Control of Network Structure by an External Field on Random Walkers

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Abstract: We propose a network model where network structure can be controlled by an external field which influences the movement of random walkers. Changes of network structure induced by the external field is detected by the observation of mean vertex-vertex distance, correlation between adjacent vertices, density of edges, frequency distribution of vertex degree, and distribution of the shortest path length from each vertex. These changes of network structure can be summarized as a phase diagram. Depending on the mechanism of the linking of vertices based on local structure of the network, various network structures can be established.

Key–Words: Network evolution, Random walker, Degree distribution, Degree correlation, Phase diagram

1 Introduction

With the discovery of the common structure of real networks and growing awareness of its importance, numerous mathematical models comprising simple principles which can describe certain features of real networks have been proposed. Studies on small world network model [1] and preferential attachments model [2] are epoch-making, both of which can reproduce small-world phenomena and scale-free degree distribution that can be seen commonly in the real world such as internet structures, biological systems, social networks, etc [3, 4].

On the other hand, recent studies have revealed that different systems sometimes have different structures which stem from their intrinsic characters. For example, it was found that degree-degree correlation between adjacent vertices can take positive (assortative mixing) or negative (disassortative mixing) values according to the kind of networked system [5]. A class of intrinsic characters of networked systems should be originated from the generating behavior of the local structure of the network. For example, considerations for connecting newly attached vertices with other vertices which are reachable by following a path can result in clustering coefficients and degree correlations commonly seen in real networks [6, 7]. These models have been proposed with characters of social network in mind.

In our recent works, we proposed a network evolution model considering the rise and fall of connections determined by random transports between elements of a system [8]. In the model, random walkers can move and leave edges behind their traces, the movements of which conform to local connections in a network. It has been found that this model can produce typical properties of complex networks such as large clustering coefficients, small mean vertex-vertex distance, and broad type degree distribution. A few number of vertices with large degree can be realized intuitively as a consequence of the effect that vertices with large degree tend to attract random walkers with its edges and gain new shortcuts to the vertex. However, properties intrinsic to this model which stem from random walker movements have not been really investigated yet and the vertices under consideration have been only passively waiting for random walkers.

The aim of this paper is to find out if we can control network structure by controlling local density of random walkers by using a certain kind of external field. This idea is based on the observation that a diffusion process of random walkers starting from the same vertices yields time-dependent changes of the network structure [9] and that vertices with large degree tend to attract random walkers. The external field introduced in this paper can control the local density of random walkers intentionally(Section 3). The changes of network structure is examined by the calculation of degree distribution, degree correlation, and vertex-vertex distance distribution (Section 4). Results obtained by the calculation can be summarized as a phase diagram (Section 5).

More detailed rules of the model are described as follows. Suppose that there are a number of random walkers in a graph. The movements of random walkers are affected by the field explained in the next section. The creation rule of edges is that a vertex where a walker currently stays and vertices where the walker has stayed one and two time-steps earlier are newly linked (or added strength by one if there already exists an edge there.). The extinction rules of edges specify that all edges except initially existed edges tend to be extinct with probability p_d per unit time.

The exception rule for the extinction means that the network maintains initial geographical connections. Consideration of the geographical structure of networks is one of the characteristics of our model, although the present paper only takes one-dimensional lattice as initial lattice for easy calculation. Selection of other initial lattices (for example, two-dimensional squared lattice) will lead to results different to those in this paper. It should be also noted that the movement of random walkers expresses special type of transports, as the transports do not exhibit birth and death and the movement of random walker is always restricted by its last movement.

2 Description of External Field

In order to describe an external field, let us introduce a potential energy function $U(|\boldsymbol{r}|)$ defined on each vertices, where index $|\boldsymbol{r}|$ means a distance measured in an initial regular lattice from the origin. Note that $|\boldsymbol{r}|$ does not mean the shortest path length from the origin in the evolved network. We assume that transition probabilities of random walkers have differences proportional to a force term $-(U(|\boldsymbol{r}_2|) - U(|\boldsymbol{r}_1|))/(|\boldsymbol{r}_2| - |\boldsymbol{r}_1|)$, where $|\boldsymbol{r}_1|$ and $|\boldsymbol{r}_2|$ are distances corresponding to a vertex where the walker currently stays and a vertex where the walker can reach in the next time-step, respectively. In other words, the transition probability $T_{x\to y}$ of random walkers from a vertex x to $y(\neq x)$ is given by

$$T_{x\to y} = \begin{cases} q_x - (U(|\boldsymbol{r}_y|) - U(|\boldsymbol{r}_x|))/(|\boldsymbol{r}_y| - |\boldsymbol{r}_x|) & \\ \qquad\qquad \text{for } |\boldsymbol{r}_x| \neq |\boldsymbol{r}_y| & \\ q_x \qquad\qquad \text{for } |\boldsymbol{r}_x| = |\boldsymbol{r}_y| & \end{cases} \tag{1}$$

where q_x is determined by a normalization.

If the initial network is a one-dimensional lattice with a constant lattice period and $U(|\boldsymbol{r}|)$ generates a uniform attractive force to the origin like $U(|\boldsymbol{r}|) = f|\boldsymbol{r}|$, then the first row of (1) can be simplified as $T_{x\to y} = q_x \pm f$. Furthermore, if vertex x is linked to H vertices with potential higher than x, linked to L vertices with potential lower than x, and linked to I vertices with the same potential as x, the normalization condition of $T_{x\to y}$ yields the next formula,

$$T_{x\to y} = \begin{cases} (1 + (H - L))/k + f & \text{for } |\boldsymbol{r}_x| > |\boldsymbol{r}_y| \\ (1 + (H - L))/k - f & \text{for } |\boldsymbol{r}_x| < |\boldsymbol{r}_y| \\ (1 + (H - L))/k & \text{for } |\boldsymbol{r}_x| = |\boldsymbol{r}_y|, \end{cases} \tag{2}$$

where k denotes degree of vertex x ($k = H + L + I$). If $(1 + (H - L))/k - f < 0$ in (2), it is treated as 0 and other probabilities are renormalized.

In the following sections, network formed by movements of random walkers following the transition rule of (2) is investigated. All calculations presented in this paper are carried out for a case of number of walkers $w = 32$. The number of walkers directly affects the increase rate of new edges per time, but may not cause a large qualitative change of the results.

3 Effects of External Field on Movements of Random Walkers

In this section, movements of a random walker modulated by the external field are examined and compared to a case without an external field. First, Figure 1 presents three samples of the movement of random walkers, which indicate typical behavior of the movemer

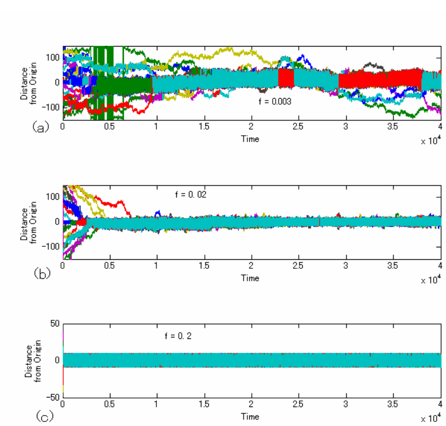

Figure 1: Typical examples of random walker's movement. The conditions are (a) $p_d = 0.15$, $f = 0.003$, (b) $p_d = 0.15$, $f = 0.02$, and (c) $p_d = 0.15$, $f = 0.2$.

Figure 1 (a) presents a case for a low field, in which walkers are slowly attracted to the origin and form a region where the walkers stationary exist but the walkers can easily escape from the region. On the other hand, the walkers illustrated in Figure 1 (b) are nearly confined in a certain region where the walkers can move freely. However the boundary fluctuates with time. Further increase in the intensity of the field make the boundary of the region hard (see Figure 1 (c)). In this stage, the walker cannot go beyond a certain point.

For cases where movements of random walkers do not generate new links, which means p_d is 1, the movements can be investigated analytically. In this case, the transition probability is simplified as $T_{x\to y} = 1/2 \pm f$, because all vertices always have degree 2. In this case, the probability can be expressed by the following equation,

$$P_x(t+1) = (\frac{1}{2}+f)P_{x+1}(t) + (\frac{1}{2}-f)P_{x-1}(t), \quad (3)$$

where the probability of walkers visiting the vertex x at time t is denoted as $P_x(t)$ and the condition $x \geq 1$ is assumed. To obtain equilibrium values of $P_x(t)$, let us substitute $P_{|x|}$ for $P_x(t)$, where the function $P_{|x|}$ does not depend on t and assumed to be symmetrical with respect to the origin $x = 0$. Then (3) becomes

$$0 = (\frac{1}{2}+f)P_{|x+1|} - P_{|x|} + (\frac{1}{2}-f)P_{|x-1|}. \quad (4)$$

By solving (4) the equilibrium probability $P_x(t)$ is obtained as

$$P_{|x|} = (\frac{\frac{1}{2}-f}{\frac{1}{2}+f})^{|x|}(1-\frac{\frac{1}{2}-f}{\frac{1}{2}+f})/(1+\frac{\frac{1}{2}-f}{\frac{1}{2}+f}). \quad (5)$$

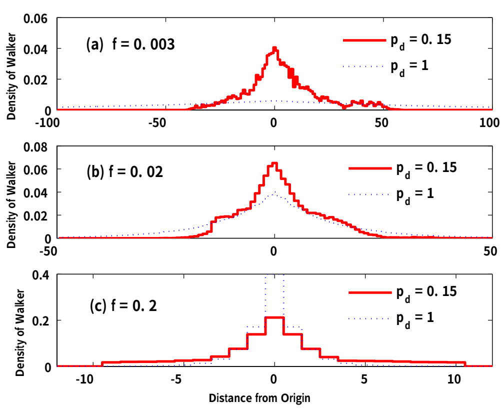

Figure 2: Frequency distribution of visits of random walkers to vertices. Calculations were carried out for movements of $50,000$ steps after $t = 100,000$ when $p_d = 0.15$. (a) $f = 0.003$. (b) $f = 0.02$. (c) $f = 0.2$. These conditions corresponds to Figure 1.

The analytical spatial distributions of random walkers for cases $p_d = 1$ are compared with observed data for cases of $p_d = 0.15$ in Figure 2. Their differences indicate the effect of creation of edges by random walkers. A noticeable feature in the figure is reversal of width of the region where the walker can move due to an increase in the field. When low field is applied, walkers gathered by an external field produce further cohesion of the walkers (see Figure 2 (a) and (b)). This cohesion is considered to originate from the restricted movement of walkers by edges made by themselves. On the other hand, a high field case illustrated in Figure 2 (c) shows an area where the walker can exist at $p_d = 0.15$ wider than at $p_d = 1$. In this case, random walkers when $p_d = 0.15$ can move along many shortcuts made by themselves, which do not exist when $p_d = 1$.

Those results demonstrate examples of a relation between random walker's movements and structure of the formed network. Another evident example is the fact that vertices not visited by walkers never gain new edges. Even vertices where random walkers stayed once may lose their edges. In the next section, results of further investigation on the changes of network structure are discussed.

4 Network Structure Induced by the External Field

In this section, changes of network structure induced by the external field are determined by the mean vertex-vertex distances, correlation between adjacent vertices, density of edges, frequency distribution of vertex degree, and distribution of the shortest path length from each vertex.

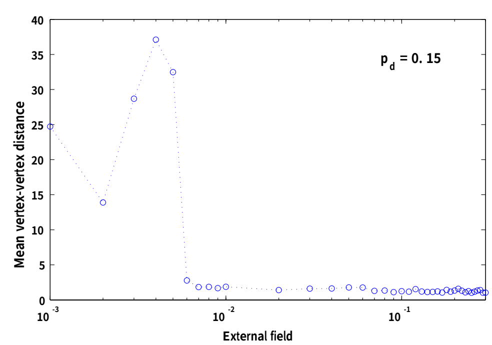

Figure 3: Mean distances between vertices with edges created by random walker's movement. Graphs after $200,000$ time steps were considered.

Figure 3 and Figure 4 presents changes of mean vertex-vertex distances, degree-degree correlation, and degree-clustering strength between adjacent vertices with respect to the external field f. In this section, note that calculations were limited on vertices

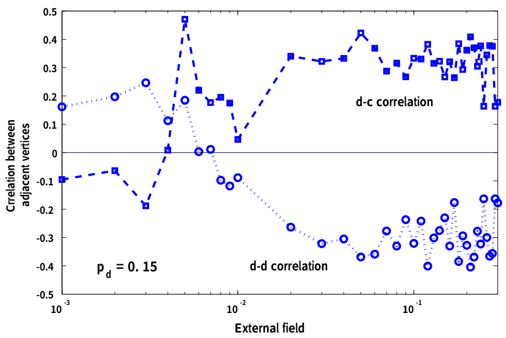

Figure 4: Degree-degree correlation and degree-clustering strength between adjacent vertices. Considered graphs are same as that in Figure 3.

that were incident to at least three edges, a part of which is created by the passing of random walkers. Large mean distance found in low field is attributable to a dispersed structure of the network by frequent escapes from random walker from the central network, which can be seen in Figure 1 (a). A point to be noted is that the field intensity indicating sudden reduction of the mean distance (Figure 3) is corresponding to an field that indicates inversion of the sign of correlation (see Figure 4). In our model, negative degree-degree correlation between adjacent vertices has been found after introducing external fields, while positive degree-degree correlation can be ordinary observed.

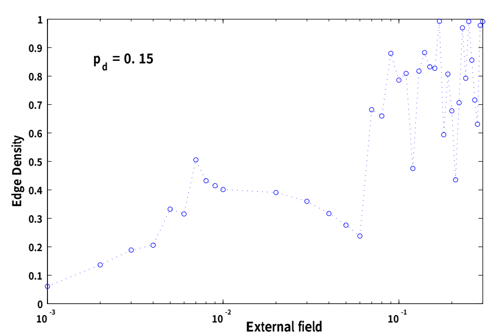

Figure 5: Edge-density $L/{}_MC_2$ with respect to f, where L and M denote number of edges by random walkers and number of vertices with degree more than three. The value is an average of the time interval $14,0000 < t < 150,000$.

Figure 5 shows changes in the edge-density with respect to f, where edge-density means a ratio of the number of edges to the number of all pair of vertices under consideration. The edge-density indicates sudden increase at a certain value of f which corresponds to the change from the fluctuation boundary of a graph to a hard boundary of graph as can be seen in Figure 2 (b) and (c). In other words, the network transits from a low density state of edges to a high density state of edges like a nearly complete graph.

The changes of network structure can be also found by the observation of degree distribution. In Figure 6 degree distributions are presented, each of which corresponds to three types of network structure discussed above. In Figure 6 (a) a large part of vertices only have a few incident edges, while a small part of vertices have a large degree in comparison. In Figure 6 (b) the number of vertices with a few incident edges is seen to decrease. However, amore important observation is that the maximum degree is identical to the number of vertices subtracted by one. In other words, vertices around the center of attraction are linked to all the other vertices that random walkers can visit. This structure is the reason for the negative degree-degree correlation mentioned above, for vertices with large degree are surely joined to other vertices with small degree. Finally, Figure 6 (c) indicates a formation of a nearly complete graph.

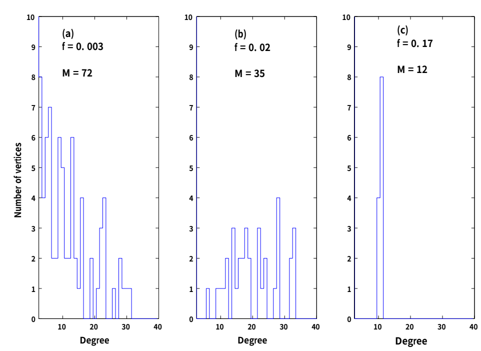

Figure 6: Degree distributions calculated for graphs after passing sufficient large times when $p_d = 0.15$. (a) $f = 0.003$. (b) $f = 0.02$. (c) $f = 0.17$. M denotes number of vertices with degree more than three.

A more detailed structure can be found from the observation of the distribution of the shortest path length from each vertex. When the field f is small, each vertex have various distances to all the other vertices (see Figure 7). In this case, a positive degree-degree correlation was observed. In Figure 8, however, it can be observed that not only are pairs of vertices more or less connected by a small path length but there is a particular area in which vertices are linked to all the other vertices. That is a coexistence of degree rich area and other areas. This structure can be found commonly when negative degree-degree correlation is observed.

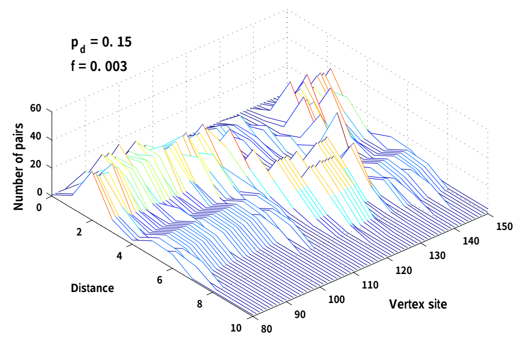

Figure 7: Observed distribution of the shortest path length from each vertex around the center of attraction for the case $p_d = 0.15$ and $f = 0.003$.

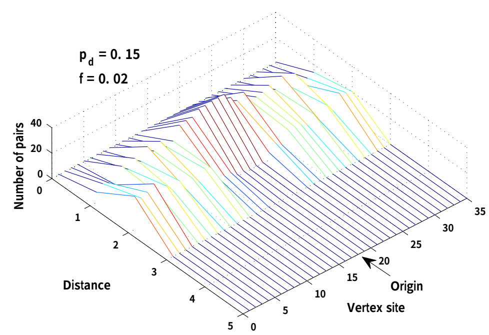

Figure 8: Observed distribution of the shortest path length from each vertex around the center of attraction for the case $p_d = 0.15$ and $f = 0.02$. The origin means a center of attractive force. Vertices around the origin are directly linked to almost all other vertices.

5 Phase diagram

According to the observation in the preceding sections, three phases of network structure can be found, each of which are unstable graphs with rare breakup; a connected graph formed by edges created by random walkers where the coexistence of a degree rich area and others area can be found, and graph of high density of edges with hard boundary. Graphs of high density of edges with hard boundary can be easily detected by observing the behavior of the boundary. Transition from an unstable graph to connected graph by the created edges were formally-verified by the observation of signs of degree-degree correlation. Figure 9 is a phase diagram obtained by this method in $p_d - f$ space.

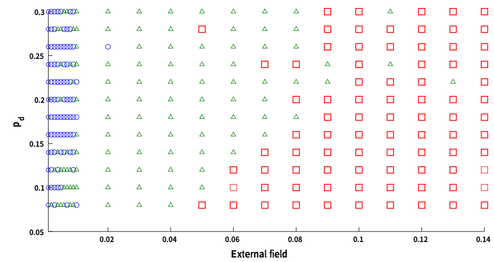

Figure 9: Phase diagram. The circles indicates a phase of unstable graphs. The triangle indicates a connected graph by created edges. The square indicates a graph with hard boundary. This result was obtained by only one observation, so statistical uncertainty is ignored.

The figure shows that the formed networks for small p_d are sensitive to the external field. This result is consistent with our previous observation that networks formed by random walkers starting from the same vertex remain in a certain area when p_d is sufficiently small (about $p_d < 0.05$). The phase of unstable graph with rare breakup is thought to cease to exist with the reduction of p_d. On the other hand, determination of the most probable phase for each condition becomes difficult as p_d tends to become large. This is because in the large p_d region in Figure 9, squares and triangles appear to overlap. Under these conditions, sudden transition from a graph with hard boundary to a graph with time fluctuated boundary can be found in extended observations of the movement of random walkers.

6 Conclusion

This paper explains a network model where the network was formed by random walker's movement and an external field that influences the movement of random walkers. The investigation are carried out for a case where the initial graph is one-dimensional lattice and the force on random walkers is constant. Three types of network structures induced by the external field can be found; namely unstable graphs with rare breakup, connected graph with time fluctuated boundary, and graph of high density of edges with hard boundary. The unstable graphs with rare breakup spread over a smaller range than width of the probability distribution of random walkers without creation edges, but the graph with hard boundary can be spread wider. These changes of network structure can be determined by the following observation; sudden reduction of the mean vertex-vertex distance, inversion of the signs of correlation between adjacent vertices, changes in the density of edges, and changes in the degree distribution. The inversion of sign of correlation can be interpreted by the coexistence of degree rich area around the center of attractive force and other areas, for vertices in the degree rich area are directly linked to all the other vertices with small degree. In our model, negative degree-degree correlation between adjacent vertices has been found after introducing external fields, while positive degree-degree correlation can be ordinary observed.

It is an important point that the mechanism of the linking of vertices based on the local structure of the network can produce various structures of network. When vertices only passively wait for visits by random walkers, it is natural that large clustering coefficients and positive degree-degree correlations are observed, for vertices with large degree tend to attract random walkers. Wandering of random walkers in a certain area can lead to large clustering coefficients. In this study, it was found that an active attraction by a vertex can produce contrasting results to the case without a field, that is negative degree-degree correlation which indicates existence of a part of vertices with overwhelming number of incident edges.

However, our calculation was limited to one-dimensional case where the random walker is generally hard to be spread because of the edges made by themselves. It should be noticed that the movement of random walkers is strongly dependent on the dimension of initial lattice. It may be interesting problem to consider other dimensions and other definitions of external field.

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