# Application of The Variational Iteration Method to The Benjamin-Bona-Mahony (BBM) Equation

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*Abstract:* - In this study, a numerical solution of the Benjamin-Bona-Mahony (BBM) equation, also known as the regularized long-wave (RLW) equation, is presented by using the variational iteration method (VIM). Series expansions of the numerical approximation are done. The available analytical results and the obtained numerical results are compared to see the accuracy of the methods.

*Key-Words:* - Benjamin-Bona-Mahony (BBM) equation, Variational iteration method (VIM).

## 1 Introduction

The Benjamin-Bona-Mahony (BBM) equation was first introduced by Benjamin et al (1972) to study many problems of mechanical sciences, and it can be stated that BBM is a generalized version of the Korteweg-De Vries (KdV) equation for shallow water waves. BBM can be applicable to drift waves as well as Rossby waves in rotating fluids (Meiss and Hurton, 1982). The general form of the BBM is given by Zhang et al (2001) for generalized long waves.

In the nonlinear partial differential equation form, it can be written as

$$u_t + \alpha u_x + \beta u u_x - \delta u_{xxt} = 0 \qquad (1)$$

where $u(x,t)$ is a function of spatial coordinate $x$ and time $t$, and the subscripts $t$ and $x$ denote partial differentiation with respect to $t$ and $x$. $\alpha, \beta, \delta$ are constants with nonlinear and dispersion coefficients, i.e. $\beta \neq 0$ and $\delta > 0$.

Because of the appearance of the $u_{xxt}$ term in the equation (1), it is not similar to KdV type of evolution partial differential equation.
Initial condition for $u$ is denoted by

$$u(x,0) = f(x) \qquad (2)$$

By using a variable coefficient balancing act method, an exact solution of the general form of the BBM equation, Eq (1), was obtained by Zhang et al (2001). In this study, the exact solution will be compared with a newly developed powerful and efficient numerical method, VIM.

## 2 Variational Iteration Method (VIM)

Let consider the differential equation

$$Lu + Nu = f(t) \qquad (3)$$

where $L$ and $N$ are linear and nonlinear operators, respectively, and $f(t)$ is the inhomogeneous term. In the references (4-8), a correction functional for Eq. (3) can be written as

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(Lu_n(\xi) + N\tilde{u}_n(\xi) - f(\xi))d\xi \qquad (4)$$

where $\lambda$ is a general Lagrange's multiplier, which can be identified optimally via the variational theory, and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. The successive approximations $u_{n+1}$, $n > 0$, of the solution $u$ will be readily obtained upon using the determined Lagrangian multiplier and any selective function $u_0$. Therefore, the solution is given by

$$u = \lim_{n\to\infty} u_n \qquad (5)$$

## 3 Application of VIM and Numerical Results

Considering the BBM equation (Eq. 1), an initial condition can be assigned with $\alpha = 1$, $\beta = 1$ and $\delta = 1$:

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

$$u(x,0) = \frac{3}{8}\sec h^2(x/6) \qquad (6)$$

and the exact solution is given by Zhang et al (2001) as in the following form

$$u(x,t)=\frac{3}{8}Sech^2(x/6-9t/48) \quad (7)$$

The VIM can be applied to equation (6) in the form

$$u_{n+1}(x,t)=u_n(x,t)$$
$$+\int_0^t \lambda(u_t(x,\xi)+\tilde{u}_x(x,\xi)+u\tilde{u}_x(x,\xi)-\tilde{u}_{xxt}(x,\xi))d\xi \quad (8)$$

where $\lambda$ yields to $-1$.One can substitute above $\lambda$ and use the initial approximation as in equation (4) to get the successive approximation by the following expressions:

$$u_0=u[0]=\frac{3}{8}\sec h^2(x/6) \quad (9a)$$

$$u_1=\frac{3}{8}\sec h^2(x/6)+\frac{1}{8}t\sec h^2(x/6)\tanh(x/6)$$
$$\frac{11t^2\cosh(x/3)\sec h^8(x/6)}{2048}+$$
$$+\frac{3}{64}t\sec h^4(x/6)\tanh(x/6) \quad (9b)$$

$$u_2=\frac{3}{8}\sec h^2(x/6)-\frac{29t^2\sec h^8(x/6)}{1536}-$$
$$\frac{t^2\cosh(2x/3)\sec h^8(x/6)}{256}$$
$$+\frac{t^2\cosh(x)\sec h^8(x/6)}{1536}-$$
$$\frac{t\sinh(x/2)\sec h^7(x/6)}{384}-\frac{5t^3\sinh(x/2)\sec h^{11}(x/6)}{12288}+$$
$$\frac{t\sec h^7(x/6)\sinh(5x/6)}{1152}$$
$$+\frac{7t^3\sec h^{11}(x/6)\sinh(5x/6)}{73728}+$$
$$\frac{t^3\sec h^{11}(x/6)\sinh(7x/6)}{36864}+$$
$$\frac{t\sec h^2(x/6)\tanh(x/6)}{8}$$
$$+\frac{3t\sec h^4(x/6)\tanh(x/6)}{64}-\frac{49t\sec h^6(x/6)\tanh(x/6)}{1152}$$
$$-\frac{5t^3\sec h^{10}(x/6)\tanh(x/6)}{4608} \quad (9c)$$

These expressions for u[n] are obtained by using the Mathematica software. To illustrate the efficiency of the VIM, only first three terms will be used to compare with the exact solution. The comparison is given in Table 1, for *0<x<1*, and for *t=0.3* sec. In the figures from 2 to 4, the variation of u with its variation both in time and space is illustrated to see the obtained approximation with the exact one (figure 1).

## 4 Conclusion

The numerical solution of the Benjamin-Bona-Mahony (BBM) was studied by using a new powerful numerical method, variational iteration method. It was observed that the use of VIM provided a very good estimation when compared to exact values. Even the first three terms was enough to get accurate results. The higher terms are not required, since it causes a decrease in the efficiency of the method.

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**TABLE 1.** Comparison of exact and numerical methods for *u(x,t)* at time *t* = 0.3

| x | Exact | VIM | % Error |
|---|---|---|---|
| 0.1 | 0.37441304809846043 | 0.37490879457737475 | 0.132 |
| 0.2 | 0.37480312882714945 | 0.3753474930952519 | 0.145 |
| 0.3 | 0.3749853519439612 | 0.37557321401047816 | 0.157 |
| 0.4 | 0.3749593128390916 | 0.37558501409151046 | 0.167 |
| 0.5 | 0.3747250693480553 | 0.3753824958262723 | 0.175 |
| 0.6 | 0.3742831414788761 | 0.3749658146647877 | 0.182 |
| 0.7 | 0.3736345089609346 | 0.3743356821947829 | 0.188 |
| 0.8 | 0.37278060663528295 | 0.3734933651596483 | 0.191 |
| 0.9 | 0.37172331772359474 | 0.3724406803001826 | 0.193 |
| 1.0 | 0.3704649650298208 | 0.37117998507386646 | 0.193 |

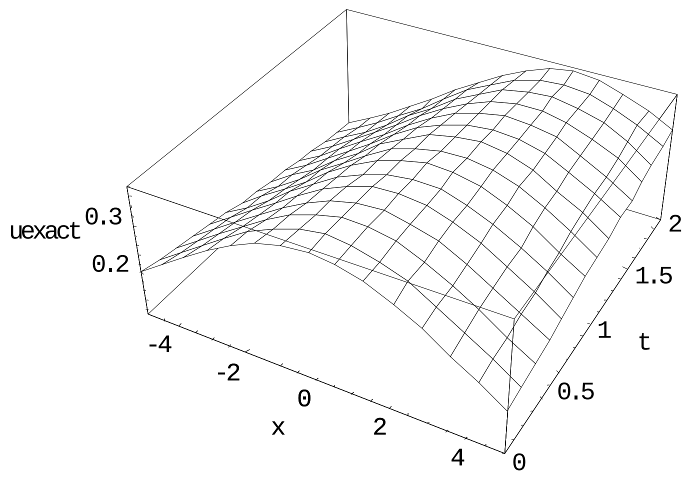


Figure 1. The exact solution of u(x,t) for $-5<x<5$ and $0<t<2$.

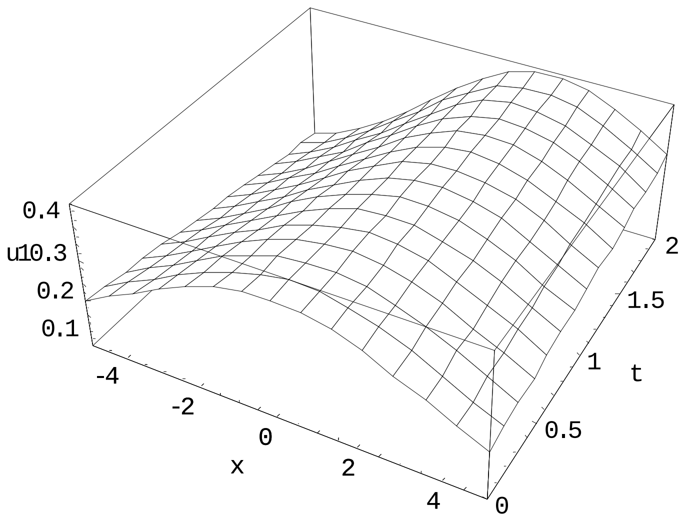


Figure 2. The VIM solution of $u_1(x,t)$ for $-5<x<5$ and $0<t<2$.

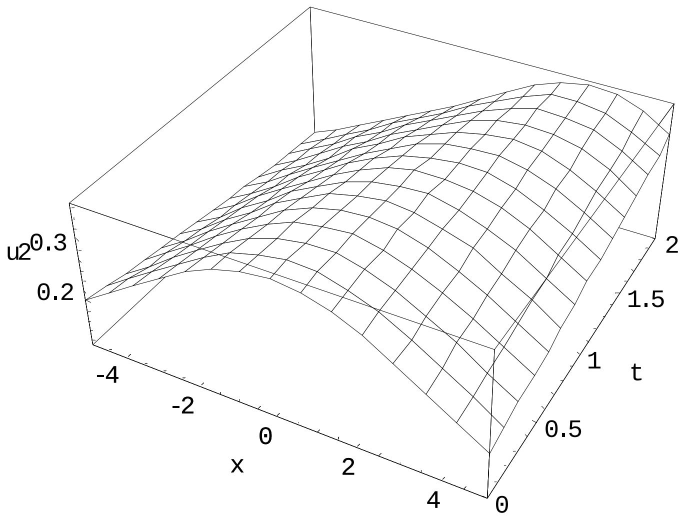


Figure 3. The VIM solution of $u_2(x,t)$ for $-5<x<5$ and $0<t<2$.

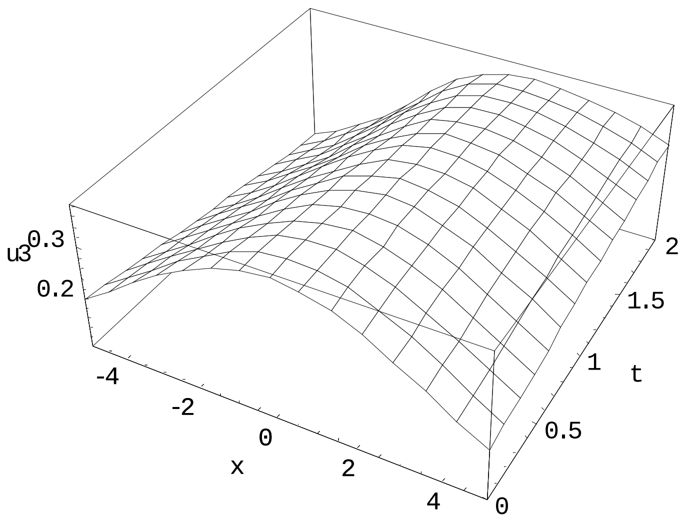


Figure 4. The VIM solution of $u_3(x,t)$ for $-5<x<5$ and $0<t<2$.