MODELLING AND SIMULATION OF SOLITONS AND DISCONTINUITIES IN ADIABATIC AND IDEAL GAS AND ELECROSTATIC PLASMA ONE DIMENSIONAL SYSTEMA

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Abstract: - Four models of gas and electrostatic plasma are studied viz ideal gas and adiabatic gas and ideal plasma and adiabatic plasma. These models are fluid equations (continuity ,momentum and energy) of gas and electrostatic plasma with initial conditions which give rise to soliton and shock formation and examine the effects of the ideal and adiabatic thermodynamic assumptions, used in each case, on the mathematical structures. Analyses of the gas and plasma hyperbolic Euler ideal/adiabatic systems exhibits shocks, contact and rarefaction waves and solitons exhibit solutions different from each other. Numerical computations of gas and plasma are derived from recent classes of high resolution schemes for hyperbolic systems *[H.Nessyahu and E.Tadmor: J.Comput.Phys.87 (1990) 408-463*] which have been successfully deployed in shock propagation computations reveal and confirm sharp difference in the two systems and ideal/adiabatic systems.

Key-Words: -Gas,electrostatic, plasma,solitons,shocks,hyperbolic,fluids,high resolution schemes.

1 Introduction

Gas and electrical plasma (ionised gas) are used widely in the electrical industry such as power systems, gas insulated substations (GIS) and the semiconductor industry [12]. Naldi et al [12] and Baboolal [2] modelled electrons and ions in a semiconductor with conservation of mass and momentum equations only. Fang and Yen [7] modelled electron and ions with charge conservation using the FCT Technique.

Mason [14] demonstrated solitons in plasma from a density hump and Cohn [6] observed the solitons experimentally. Biskamp et al [4] studied shocks but he treated electrons as a massless Boltzman fluid. This study is to model and simulate solitons and discontinuities in adiabatic and ideal gas and plasma systems and to demonstrate differences and similarities in their nonlinear structures. Modelling the gas enables understanding of the charges in the shocks and solitons. The adiabatic and ideal plasma and gases can be written in conservation form

$$\frac{\partial U}{\partial t}+\frac{\partial F(U)}{\partial x}=G \qquad (1)$$

where U is the field variable, F is the flux variable and G is the continuous source term and the system is hyperbolic i.e. the Jacobian has distinct eigenvalues. The gas and electrostatic plasma is Euler in that all the eigenvectors are real and distinct [8].

The solutions are generated from Riemann problem of a shock tube using Rim 3 initial conditions from which discontinuous solutions were generated for a heat transport gas systems Pember [15]. The Riemann shock tube is a piecewise constant initial data written as [10]:

$$U(x,0)=\begin{cases}U_l & x<0\\ U_r & x>0\end{cases} \qquad (2)$$

The Rim3 conditions:

$$\rho_1 = 2.5, u_1 = 1.0, p_1 = 1.0, E_1 = 3.25$$

$$\rho_2 = 1.0, u_2 = 0.4, p_2 = 0.568, E_2 = 1.5$$

where ρ, p and E are the density, pressure and energy respectively. The speed of the discontinuity is given by the Rankine-Hugoniot jump condition

$$F(u_L) - F(u_R) = s(u_L - u_R) \text{ [3]} \quad (3)$$

where u_L, u_R and s are left state, right state and the wave speed respectively

For the Euler system the Riemann shock tube depicts solutions as a shock, contact and a smooth rarefaction. A discontinuous wave is shock (compressive) if it satisfies (3) and speed relationship i.e.

$\lambda_i(U_L) > s_i > \lambda_i(U_R)$ where λ is the eigenvalue of the Jacobean of F. The contact discontinuity satisfies the relations $\lambda_i(U_L) = S_i = \lambda_i(U_R)$ and (3). The rarefaction wave satisfies only $\lambda_i(U_L) < \lambda_i(U_R)$ and is considered as a smooth transition. LeVeque [10] informs us that the shock and rarefactions are nonlinear degenerate ($\nabla\lambda_p(u).r_p(u) \neq 0 \forall u$) whilst the contact discontinuities are linear degenerate $\nabla\lambda_p(u).r_p(u) = 0 \forall u$. It is easy to verify using mathematica 6.02 software that gas and plasma system exhibits distinct eigenvector values and that shock, contact and rarefaction waves are present. Furthermore mathematica verifies that shock and rarefactions are genuinely nonlinear and the contact is linear degenerate. The Lax [10] condition advocates the shock and contact to satisfy the condition

$[u_L, u_R]$ is admissible if

$\lambda_p(u_L) \geq s \geq \lambda_p(u_R)$. The rarefaction condition is that

$\lambda_p(u_L) < s < \lambda_p(u_R)$ [3].

For smooth solution or solitons we consider the initial perturbation

$$U(x,0) = C + C_0 e^{-0.5x^2} \quad (4)$$

where C, C_0 are constants [4].

2 Problem Formulation

GAS SYSTEM:

IDEAL GAS: The conserved quantities and fluxes of the ideal gas system with energy can be written as

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho E u + p u \end{pmatrix} \quad (5).$$

The above system is closed by $p = (\gamma - 1)\rho e$ The Jacobian system is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ (\gamma-3)\frac{u^2}{2} & (3-\gamma)u & \gamma-1 \\ (\gamma-1)u^3 - \gamma u E & -\frac{3}{2}(\gamma-1)u^2 + \gamma E & \gamma u \end{bmatrix} \quad (6)$$

The eigenvalues are:

$$\lambda = u, u - \sqrt{\frac{-\gamma(\gamma-1)(u^2-2E)}{2}}, u + \sqrt{\frac{-\gamma(\gamma-1)(u^2-2E)}{2}} \quad (7)$$

To calculate the shock speeds s we use the Rankine-Hugoniot conditions below:

$$m_1 - m_2 = s(\rho_1 - \rho_2)$$

$$\left(\rho_1 u_1^2 + p_1\right) - \left(\rho_2 u_2^2 + p_2\right) = s(m_1 - m_2) \quad (8)$$

$$u_1(E_1 + p_1) - u_2(E_2 + p_2) = s(E_1 - E_2)$$

The above system was solved by eliminating m_2, p_2, The speeds calculated $s_1 = 0.981, 1.827, 0.650$ .We verify Lax condition on the shock wave only by calculations below:

$$\lambda(u_L) = u_L + \sqrt{\frac{-\gamma(\gamma-1)(u_L^2 - 2\varepsilon_L)}{2}} = 2.240 > s = 1.827 > \lambda(u_R) = u_R + \sqrt{\frac{-\gamma(\gamma-1)(u_R^2 - 2\varepsilon_R)}{2}} = 1.148 \quad (9)$$

ADIABATIC GAS:

Now we consider the system closed adiabatically by using the equation $p\rho^{-\gamma} = C$. Again we consider Rim 3 conditions with left initial conditions where $P_0 = 1.0, \rho_0 = 2.5$ .Since $P\rho^{-\gamma} = P_0\rho_0^{-\gamma} = 0.277$ where $\gamma = 1.4$. Therefore $p = 0.277\rho^{\gamma}$.

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix}$$

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + 0.277\rho^\gamma \\ \rho E u + 0.277\rho^\gamma u \end{pmatrix} \qquad (10)$$

The Jacobian matrix is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ -\frac{m^2}{\rho^2} + 0.277\gamma\rho^{\gamma-1} & \frac{2m}{\rho} & 0 \\ \frac{-\varepsilon m}{\rho^2} + 0.277(\gamma+1)m\rho^\gamma & \frac{\varepsilon}{\rho} + 0.277\rho^{\gamma+1} & \frac{m}{\rho} \end{bmatrix}$$

(11)

$$m = \rho u$$

The eigenvalues of the adiabatic system are given by

$$\lambda_1 = \frac{m}{\rho}, \lambda_2 = \frac{m}{\rho} - 0.52631\sqrt{\gamma}\rho^{0.5(\gamma-1)},$$

$$\lambda_3 = \frac{m}{\rho} + 0.52631\sqrt{\gamma}\rho^{0.5(\gamma-1)}$$

(12)

Again using (8) yields

$$m_1 - m_2 = s(\rho_1 - \rho_2)$$

$$\left(\frac{m_1^2}{\rho_1} + 0.277\rho_1^\gamma\right) - \left(\frac{m_2^2}{\rho_2} + 0.277\rho_2^\gamma\right) = s(m_1 - m_2)$$

$$\left(\varepsilon_1 + 0.277\rho_1^\gamma\right)\frac{m_1}{\rho_1} - \left(\varepsilon_2 + 0.277\rho_2^\gamma\right)\frac{m_2}{\rho_2} = s(\varepsilon_1 - \varepsilon_2)$$

(13)

The speeds are given by $s = 0.522, 1.460, 0.22$. The shock wave satisfies the Lax theorem (not shown here)

THE 3-FLUID SYSTEM ELECTROSTATIC EQUATIONS

We set up the system of plasma equations taking into account conservation of mass, continuity and energy.

IDEAL PLASMA SYSTEM

Continuity:

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v_e) = 0 \qquad (14)$$

$$\frac{\partial n_i}{\partial t} + \frac{\partial}{\partial x}(n_i v_i) = 0 \qquad (15)$$

Momentum:

$$\frac{\partial}{\partial t}(n_e v_e) + \frac{\partial}{\partial x}\left[\frac{(\alpha-1)}{Rm}E_e - \frac{1}{2}(\alpha-3)\frac{(n_e v_e)^2}{n_e}\right] = \frac{-e}{Rm}n_e \mathrm{E}$$

(16)

$$\frac{\partial}{\partial t}(n_i v_i) + \frac{\partial}{\partial x}\left[(\alpha-1)E_i - \frac{1}{2}(\alpha-3)\frac{(n_i v_i)^2}{n_i}\right] = e n_i \mathrm{E}$$

(17)

Energy:

$$\frac{\partial E_e}{\partial t} + \frac{\partial}{\partial n}\left[\frac{n_e v_e}{n_e}\alpha E_e - \frac{1}{2}(\alpha-1)R_m\frac{(n_e v_e)^3}{n_e^2}\right] = -e n_e v_e \mathrm{E}$$

(18)

$$\frac{\partial Ei}{\partial t} + \frac{\partial}{\partial x}\left[\alpha\frac{n_i v_i}{n_i}E_i - \frac{1}{2}(\alpha-1)\left(\frac{(n_i v_i)^3}{n_i^2}\right)\right] = e n_i v_i \mathrm{E}$$

(19)

The electric field (E) and potential field (ϕ) is calculated using

$$E = -\frac{\partial\phi}{\partial x},$$

(20)

$$\frac{\partial^2\phi}{\partial x^2} = -4\pi\sum_{k=e,i} q_k n_k = n_e - n_i \equiv S(U)$$

(21)

The electric field and potential energy contribute to the source terms in (1) and it is assume continuous across a thermodynamic discontinuity. The variables are respectively

$n_e, n_i, v_e, v_i, m_e, m_i, P_e, P_i, E_e, E_i$

are electron/ion densities, electron/ion velocities, mass electron/ion, Pressure electron/ion and energy electron/ion respectively. E and ϕ are electric field and potential respectively. Relative mass is $R_m = \frac{m_e}{m_i}$ =0.02 and

$\rho_k = m_k n_k, k = i, e$

Separate blocks for electron and ion subsystem can be written as the field, flux and source vectors as follows:

$$U = \begin{pmatrix} n_e \\ n_e v_e \\ E_e \\ n_i \\ n_i v_i \\ E_i \end{pmatrix}$$

$$F = \begin{pmatrix} n_e v_e \\ \frac{(\alpha-1)}{Rm} E_e - \frac{1}{2}(\alpha-3)\frac{(n_e v_e)^2}{n_e} \\ \frac{n_e v_e}{n_e}\alpha E_e - \frac{1}{2}(\alpha-1)R_m \frac{(n_e v_e)^3}{n_e^2} \\ n_i v_i \\ (\alpha-1)E_i - \frac{1}{2}(\alpha-3)\frac{(n_i v_i)^2}{n_i} \\ \alpha\frac{n_i v_i}{n_i}E_i - \frac{1}{2}(\alpha-1)\left(\frac{(n_i v_i)^3}{n_i^2}\right) \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ -e n_e E \\ -e n_e v_e E \\ 0 \\ e n_i E \\ e n_i v_i E \end{pmatrix} \qquad (22)$$

The Jacobian matrix is given by

(23)

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{(\gamma-3)m_1^2}{2\rho_1^2} & -\frac{(\gamma-3)m_1}{\rho_1} & \frac{(\gamma-1)}{R_m} & 0 & 0 & 0 \\ -\frac{m_1}{\rho_1^2}\gamma E_1 + (\gamma-1)R_m\frac{m_1^3}{\rho_1^3} & \frac{\gamma E_1}{\rho_1} - \frac{3}{2}(\gamma-1)R_m\frac{m_1^2}{\rho_1^2} & \gamma\frac{m_1}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{(\gamma-3)m_2^2}{2\rho_2^2} & -\frac{(\gamma-3)m_2}{\rho_2} & (\gamma-1) \\ 0 & 0 & 0 & -\frac{m_2}{\rho_2^2}\gamma E_2 + (\gamma-1)\frac{m_2^3}{\rho_2^3} & \frac{\gamma E_2}{\rho_2} - \frac{3}{2}(\gamma-1)\frac{m_2^2}{\rho_2^2} & \gamma\frac{m_2}{\rho_2} \end{pmatrix}$$

The eigenvalues for the ideal electron subsystem are:

$$\lambda = u_e, u_e - \sqrt{\frac{-\gamma(\gamma-1)(m_e^2 R_m - 2E_e\rho_e)}{2R_m\rho_e}}, u_e + \sqrt{\frac{-\gamma(\gamma-1)(m_e^2 R_m - 2E_e\rho_e)}{2R_m\rho_e}},$$

(24)

Using (3) again the system reduces to:

$$n_{e1}v_{e1} - n_{e2}v_{e2} = s(n_{e1} - n_{e2})$$

$$\left(\frac{\gamma-1}{R_m}E_{e1} - \frac{1}{2}(\gamma-3)\frac{(n_{e}v_{e1})^2}{n_{e1}}\right) - \left(\frac{\gamma-1}{R_m}E_{e2} - \frac{1}{2}(\gamma-3)\frac{(n_{e}v_{e2})^2}{n_{e2}}\right) = s(n_{e1}v_{e1} - n_{e2}v_{e2})$$

$$\left(\gamma v_{e1}E_{e1} - \frac{1}{2}R_m n_{e1}v_{e1}^3\right) - \left(\gamma v_{e2}E_{e2} - \frac{1}{2}R_m n_{e2}v_{e2}^3\right) = s(E_{e1} - E_{e2})$$

(25)

$$(4.5 - 0.5R_m) - \left(3.5E_{e2} - 0.2R_m\left(\frac{1.625}{R_m} + 2.5 - 1.875s^2 - 0.5\frac{E_{e2}}{R_m}\right)^{\frac{3}{2}}\right) = s(3.25 - E_{e2})$$

(26)

Using mathematica 6.02 speeds have two complex values and one real value i.e. $s = 0.424$.

The eigenvalues for the ideal ion subsystem are:

$$\lambda = u_i, u_i - \sqrt{\frac{-\gamma(\gamma-1)(m_i^2 - 2E_i\rho_i)}{2\rho i}},$$

$$u_i + \sqrt{\frac{-\gamma(\gamma-1)(m_i^2 - 2E_i\rho_i)}{2\rho_i}},$$

(27)

The eigenvalues are for the ions are same for the ideal gas system in (8). Hence the ion shock will also satisfy the Lax condition.

ADIABATIC PLASMA SYSTEM

Continuity:

$$\frac{\partial n_e}{\partial t}+\frac{\partial}{\partial x}\left(n_e v_e\right)=0 \tag{28}$$

$$\frac{\partial n_i}{\partial t}+\frac{\partial}{\partial x}\left(n_i v_i\right)=0 \tag{29}$$

Momentum:

$$\frac{\partial\left(n_e v_e\right)}{\partial t}+\frac{\partial\left(n_e v_e^2+\frac{P_e}{m_e}\right)}{\partial x}=-\frac{e n_e E}{m_e} \tag{30}$$

$$\frac{\partial}{\partial t}\left(n_i v_i\right)+\frac{\partial}{\partial x}\left(n_i v_i{}^2+\frac{P_i}{m_i}\right)=\frac{e n_i E}{m_i} \tag{31}$$

Energy:

$$\frac{\partial E_e}{\partial t}+\frac{\partial\left(E_e+p_e\right)v_e}{\partial x}=-e n_e v_e E \tag{32}$$

$$\frac{\partial E_i}{\partial t}+\frac{\partial\left(E_i+p_i\right)v_i}{\partial x}=-e n_i v_i E \tag{33}$$

We now close the above equation system using the adiabatic condition $p=0.277\rho^{\gamma}$ yielding the following system of equations

$$U=\begin{pmatrix} n_e \\ n_e v_e \\ E_e \\ n_i \\ n_i v_i \\ E_i \end{pmatrix}$$

$$F=\begin{pmatrix} n_e v_e \\ n_e v_e^2+0.277\,R_m^{\gamma-1} n_e^{\gamma} \\ \left(E_e+0.277\,R_m^{\gamma} n_e^{\gamma}\right)\frac{n_e v_e}{n_e} \\ n_i v_i \\ n_i v_i^2+0.277\,n_i^{\gamma} \\ \left(E_i+0.277\,n_i^{\gamma}\right)\frac{n_i v_i}{n_i} \end{pmatrix}$$

$$G=\begin{pmatrix} 0 \\ -e n_e E \,/\, Rmass \\ -e n_e v_e E \\ 0 \\ e n_i E \\ e n_i v_i E \end{pmatrix} \tag{34}$$

The Jacobian matrix is given by

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-m_1^2}{\rho_1^2}+\gamma 0.277 R_m^{\gamma-1}\rho_1^{\gamma-1} & \frac{2m_1}{\rho_1} & 0 & 01 & 0 & 0 \\ -\left(E_1+0.277\rho_1^{\gamma}\right)\frac{m_1}{\rho_1^2}+\frac{m_1}{\rho_1}\gamma 0.277\rho_1^{\gamma-1} & \frac{\left(E_1+0.277\rho_1^{\gamma}\right)}{\rho_1} \; 0 & \frac{m_1}{\rho_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \\ & 0 & \frac{-m_2^2}{\rho_2^2}+\gamma 0.277\rho_2^{\gamma-1} & \frac{2m_2}{\rho_2} & 0 & \\ 0 & 0 & -\left(E_2+0.277\rho_2^{\gamma}\right)\frac{m_2}{\rho_2^2}+\frac{m_{21}}{\rho_{21}}\gamma 0.277\rho_2^{\gamma-1} & \frac{\left(E_{21}+0.277\rho_2^{\gamma}\right)}{\rho_{21}} & \frac{m_2}{\rho_2} & \end{pmatrix}$$

The eigenvalues for the adiabatic electron subsystem:

$$\lambda_e=u_e,\, u_e-0.5261\sqrt{\gamma R_m^{\gamma-1}}\,\rho_e^{0.5(\gamma-1)},\; u_e+0.5261\sqrt{\gamma R_m^{\gamma-1}}\,\rho_e^{0.5(\gamma-1)} \tag{36}$$

Using (3) on the adiabatic electron subsystem:

$$\begin{aligned} & n_{e1}v_{e1}-n_{e2}v_{e2}=s\left(n_{e1}-n_{e2}\right) \\ & n_{e1}v_{e1}^2+0.277R_m^{\gamma-1}n_{e1}^{\gamma}-n_{e1}v_{e2}^2-0.277R_m^{\gamma-1}n_{e2}^{\gamma}=s\left(n_{e1}v_{e1}-n_{e2}v_{e2}\right) \\ & \left(E_{e1}+0.277n_{e1}^{\gamma}R^{\gamma}\right)v_{e1}-\left(E_{e2}+0.277n_{e2}^{\gamma}R^{\gamma}\right)v_{e2}=s\left(E_{e1}-E_{e2}\right) \end{aligned} \tag{37}$$

The speeds are 1.2, 0.799 and -0.713. The shock speed satisfies the Lax

condition.
The eigenvalues for the adiabatic ion subsystem indicate left and right shockwaves.

$$\lambda_i = u_i, u_i - 0.5261\sqrt{\gamma}\rho_i^{0.5(\gamma-1)}, \quad u_i + 0.5261\sqrt{\gamma}\rho_i^{0.5(\gamma-1)} \tag{38}$$

The adiabatic ion wave will have the same values as the adiabatic gas in (13).

3 Problem Solution

3.1THE NUMERICAL MODEL

To capture the Riemann shock tube waves and solitons in system (1) the NNT numerical scheme [11] was used. The Nessyahu and Tadmor numerical scheme [11] as modified in [13] numerically models shocks and smooth solutions of (1) with great accuracy. The second order formula is given as follows:

$$\bar{u}_j^{n+1} = \frac{1}{4}\left[\bar{u}_{j+1}^n + 2\bar{u}_j^n + \bar{u}_{j-1}^n\right] - \frac{1}{16}\left[u_{xj+1}^n - u_{xj-1}^n\right] - \frac{1}{8}\left[u_{xj+\frac{1}{2}}^{n+1} - u_{xj-\frac{1}{2}}^{n+1}\right] + \frac{\Delta t}{8}\left[g\left(u_{j+1}^n\right) + 2g\left(u_j^n\right) + g\left(u_{j-1}^n\right)\right] + \frac{\Delta t}{8}\left[g\left(u_{j+1}^{n+1}\right) + 2g\left(u_j^{n+1}\right) + g\left(u_{j-1}^{n+1}\right)\right] - \frac{\lambda}{4}\left[\left(f_{j+1}^n - f_{j-1}^n\right) + \left(f_{j+1}^{n+1} - f_{j-1}^{n+1}\right)\right] \tag{39}$$

As in shock calculations we shall employ the min-mod derivatives [11]:

$$u_{xj-\frac{1}{2}}^{n+1} = MM\left(\Delta u_j^{n+1}, \Delta u_{j-1}^{n+1}\right), u_{xj+\frac{1}{2}}^{n+1} = MM\left(\Delta u_{j+1}^{n+1}, \Delta u_j^{n+1}\right) \tag{40}$$

Where the nonlinear limiter MM is defined by

$$\mathrm{MM}(s_1, s_2, \ldots) = \begin{cases} \min\{s_j\} \text{if } s_j > 0 \forall j \\ \max\{s_j\} \text{if } s_j < 0 \forall j \\ 0 \text{ otherwise} \end{cases} \tag{41}$$

And where after some simplification:

$$\Delta u_j^{n+1} = \bar{u}_{j+\frac{1}{2}}^{n+1} - \bar{u}_{j-\frac{1}{2}}^{n+1} = \frac{1}{2}\left[\bar{u}_{j+1}^n - \bar{u}_{j-1}^n\right] - \frac{1}{8}\left[u_{xj+1}^n - 2u_{xj}^n + u_{xj-1}^n\right] - \frac{\lambda}{2}\left[f_{j+1}^{n+1} - 2f_j^{n+1} + f_{j-1}^{n+1} + f_{j+1}^n - 2f_j^n + f_{j-1}^n\right] + = + \frac{\Delta t}{4}\left[g\left(u_{j+1}^{n+1}\right) - g\left(u_{j-1}^{n+1}\right) + g\left(u_{j+1}^n\right) - g\left(u_{j-1}^n\right)\right] \tag{42}$$

Here again the

$$u_{xj} = \mathrm{MM}\left(u_{j+1} - u_j, u_j - u_{j-1}\right) \tag{43}$$

Also useful in our applications is the more accurate UNO derivative [11]

$$u_{xj} = \mathrm{MM}\Big(u_j - u_{j-1} + \frac{1}{2}\mathrm{MM}\left(u_j - 2u_{j-1} + u_{j-2}, u_{j+1} - 2u_j + u_{j-1}\right), u_{j+1} - u_j - \frac{1}{2}\mathrm{MM}\left(u_{j+1} - 2u_j + u_{j-1}, u_{j+2} - 2u_{j+1} + u_j\right)\Big). \tag{44}$$

To apply the scheme, which is implicit in time, we use the predictor given in [11]

$$u^{n+1} = u^n + \Delta t\left[g\left(u^n\right) - \frac{1}{\Delta x} f_x^n\right] \tag{45}$$

Where the flux derivatives $\frac{1}{\Delta x} f_x^n$ are at the indicated time level n and can be evaluated using the MM function or calculated from the explicit form of $f(u)$.
The stability has been demonstrated in [13] using Roe's stability formula. The linear stability analysis of the NNT scheme indicates that it should remain stable under CFL condition

$\Lambda_m \dfrac{\Delta t}{\Delta x} \leq 0.01$, where Λ_m is the spectral radius of the flux Jacobian. This is stronger condition than that used in [13] for electrostatic simulations.

Here $\Delta x = 0.1, \Delta \mathrm{t} = 0.001.$ The CFL value can be calculated as

$$CFL = \frac{\Delta t}{\Delta x} = \frac{0.001}{0.1} = 0.01$$

All boundary conditions were reflective Neumann homogeneous $(\dfrac{\partial u}{\partial x} = 0)$ given in [1].

The central difference scheme was used for the Poisson equation:

$$\phi_{m-1} - 2\phi_m + \phi_{m+1} = \Delta x^2 S_m, m = 1,2,...M-1 \quad (46)$$

The electric field was solved as a tridiagonal linear system with right hand side vector

$$\left[\Delta x^2 S_1 - \phi_0, \Delta x^2 S_2, ..., \Delta x^2 S_{M-2}, \Delta x^2 S_{M-1} - \phi_M\right] \quad (47)$$

This is solved iteratively due to the simi implicit scheme in [2].

SOLITON AND SHOCK WAVES COMPUTATIONS

The initial values for gas equations generated using (4):

$\rho = 1.0 + e^{-0.5x^2}$ For 40 grids at the centre

$\rho = 1.0$ Elsewhere

$\rho u = 0$ For all grids

$\varepsilon = 2.5 + 0.5e^{-0.5x^2}$ For 40 grids at the centre

$\varepsilon = 2.5$ Elsewhere

The initial values are for the plasma equations are taken as follows:

$$n_k(x,0) = 1.0 + 2.0\exp\left(-0.5x^2\right), \left(n_k = e,i\right)$$

$$En_k(x,0) = 1.0 + 2.0\exp\left(-0.5x^2\right), \left(n_k = e,i\right)$$

$$v_k = 0, \ \left(n_k = e,i\right)$$

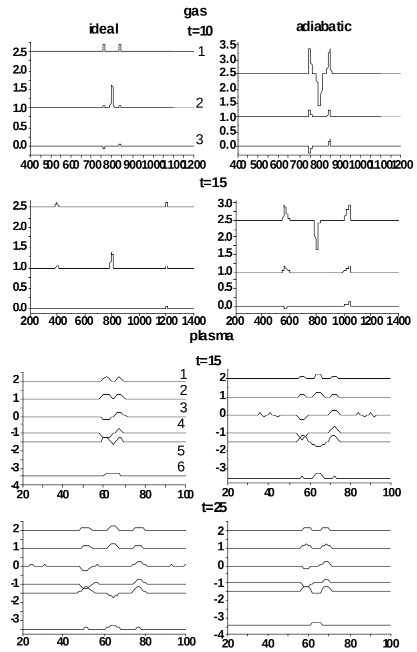

Fig 1 Gas and plasma ideal and adiabatic smooth graphs.In gas graphs 1,2 and 3 represents energy,density and momentum respectively.The gas CFL values are δx=0.3,δt=0.1.In the plasma graphs 1~n_e+1, 2~n_i, 3~$n_e v_e$,4~$n_i v_i$-1,5~ε_e-1.5,6~ε_i-3.The plasma CFL values are δx=0.5 and δt=0.005.

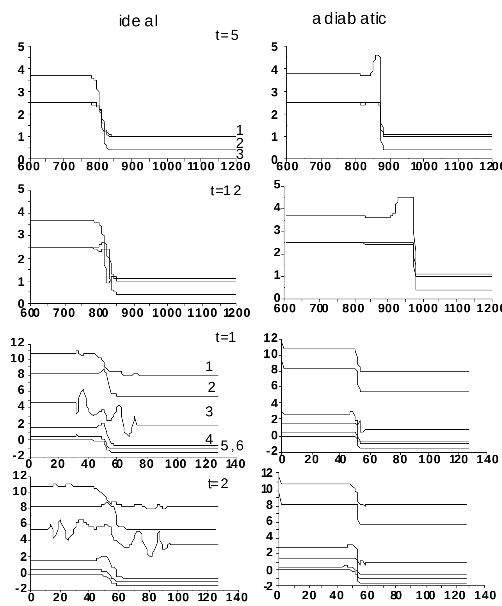

Fig2 Gas and plasma ideal and adiabatic shock graphs. In the gas system 1~energy,2~density and 3~momentum.The CFL values are δt=0.1 and δx=0.3. In the plasma system 1/2~energy$_{e/i}$, 3/4~momentum$_{e/i}$,5/6~density$_{e/i}$.The CFL values are δt=0.001 and δx=0.1

Conclusion

For ideal gas and ideal plasma two solitons travel to the left and right with equal velocity and equal amplitude. In both models a stationary soliton at the centre is exhibited. The adiabatic gas and adiabatic plasma yields two density solitons with the energy waves showing a negative growth at the centre. In the adiabatic models solitons tend to travel slower than the ideal models.
The adiabatic shockwaves for both the models exhibits all three Riemann solutions whilst the ideal gas shockwaves tend produce oscillations at the contact. The ideal plasma electron subsystem produces contact discontinuity only while the ideal ion plasma subsystem produces all three Riemann solutions..

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