g\cos\theta_1\} \end{aligned}$$

$$\begin{aligned} D_u = {} & X_{\delta e}(M_\alpha + M_{T_\alpha})g\sin\theta_1 - Z_{\delta e}M_\alpha g\cos\theta_1 M_{\delta e}(Z_\alpha g\cos\theta_1) \\ & - X_\alpha g\sin\theta_1 \end{aligned}$$

3.2.2 $\dfrac{\alpha(s)}{\delta_e(s)} = \dfrac{N_\alpha}{\overline{D}_1}$

$$\frac{N_\alpha}{\overline{D}_1} = \frac{\begin{vmatrix} (s - X_u - X_{T_u}) & X_{\delta e} & g\cos\theta_1 \\ -Z_u & Z_{\delta e} & \{-(Z_q + U_1)s + g\sin\theta_1\} \\ -(M_u + M_{T_u}) & M_{\delta e} & (s^2 - M_q s) \end{vmatrix}}{\left|\overline{D}_1\right|} \tag{6}$$

$$N_\alpha = A_\alpha s^3 + B_\alpha s^2 + C_\alpha s + D_\alpha$$

where;

$$A_\alpha = Z_{\delta e}$$

$$B_\alpha = X_{\delta e}Z_u + Z_{\delta e}\{-M_q - (X_u + X_{T_u})\} + M_{\delta e}(U_1 + Z_q)$$

$$\begin{aligned} C_\alpha = {} & X_{\delta e}\{(U_1 + Z_q)(M_u + M_{T_u}) - M_q Z_u\} \\ & + Z_{\delta e}M_q(X_u + X_{T_u}) \\ & + M_{\delta e}\{-g\sin\theta_1 - (U_1 + Z_q)(X_u + X_{T_u})\} \end{aligned}$$

$$\begin{aligned} D_\alpha = {} & -X_{\delta e}(M_u + M_{T_u})g\sin\theta_1 - Z_{\delta e}(M_u + M_{T_u})g\cos\theta_1 \\ & + M_{\delta e}(X_u + X_{T_u})g\sin\theta_1 - Z_u g\cos\theta_1\} \end{aligned}$$

3.2.3 $\dfrac{\theta(s)}{\delta_e(s)} = \dfrac{N_\theta}{\overline{D}_1}$

$$\frac{N_\theta}{\overline{D}_1} = \frac{\begin{vmatrix} (s - X_u - X_{T_u}) & -X_\alpha & X_{\delta e} \\ -Z_u & \{s(U_1 - Z_{\dot\alpha}) - Z_\alpha\} & Z_{\delta e} \\ -(M_u + M_{T_u}) & -\{M_{\dot\alpha}s + M_\alpha + M_{T_\alpha}\} & M_{\delta e} \end{vmatrix}}{\left|\overline{D}_1\right|} \tag{7}$$

$$N_\theta = A_\theta s^2 + B_\theta s + C_\theta$$

where;

$$A_\theta = Z_{\delta e}M_{\dot\alpha} + M_{\delta e}(U_1 - Z_{\dot\alpha})$$

$$\begin{aligned} B_\theta = {} & X_{\delta e}\{Z_u M_{\dot\alpha} + (U_1 - Z_{\dot\alpha})(M_u + M_{T_u})\} \\ & + Z_{\delta e}\{(M_\alpha + M_{T_\alpha}) - M_{\dot\alpha}(X_u + X_{T_u})\} \\ & + M_{\delta e}\{-Z_\alpha - (U_1 - Z_{\dot\alpha})(X_u + X_{T_u})\} \end{aligned}$$

$$\begin{aligned} C_\theta = {} & X_{\delta e}\{(M_\alpha + M_{T_u})Z_u - Z_\alpha(M_u + M_{T_u})\} \\ & + Z_{\delta e}\{-(M_\alpha + M_{T_u})(X_u + X_{T_u}) + X_\alpha(M_u + M_{T_u})\} \\ & + M_{\delta e}\{Z_\alpha(X_u + X_{T_u}) - X_\alpha Z_u\} \end{aligned}$$

From these equations, the following transfer functions can be obtained:

$$\frac{u(S)}{\delta_\varepsilon(S)} = \frac{-309{,}8021\,S^3 - 989{,}4632\,S^2 + 18635{,}1484\,S + 26624{,}3810}{491{,}5451\,S^4 + 731{,}3739\,S^3 + 1533{,}7677\,S^2 + 15{,}5175\,S + 20{,}5709} \tag{8}$$

$$\frac{\alpha(S)}{\delta_\varepsilon(S)} = \frac{-30{,}6727\,S^3 - 1336{,}0986\,S^2 - 6{,}4107\,S - 13{,}5486}{491{,}5451\,S^4 + 731{,}3739\,S^3 + 1533{,}7677\,S^2 + 15{,}5175\,S + 20{,}5709} \tag{9}$$

$$\frac{\theta(S)}{\delta_\varepsilon(S)} = \frac{-1335{,}2592\,S^2 - 840{,}5910\,S - 14{,}5224}{491{,}5451\,S^4 + 731{,}3739\,S^3 + 1533{,}7677\,S^2 + 15{,}5175\,S + 20{,}5709} \tag{10}$$

4 Frequency Domain Analysis

Following bode plots were drawn by using Matlab to investigate aircraft longitudinal stability in depth as discussed in ref. [6].

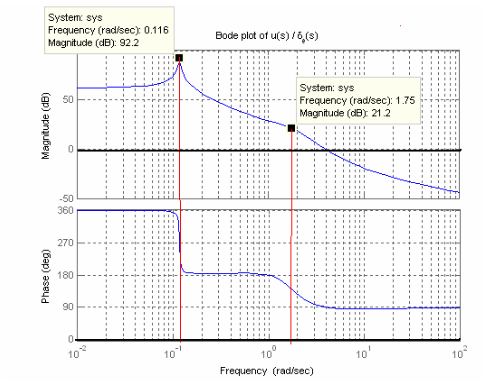

Fig.8 Bode Plot of $\dfrac{u(s)}{\delta_e(s)}$

Fig.8 indicates that the amplitude of the $\dfrac{u(s)}{\delta_e(s)}$ responce is very small at the natural frequency of the short period oscilation.

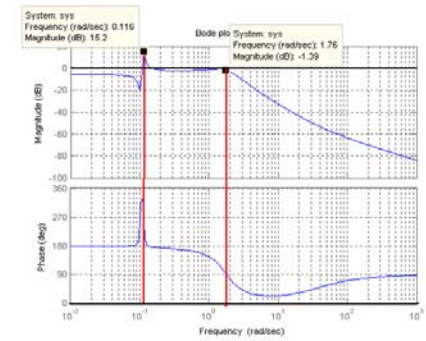

Fig.9 Bode Plot of $\dfrac{\alpha(s)}{\delta_e(s)}$

Fig.9 demonstrates that the earlier statement that the phugoid oscillation takes place at almost constant angle of attack.

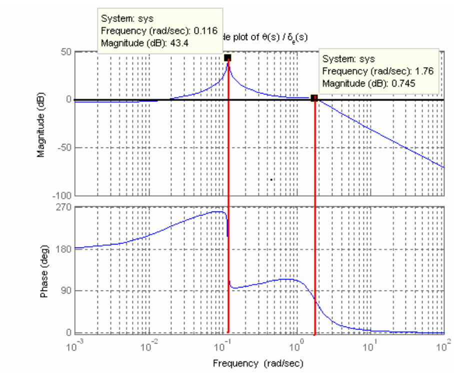

Fig.10 Bode Plot of $\dfrac{\theta(s)}{\delta_e(s)}$

It can be seen from the Bode plots, the longitudinal flight is obviously imposed in phugoind mode owning to a given elevator deflection as anticipated from the natural freqencies and damping ratios of the aircraft.

5 Transient Response of the Aircraft

In this section, the open loop time domain responses to an elevator deflection are plotted.

It is now necessary to define the positive deflection of the elevator. Down elevator (stick forward) is defined as “positive elevator” by NACA convention [7]. In the simulation elevator deflection was held 5 deg between 100[th] and 105[th] second.

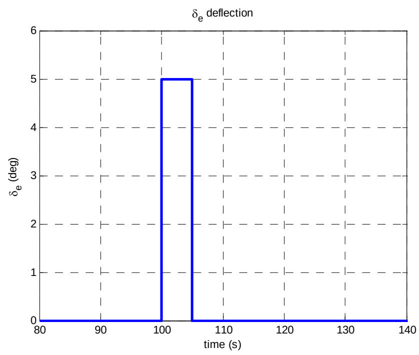

Fig.11 Elevator Deflection

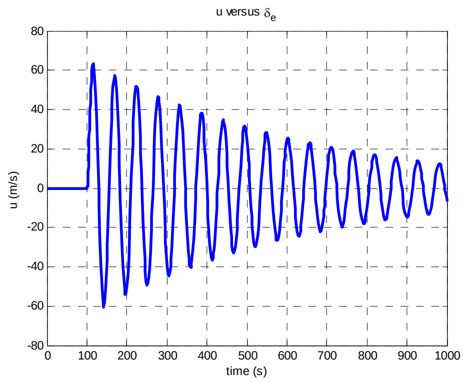

Fig.12 Time response u vs. t

The short- period mode primarily consists of variations in α and θ with very little change in the forward velocity[7,8]. As seen in Fig. 12, the short-period influence on u can not seen obviously, on the contrary the short-period influence on α and θ is noticeable in Figs. 14,16.

The phugoid mode oscillition primarily consist of variations of θ and u with almost a constant α [7,8]. In accordance with this, in the Fig.12, phugoid mode oscilition is dominant.

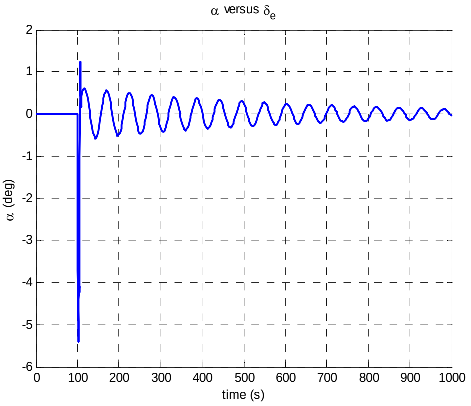

Fig.13 Time response α vs. t

Short-period oscillation can be seen more clearly in Fig.14.

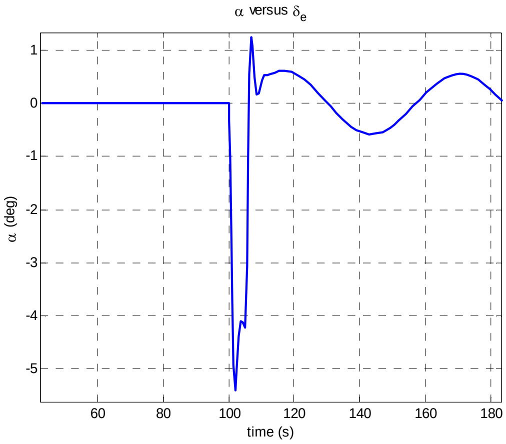

Fig.14 Short-period mode influence on α

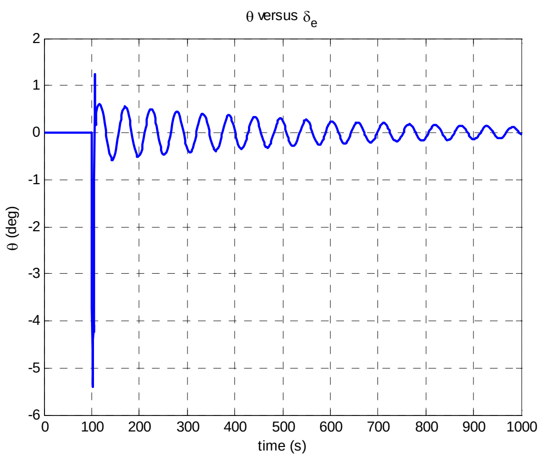

Fig.15 Time response vs. t

Short-period oscillation can be seen more clearly in Figure 16.

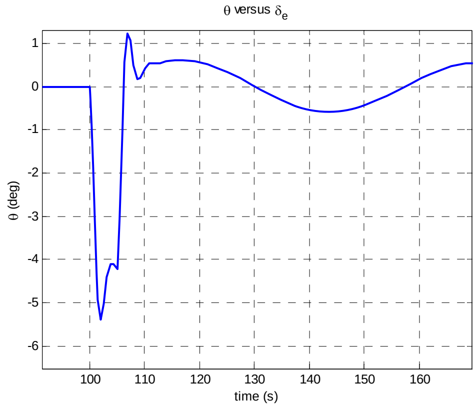

Fig.16 Short-period mode influence on θ

4 Conclusion

In this study, the longitudinal stability derivatives of a transport aircraft have been calculated by modeling the aircraft configuration and a flight condition at Advanced Aircraft Analysis (AAA) program. A longitudinal motion of aircraft is investigated and the transfer functions of the linearized equations have been obtained. The phugoid and short periods modes and frequency and transient-responses were analysed respectively, which show that there are the low damping and the long lasting oscillations in the longitudinal flight. For this reason, the flight dynamics of the aircraft needs automatic control systems design.

References:

[1] Advanced Aircraft Analysis Version 2.5, *User's Manual*, DAR Corporation.

[2] Roskam, J., *Airplane Flight Dynamics and Automatic Flight Controls*, Part I and II, DAR Corporation, 2001.

[3] Lan, C. E. and Roskam, J., *Airplane Aerodynamics and Performance*, Roskam Aviation and Engineering, 1980.

[4] Roskam J. , *Airplane Design, Parts I-VIII*, DAR Corporation, 1989.

[5] Yechout, T.R., Morris S.L., Bossert D.E. and Hallgren W.F., Introduction *to aircraft flight mechanics: performance, static stability, dynamic stability, and classical feedback control*, AIAA, 2003

[6] Ogata, K., *Modern Control Engineering*, Prentice Kall, Fourth Edition, 2002.

[7]Blakelock, J.H., *Automatic Control of Aircraft and Missiles*, Wiley, 1991.

[8] Turkoglu, K. and Jararov, E. M., Application of H inf. Lool Shaping Robust Control System Design on Longitudinal Dynamics of Hezarfen UAV with Classical PI(D) and Pole Placement Methods: A Comparison Analysis – (S/T), *WSEAS Transactions on Systems*, Issue 1,Vol 6,2007,pp.206-213.