

A New Technique of Multi-layer Thermal Analysis for VLSI Chips

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Abstract: - We propose a new technique of two dimensional, steady-state thermal analysis for VLSI chips. It can be applied to an analysis model described by a combination of Laplace and Poisson equations for multiple layers of materials. We explain the technique using a simplified model of four layer materials. Our experimental results show that it achieved 70 times speed-up, 3 times memory usage reduction, and an order of magnitude lower residual as compared to the most efficient direct method of LU decomposition. They also reveal the superiority of our technique over the most effective iterative method of ICCG with up to 3 times speed-up and 1.8 times memory usage reduction.

Key-Words: - thermal analysis, Laplace equation, Poisson equation, direct method, LU decomposition, ICCG.

1 Introduction

With ever increasing improvements in feature size reduction, speed-up, and power reduction, thermal analysis of VLSI chips is becoming more and more important [9]. Since the late 1990s, much attention has been paid to the effect of heat generation and thermal conduction within silicon substrate and from metal layers on the characteristics of VLSI chips in the post-90 nm era. There have been two major approaches to thermal analysis [17].

The first approach starts with a physical equation for describing thermal conduction such as a Laplace and Poisson equation. The equation is discretized and a large scale system of linear equations is generated. Such a system is then solved by iterative methods such as Incomplete Cholesky Conjugate Gradient (ICCG) method [7][20] or direct methods such as LU decomposition (LUD) method as well as a special purpose SPICE-like algorithm [1] or a simple tridiagonal band matrix solver, called the Thomas algorithm [16][22].

The second approach utilizes analogy between heat and electricity to model heat transfer mechanism as RC networks [2][8] and analyzes them using SPICE-like [1][11] or a full-chip-scale circuit simulator [21]. Recent work on the use of Green function to describe thermal conduction [24] could be considered as a third approach.

We propose a new multi-layer thermal analysis technique along the line of the first approach. It

utilizes a new direct method that requires less time and memory than even most iterative methods.

The application of finite difference methods to two or three dimensional Laplace and/or Poisson equations generates large scale systems of linear equations. The coefficient matrices of such systems have a special structure, called block tridiagonal band matrices. Most of the methods and tools thus far proposed use iterative methods for solving the tridiagonal systems of linear equations [5][10][20][25] since such methods require less time and memory than direct methods. However, the former cannot obtain as accurate solutions as the latter due to an error caused by forced termination of computation [7][19]. Thus, the direct methods are still preferred in certain situations. Furthermore, in the case of transient thermal analysis, LUD is most often used for easy substitution iteration.

As mentioned above, our technique uses a direct method for block tridiagonal band coefficient matrices. It was derived from a general case linear system solver, called Partial Solution Method (PSM) [15] and is known as Symbolic PSM (S-PSM) in the area of computational fluid dynamics [3][4]. It was applied to thermal conduction analysis for two and four adjacent materials, where thermal conduction was described by Laplace equations [12][13].

We demonstrate that S-PSM can be applied to multiple layer materials whose thermal conduction is described by a combination of Laplace and Poisson equations. In particular, we consider the case of four

layer materials of different thermal conductivities. We first use finite difference method to discretize a combination of Poisson and Laplace equations and generate large scale systems of linear equations. We then apply S-PSM, a block LUD, and ICCG to the linear systems. The experimental results reveal that our method ran 70 times faster, required 3 times less memory, and an order of magnitude smaller residual than LUD. They also show that the technique achieved up to 3 times speed-up and used 1.8 times smaller memory than ICCG.

In the next section, we provide a brief overview of S-PSM using a Poisson equation and its extension to the four layer case. Section III describes our experiments and compares the results. We conclude the paper in Section IV.

2 S-PSM and Poisson Equations

We describe S-PSM using a Poisson equation as a target equation. Given a large scale system of linear equations derived by finite difference method, the S-PSM decomposes it into its subsystems and finds the values of the variables shared by each pair of adjacent subsystems. Fig. 2.1 shows an overall flow of the major algebraic computations to take place at each subsystem with their relevant equations and solutions specified.

Note from the figure that the S-PSM-based solution process goes through many levels of repeated operations of decomposition and merging. In the following, the level information is attached to variable vectors and coefficient submatrices as their superscripts with parentheses such as (0) and (e).

Consider the following Poisson equation that describes heat diffusion.

$$k_u \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \rho^u \quad (2-1)$$

where k_u is the thermal conductivity and ρ^u is the heat generation per unit volume of the material u under consideration and u denotes the variable associated with the material u . In our steady-state heat conduction analysis, the bold lower case letter u represents one of the four materials $p, q, r,$ and s and the lower case letters, $p, q, r,$ and s are the names of variables associated with each of the materials.

When we apply finite difference method to Eq. (2-1), each domain is divided into grid points, which are numbered as depicted in Fig. 2.2. Note that there are $(n_u + 2)$ rows of grid points instead of n_u . As shown below, we need a single extra row of grid points at the

top and bottom boundaries of each of the rectangle domains so as to define the equivalent thermal conductivity of the connection boundary between adjacent materials.

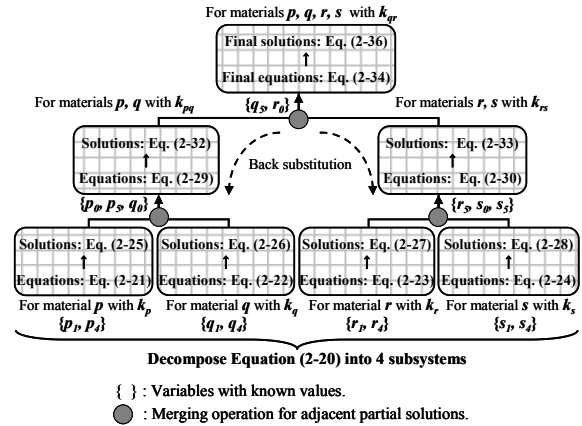


Fig. 2.1: An overall computation flow with relevant equation numbers of a S-PSM-based solution process for four materials.

In the following, whenever two indices i and j are used as subscripts for interior grid point variables and their relevant vectors, their respective ranges are $i = 1, 2, \dots, n_u$ and $j = 1, 2, \dots, m$. Since the same thermal conductivity prevails within each region of the same material, the following discretized form of Eq. (2-1) is obtained:

$$k_u \left(\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{\Delta y^2} \right) = \rho^u_{i,j} \quad (2-2)$$

where $\rho^u_{i,j}$ denotes the heat generation at grid point (i,j) .

Assuming that $\Delta x = \Delta y$, we have

$$u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1} - 4u_{i,j} = \Delta x^2 \frac{\rho^u_{i,j}}{k_u} \quad (2-3)$$

The left and right boundary conditions for each material u is given as,

$$\mathbf{f}_{u,j}^{(0)} = (-u_{i,0} + \Delta x^2 \frac{\rho^u_{i,1}}{k_u} \quad \Delta x^2 \frac{\rho^u_{i,2}}{k_u} \quad \dots \quad \Delta x^2 \frac{\rho^u_{i,m-1}}{k_u} \quad -u_{i,m+1} + \Delta x^2 \frac{\rho^u_{i,m}}{k_u}) \quad (2-4)$$

The top and bottom boundary conditions for the combined four layers of materials are given as

$$\mathbf{v}^{(0)} = (v_{0,1} \quad v_{0,2} \quad \dots \quad v_{0,m-1} \quad v_{0,m})^T \quad (2-5)$$

$$\mathbf{w}^{(0)} = (w_{0,1} \quad w_{0,2} \quad \dots \quad w_{0,m-1} \quad w_{0,m})^T \quad (2-6)$$

Note that the superscripts (0) for the above vectors indicate that their element values are given at the start

of the S-PSM process. For the sake of simplicity, we assume that

$$n_p = 2^{e_1+2}, n_q = 2^{e_2+2}, n_r = 2^{e_3+2}, n_s = 2^{e_4+2} \quad (2-7)$$

where e_1, e_2, e_3 , and e_4 are positive integers.

We define the equivalent thermal conductivity of the connection boundary between adjacent materials using the extra rows of grid points as mentioned above. For materials, say q and r of thermal conductivities k_q and k_r , we have

$$k_{q,r} = \frac{2k_q k_r}{k_q + k_r} \quad (2-8)$$

For each material domain, there are two boundaries, upper and lower ones. At each boundary, we use the first order approximation for heat conduction. As such, from the point of view of material q , the following difference equation holds at its boundary with material

$$p: \quad k_{p,q}(p_{n_p+1,j} - q_{0,j}) + k_q(q_{0,j-1} - q_{0,j}) + k_q(q_{0,j+1} - q_{0,j}) = \Delta x^2 \rho_{0,j}^q \quad (2-9)$$

Similarly, at its boundary with material r , we have

$$k_q(q_{n_q,j} - q_{n_q+1,j}) + k_q(q_{n_q+1,j-1} - q_{n_q+1,j}) + k_q(q_{n_q+1,j+1} - q_{n_q+1,j}) = \Delta x^2 \rho_{n_q+1,j}^q \quad (2-10)$$

Thus, using the variable vector notation

$$\mathbf{u}_i^{(0)} = (u_{i,1} \quad u_{i,2} \quad u_{i,3} \quad \dots \quad u_{i,m-1} \quad u_{i,m})^T \quad (2-11)$$

we have the following four equations for the pair of adjacent materials, q and r .

$$\left. \begin{aligned} k_{p,q} \mathbf{p}_{n_p+1}^{(e_1)} + A_{q1}^{(0)} \mathbf{q}_0^{(e_2)} + k_q \mathbf{q}_1^{(e_2)} &= k_q \mathbf{f}_{q,0}^{(0)} \\ k_q \mathbf{q}_{n_q}^{(e_2)} + A_{q2}^{(0)} \mathbf{q}_{n_q+1}^{(e_2)} + k_{q,r} \mathbf{r}_0^{(e_3)} &= k_q \mathbf{f}_{q,n_q+1}^{(0)} \\ k_{q,r} \mathbf{q}_{n_q+1}^{(e_2)} + A_{r1}^{(0)} \mathbf{r}_0^{(e_3)} + k_r \mathbf{r}_1^{(e_3)} &= k_r \mathbf{f}_{r,0}^{(0)} \\ k_r \mathbf{r}_{n_r}^{(e_3)} + A_{r2}^{(0)} \mathbf{r}_{n_r+1}^{(e_3)} + k_{r,s} \mathbf{s}_0^{(e_4)} &= k_r \mathbf{f}_{r,n_r+1}^{(0)} \end{aligned} \right\} \quad (2-12)$$

where

$$\left. \begin{aligned} \mathbf{p}_{n_p+1}^{(e_1)} &= \mathbf{p}_5^{(e_1)}, \quad \mathbf{q}_{n_q}^{(e_2)} = \mathbf{q}_4^{(e_2)}, \quad \mathbf{q}_{n_q+1}^{(e_2)} = \mathbf{q}_5^{(e_2)} \\ \mathbf{r}_{n_r}^{(e_3)} &= \mathbf{r}_4^{(e_3)}, \quad \mathbf{r}_{n_r+1}^{(e_3)} = \mathbf{r}_5^{(e_3)} \end{aligned} \right\} \quad (2-13)$$

$$A_{q1}^{(0)} = \begin{pmatrix} -(k_{p,q}+3k_q) & k_q & & & \\ k_q & -(k_{p,q}+3k_q) & k_q & & \\ & k_q & -(k_{p,q}+3k_q) & k_q & \\ & & \bullet & \bullet & \\ & & & k_q & -(k_{p,q}+3k_q) \end{pmatrix} \quad (2-14)$$

$$A_{q2}^{(0)} = \begin{pmatrix} -(3k_q+k_{q,r}) & k_q & & & \\ k_q & -(3k_q+k_{q,r}) & k_q & & \\ & k_q & -(3k_q+k_{q,r}) & k_q & \\ & & \bullet & \bullet & \bullet \\ & & & k_q & -(3k_q+k_{q,r}) \end{pmatrix} \quad (2-15)$$

$$A_{r1}^{(0)} = \begin{pmatrix} -(k_{q,r}+3k_r) & k_r & & & \\ k_r & -(k_{q,r}+3k_r) & k_r & & \\ & k_r & -(k_{q,r}+3k_r) & k_r & \\ & & \bullet & \bullet & \\ & & & k_r & -(k_{q,r}+3k_r) \end{pmatrix} \quad (2-16)$$

$$A_{r2}^{(0)} = \begin{pmatrix} -(3k_r+k_{r,s}) & k_r & & & \\ k_r & -(3k_r+k_{r,s}) & k_r & & \\ & k_r & -(3k_r+k_{r,s}) & k_r & \\ & & \bullet & \bullet & \bullet \\ & & & k_r & -(3k_r+k_{r,s}) \end{pmatrix} \quad (2-17)$$

$$\begin{matrix} v_{0,0} & v_{0,1} & v_{0,2} & v_{0,3} & v_{0,m} & v_{0,m+1} \\ p_{0,0} & p_{0,1} & p_{0,2} & p_{0,3} & p_{0,m} & p_{0,m+1} \\ p_{1,0} & p_{1,1} & p_{1,2} & p_{1,3} & p_{1,m} & p_{1,m+1} \\ p_{2,0} & p_{2,1} & p_{2,2} & p_{2,3} & p_{2,m} & p_{2,m+1} \\ & & & \mathbf{k}_p & & \\ p_{n_p+1,0} & p_{n_p+1,1} & p_{n_p+1,2} & p_{n_p+1,3} & p_{n_p+1,m} & p_{n_p+1,m+1} \\ q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,m} & q_{0,m+1} \end{matrix}$$

$$\begin{matrix} p_{n_p+1,0} & p_{n_p+1,1} & p_{n_p+1,2} & p_{n_p+1,3} & p_{n_p+1,m} & p_{n_p+1,m+1} \\ q_{0,0} & q_{0,1} & q_{0,2} & q_{0,3} & q_{0,m} & q_{0,m+1} \\ q_{1,0} & q_{1,1} & q_{1,2} & q_{1,3} & q_{1,m} & q_{1,m+1} \\ q_{2,0} & q_{2,1} & q_{2,2} & q_{2,3} & q_{2,m} & q_{2,m+1} \\ & & & \mathbf{k}_q & & \\ q_{n_q+1,0} & q_{n_q+1,1} & q_{n_q+1,2} & q_{n_q+1,3} & q_{n_q+1,m} & q_{n_q+1,m+1} \\ r_{0,0} & r_{0,1} & r_{0,2} & r_{0,3} & r_{0,m} & r_{0,m+1} \end{matrix}$$

$$\begin{matrix} q_{n_q+1,0} & q_{n_q+1,1} & q_{n_q+1,2} & q_{n_q+1,3} & q_{n_q+1,m} & q_{n_q+1,m+1} \\ r_{0,0} & r_{0,1} & r_{0,2} & r_{0,3} & r_{0,m} & r_{0,m+1} \\ r_{1,0} & r_{1,1} & r_{1,2} & r_{1,3} & r_{1,m} & r_{1,m+1} \\ r_{2,0} & r_{2,1} & r_{2,2} & r_{2,3} & r_{2,m} & r_{2,m+1} \\ & & & \mathbf{k}_r & & \\ r_{n_r+1,0} & r_{n_r+1,1} & r_{n_r+1,2} & r_{n_r+1,3} & r_{n_r+1,m} & r_{n_r+1,m+1} \\ s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} & s_{0,m} & s_{0,m+1} \end{matrix}$$

$$\begin{matrix} r_{n_r+1,0} & r_{n_r+1,1} & r_{n_r+1,2} & r_{n_r+1,3} & r_{n_r+1,m} & r_{n_r+1,m+1} \\ s_{0,0} & s_{0,1} & s_{0,2} & s_{0,3} & s_{0,m} & s_{0,m+1} \\ s_{1,0} & s_{1,1} & s_{1,2} & s_{1,3} & s_{1,m} & s_{1,m+1} \\ s_{2,0} & s_{2,1} & s_{2,2} & s_{2,3} & s_{2,m} & s_{2,m+1} \\ & & & \mathbf{k}_s & & \\ s_{n_s+1,0} & s_{n_s+1,1} & s_{n_s+1,2} & s_{n_s+1,3} & s_{n_s+1,m} & s_{n_s+1,m+1} \\ w_{0,0} & w_{0,1} & w_{0,2} & w_{0,3} & w_{0,m} & w_{0,m+1} \end{matrix}$$

Fig. 2.2: The arrangement of interior grid and boundary points for the four material domains.

As mentioned at the beginning of this section, many levels of decomposition and merging operations take place but most of those operations occur within each material. (For detailed discussions, see [14].) Thus as seen in the above equations and those to follow, the superscripts attached to vectors are at the last levels such as (e) and $(e+1)$. Note that the superscript (0)

indicates that the values given at the beginning of the S-PSM process hold until the end of the process. Note also that Eq. (2-13) gives the correspondence between the variable names of each of the two materials. (For the reasoning to derive such equations, see [14].)

We now obtain the following equations for the first and second materials.

$$\left. \begin{aligned} \mathbf{q}_1^{(e_2)} &= \mathbf{f}_{q,1}^{(e_2+1)} - A_1^{(e_2+1)} \mathbf{q}_0^{(e_2)} - A_2^{(e_2+1)} \mathbf{q}_5^{(e_2)} \\ \mathbf{q}_4^{(e_2)} &= \mathbf{f}_{q,2}^{(e_2+1)} - A_2^{(e_2+1)} \mathbf{q}_0^{(e_2)} - A_1^{(e_2+1)} \mathbf{q}_5^{(e_2)} \end{aligned} \right\} \quad (2-18)$$

$$\left. \begin{aligned} \mathbf{r}_1^{(e_3)} &= \mathbf{f}_{r,1}^{(e_3+1)} - A_1^{(e_3+1)} \mathbf{r}_0^{(e_3)} - A_2^{(e_3+1)} \mathbf{r}_5^{(e_3)} \\ \mathbf{r}_4^{(e_3)} &= \mathbf{f}_{r,2}^{(e_3+1)} - A_2^{(e_3+1)} \mathbf{r}_0^{(e_3)} - A_1^{(e_3+1)} \mathbf{r}_5^{(e_3)} \end{aligned} \right\} \quad (2-19)$$

We repeatedly apply the above procedure for derivation of Eqs. (2-12), (2-13), (2-18), and (2-19) to the remaining material boundaries. We then obtain the following matrix-vector form for the three boundaries:

$$\begin{pmatrix} A_{p1}^{(0)} & I & O & O \\ A_{p1}^{(e_1+1)} & I & O & A_2^{(e_1+1)} \\ A_{p2}^{(e_1+1)} & O & I & A_1^{(e_1+1)} \\ O & O & k_p I & A_{p2}^{(0)} \end{pmatrix} \begin{pmatrix} \mathbf{p}_0^{(e_1)} \\ \mathbf{p}_1^{(e_1)} \\ \mathbf{p}_4^{(e_1)} \\ \mathbf{p}_5^{(e_1)} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0^{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{p,q} \mathbf{q}_0^{(e_2)} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{p,0}^{(0)} \\ \mathbf{f}_{p,1}^{(e_1+1)} \\ \mathbf{f}_{p,2}^{(e_1+1)} \\ k_p \mathbf{f}_{p,p,p+1}^{(0)} \end{pmatrix} \quad (2-20)$$

We use S-PSM to decompose the above system of equations into four subsystems, where each subsystem corresponds to one of the materials. We then have the following four sets of equations,

$$\begin{pmatrix} A_{p1}^{(0)} & I & O & O \\ A_1^{(e_1+1)} & I & O & A_2^{(e_1+1)} \\ A_2^{(e_1+1)} & O & I & A_1^{(e_1+1)} \\ O & O & k_p I & A_{p2}^{(0)} \end{pmatrix} \begin{pmatrix} \mathbf{p}_0^{(e_1)} \\ \mathbf{p}_1^{(e_1)} \\ \mathbf{p}_4^{(e_1)} \\ \mathbf{p}_5^{(e_1)} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_0^{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{p,q} \mathbf{q}_0^{(e_2)} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{p,0}^{(0)} \\ \mathbf{f}_{p,1}^{(e_1+1)} \\ \mathbf{f}_{p,2}^{(e_1+1)} \\ k_p \mathbf{f}_{p,p,p+1}^{(0)} \end{pmatrix} \quad (2-21)$$

$$\begin{pmatrix} A_{q2}^{(0)} & k_q I & O & O \\ A_4^{(e_2+1)} & I & O & A_2^{(e_2+1)} \\ A_2^{(e_2+1)} & O & I & A_1^{(e_2+1)} \\ O & O & k_q I & A_{q1}^{(0)} \end{pmatrix} \begin{pmatrix} \mathbf{q}_5^{(e_2)} \\ \mathbf{q}_4^{(e_2)} \\ \mathbf{q}_1^{(e_2)} \\ \mathbf{q}_0^{(e_2)} \end{pmatrix} = \begin{pmatrix} k_{q,r} \mathbf{r}_0^{(e_3)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{p,q} \mathbf{p}_5^{(e_1)} \end{pmatrix} + \begin{pmatrix} k_q \mathbf{f}_{q,q,q+1}^{(0)} \\ \mathbf{f}_{q,2}^{(e_2+1)} \\ \mathbf{f}_{q,1}^{(e_2+1)} \\ k_q \mathbf{f}_{q,q,0}^{(0)} \end{pmatrix} \quad (2-22)$$

$$\begin{pmatrix} A_{r1}^{(0)} & k_r I & O & O \\ A_4^{(e_3+1)} & I & O & A_2^{(e_3+1)} \\ A_2^{(e_3+1)} & O & I & A_1^{(e_3+1)} \\ O & O & k_r I & A_{r2}^{(0)} \end{pmatrix} \begin{pmatrix} \mathbf{r}_0^{(e_3)} \\ \mathbf{r}_1^{(e_3)} \\ \mathbf{r}_4^{(e_3)} \\ \mathbf{r}_5^{(e_3)} \end{pmatrix} = \begin{pmatrix} k_{q,r} \mathbf{q}_5^{(e_2)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{r,s} \mathbf{s}_0^{(e_4)} \end{pmatrix} + \begin{pmatrix} k_r \mathbf{f}_{r,r,0}^{(0)} \\ \mathbf{f}_{r,1}^{(e_3+1)} \\ \mathbf{f}_{r,2}^{(e_3+1)} \\ k_r \mathbf{f}_{r,r,n+1}^{(0)} \end{pmatrix} \quad (2-23)$$

$$\begin{pmatrix} A_{s2}^{(0)} & I & O & O \\ A_4^{(e_4+1)} & I & O & A_2^{(e_4+1)} \\ A_2^{(e_4+1)} & O & I & A_1^{(e_4+1)} \\ O & O & k_s I & A_{s1}^{(0)} \end{pmatrix} \begin{pmatrix} \mathbf{s}_5^{(e_4)} \\ \mathbf{s}_4^{(e_4)} \\ \mathbf{s}_1^{(e_4)} \\ \mathbf{s}_0^{(e_4)} \end{pmatrix} = \begin{pmatrix} \mathbf{w}_0^{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{r,s} \mathbf{r}_5^{(e_3)} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{s,n_s+1}^{(0)} \\ \mathbf{f}_{s,2}^{(e_4+1)} \\ \mathbf{f}_{s,1}^{(e_4+1)} \\ k_s \mathbf{f}_{s,0}^{(0)} \end{pmatrix} \quad (2-24)$$

Using the inverse matrix of each coefficient matrix, these equations are rewritten as follows. (For the correspondence between the blocks of the two matrices, see [14].)

$$\begin{pmatrix} \mathbf{p}_0^{(e_1)} \\ \mathbf{p}_1^{(e_1)} \\ \mathbf{p}_4^{(e_1)} \\ \mathbf{p}_5^{(e_1)} \end{pmatrix} = \begin{pmatrix} B_{1,1}^p & B_{1,2}^p & B_{1,3}^p & B_{1,4}^p \\ B_{2,1}^p & B_{2,2}^p & B_{2,3}^p & B_{2,4}^p \\ B_{3,1}^p & B_{3,2}^p & B_{3,3}^p & B_{3,4}^p \\ B_{4,1}^p & B_{4,2}^p & B_{4,3}^p & B_{4,4}^p \end{pmatrix} \begin{pmatrix} \mathbf{v}_0^{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{p,q} \mathbf{q}_0^{(e_2)} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{p,0}^{(0)} \\ \mathbf{f}_{p,1}^{(e_1+1)} \\ \mathbf{f}_{p,2}^{(e_1+1)} \\ k_p \mathbf{f}_{p,p,p+1}^{(0)} \end{pmatrix} \quad (2-25)$$

$$\begin{pmatrix} \mathbf{q}_5^{(e_2)} \\ \mathbf{q}_4^{(e_2)} \\ \mathbf{q}_1^{(e_2)} \\ \mathbf{q}_0^{(e_2)} \end{pmatrix} = \begin{pmatrix} B_{1,1}^q & B_{1,2}^q & B_{1,3}^q & B_{1,4}^q \\ B_{2,1}^q & B_{2,2}^q & B_{2,3}^q & B_{2,4}^q \\ B_{3,1}^q & B_{3,2}^q & B_{3,3}^q & B_{3,4}^q \\ B_{4,1}^q & B_{4,2}^q & B_{4,3}^q & B_{4,4}^q \end{pmatrix} \begin{pmatrix} k_{q,r} \mathbf{r}_0^{(e_3)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{p,q} \mathbf{p}_5^{(e_1)} \end{pmatrix} + \begin{pmatrix} k_q \mathbf{f}_{q,q,q+1}^{(0)} \\ \mathbf{f}_{q,2}^{(e_2+1)} \\ \mathbf{f}_{q,1}^{(e_2+1)} \\ k_q \mathbf{f}_{q,q,0}^{(0)} \end{pmatrix} \quad (2-26)$$

$$\begin{pmatrix} \mathbf{r}_0^{(e_3)} \\ \mathbf{r}_1^{(e_3)} \\ \mathbf{r}_4^{(e_3)} \\ \mathbf{r}_5^{(e_3)} \end{pmatrix} = \begin{pmatrix} B_{1,1}^r & B_{1,2}^r & B_{1,3}^r & B_{1,4}^r \\ B_{2,1}^r & B_{2,2}^r & B_{2,3}^r & B_{2,4}^r \\ B_{3,1}^r & B_{3,2}^r & B_{3,3}^r & B_{3,4}^r \\ B_{4,1}^r & B_{4,2}^r & B_{4,3}^r & B_{4,4}^r \end{pmatrix} \begin{pmatrix} k_{q,r} \mathbf{q}_5^{(e_2)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{r,s} \mathbf{s}_0^{(e_4)} \end{pmatrix} + \begin{pmatrix} k_r \mathbf{f}_{r,r,0}^{(0)} \\ \mathbf{f}_{r,1}^{(e_3+1)} \\ \mathbf{f}_{r,2}^{(e_3+1)} \\ k_r \mathbf{f}_{r,r,n+1}^{(0)} \end{pmatrix} \quad (2-27)$$

$$\begin{pmatrix} \mathbf{s}_5^{(e_4)} \\ \mathbf{s}_4^{(e_4)} \\ \mathbf{s}_1^{(e_4)} \\ \mathbf{s}_0^{(e_4)} \end{pmatrix} = \begin{pmatrix} B_{1,1}^s & B_{1,2}^s & B_{1,3}^s & B_{1,4}^s \\ B_{2,1}^s & B_{2,2}^s & B_{2,3}^s & B_{2,4}^s \\ B_{3,1}^s & B_{3,2}^s & B_{3,3}^s & B_{3,4}^s \\ B_{4,1}^s & B_{4,2}^s & B_{4,3}^s & B_{4,4}^s \end{pmatrix} \begin{pmatrix} \mathbf{w}_0^{(0)} \\ \mathbf{0} \\ \mathbf{0} \\ k_{r,s} \mathbf{r}_5^{(e_3)} \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{s,n_s+1}^{(0)} \\ \mathbf{f}_{s,2}^{(e_4+1)} \\ \mathbf{f}_{s,1}^{(e_4+1)} \\ k_s \mathbf{f}_{s,0}^{(0)} \end{pmatrix} \quad (2-28)$$

By merging the pairs of the first and second and of the third and fourth equations, we derive the following equations (2-29) and (2-30), respectively.

$$\begin{pmatrix} I & O & k_{p,q} B_{1,4}^p & O \\ O & I & k_{p,q} B_{4,4}^p & O \\ O & k_{p,q} B_{4,4}^p & I & O \\ O & k_{p,q} B_{1,4}^p & O & I \end{pmatrix} \begin{pmatrix} \mathbf{p}_0^{(e_1)} \\ \mathbf{p}_5^{(e_1)} \\ \mathbf{q}_0^{(e_2)} \\ \mathbf{q}_5^{(e_2)} \end{pmatrix} = \begin{pmatrix} B_{1,1}^p \mathbf{v}_0^{(0)} \\ B_{4,1}^p \mathbf{v}_0^{(0)} \\ k_{q,r} B_{4,1}^p \mathbf{r}_0^{(e_3)} \\ k_{q,r} B_{1,1}^p \mathbf{r}_0^{(e_3)} \end{pmatrix} + \begin{pmatrix} \mathbf{ff}_{p,0} \\ \mathbf{ff}_{p,5} \\ \mathbf{ff}_{q,0} \\ \mathbf{ff}_{q,5} \end{pmatrix} \quad (2-29)$$

$$\begin{pmatrix} I & O & k_{r,s} B_{1,4}^r & O \\ O & I & k_{r,s} B_{4,4}^r & O \\ O & k_{r,s} B_{4,4}^r & I & O \\ O & k_{r,s} B_{1,4}^r & O & I \end{pmatrix} \begin{pmatrix} \mathbf{r}_0^{(e_3)} \\ \mathbf{r}_5^{(e_3)} \\ \mathbf{s}_0^{(e_4)} \\ \mathbf{s}_5^{(e_4)} \end{pmatrix} = \begin{pmatrix} k_{q,r} B_{1,1}^r \mathbf{q}_5^{(e_2)} \\ k_{q,r} B_{4,1}^r \mathbf{q}_5^{(e_2)} \\ B_{4,1}^s \mathbf{w}_0^{(0)} \\ B_{1,1}^s \mathbf{w}_0^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{ff}_{r,0} \\ \mathbf{ff}_{r,5} \\ \mathbf{ff}_{s,0} \\ \mathbf{ff}_{s,5} \end{pmatrix} \quad (2-30)$$

where

$$\left. \begin{aligned} \mathbf{ff}_{p,0} &= B_{1,1}^p \mathbf{f}_{p,0}^{(0)} + B_{1,2}^p \mathbf{f}_{p,1}^{(e_1+1)} + B_{1,3}^p \mathbf{f}_{p,2}^{(e_1+1)} + k_p B_{1,4}^p \mathbf{f}_{p,p,p+1}^{(0)} \\ \mathbf{ff}_{p,5} &= B_{4,1}^p \mathbf{f}_{p,0}^{(0)} + B_{4,2}^p \mathbf{f}_{p,1}^{(e_1+1)} + B_{4,3}^p \mathbf{f}_{p,2}^{(e_1+1)} + k_p B_{4,4}^p \mathbf{f}_{p,p,p+1}^{(0)} \\ \mathbf{ff}_{q,0} &= k_q B_{4,1}^q \mathbf{f}_{q,q,q+1}^{(0)} + B_{4,2}^q \mathbf{f}_{q,2}^{(e_2+1)} + B_{4,3}^q \mathbf{f}_{q,1}^{(e_2+1)} + k_q B_{4,4}^q \mathbf{f}_{q,q,0}^{(0)} \\ \mathbf{ff}_{q,5} &= k_q B_{1,1}^q \mathbf{f}_{q,q,q+1}^{(0)} + B_{1,2}^q \mathbf{f}_{q,2}^{(e_2+1)} + B_{1,3}^q \mathbf{f}_{q,1}^{(e_2+1)} + k_q B_{1,4}^q \mathbf{f}_{q,q,0}^{(0)} \\ \mathbf{ff}_{r,0} &= k_r B_{1,1}^r \mathbf{f}_{r,0}^{(0)} + B_{1,2}^r \mathbf{f}_{r,1}^{(e_3+1)} + B_{1,3}^r \mathbf{f}_{r,2}^{(e_3+1)} + k_r B_{1,4}^r \mathbf{f}_{r,r,n+1}^{(0)} \\ \mathbf{ff}_{r,5} &= k_r B_{4,1}^r \mathbf{f}_{r,0}^{(0)} + B_{4,2}^r \mathbf{f}_{r,1}^{(e_3+1)} + B_{4,3}^r \mathbf{f}_{r,2}^{(e_3+1)} + k_r B_{4,4}^r \mathbf{f}_{r,r,n+1}^{(0)} \\ \mathbf{ff}_{s,0} &= B_{4,1}^s \mathbf{f}_{s,n_s+1}^{(0)} + B_{4,2}^s \mathbf{f}_{s,2}^{(e_4+1)} + B_{4,3}^s \mathbf{f}_{s,1}^{(e_4+1)} + k_s B_{4,4}^s \mathbf{f}_{s,0}^{(0)} \\ \mathbf{ff}_{s,5} &= B_{1,1}^s \mathbf{f}_{s,n_s+1}^{(0)} + B_{1,2}^s \mathbf{f}_{s,2}^{(e_4+1)} + B_{1,3}^s \mathbf{f}_{s,1}^{(e_4+1)} + k_s B_{1,4}^s \mathbf{f}_{s,0}^{(0)} \end{aligned} \right\} \quad (2-31)$$

With the use of the inverse matrix of each coefficient

matrix, the above equations are rewritten as follows (see [14]).

$$\begin{pmatrix} \mathbf{p}_0^{(e_1)} \\ \mathbf{p}_5^{(e_1)} \\ \mathbf{q}_0^{(e_2)} \\ \mathbf{q}_5^{(e_2)} \end{pmatrix} = \begin{pmatrix} I & B_{1,1}^{pq} & B_{2,1}^{pq} & O \\ O & B_{1,2}^{pq} & B_{2,2}^{pq} & O \\ O & B_{1,3}^{pq} & B_{2,3}^{pq} & O \\ O & B_{1,4}^{pq} & B_{2,4}^{pq} & I \end{pmatrix} - \begin{pmatrix} B_{1,1}^p \mathbf{v}_0^{(0)} \\ B_{4,1}^p \mathbf{v}_0^{(0)} \\ k_{q,r} B_{4,1}^q \mathbf{r}_0^{(e_3)} \\ k_{q,r} B_{1,1}^q \mathbf{r}_0^{(e_3)} \end{pmatrix} + \begin{pmatrix} \mathbf{ff}_{p0} \\ \mathbf{ff}_{p5} \\ \mathbf{ff}_{q0} \\ \mathbf{ff}_{q5} \end{pmatrix} \quad (2-32)$$

$$\begin{pmatrix} \mathbf{r}_0^{(e_3)} \\ \mathbf{r}_5^{(e_3)} \\ \mathbf{s}_0^{(e_4)} \\ \mathbf{s}_5^{(e_4)} \end{pmatrix} = \begin{pmatrix} I & B_{1,1}^{rs} & B_{2,1}^{rs} & O \\ O & B_{1,2}^{rs} & B_{2,2}^{rs} & O \\ O & B_{1,3}^{rs} & B_{2,3}^{rs} & O \\ O & B_{1,4}^{rs} & B_{2,4}^{rs} & I \end{pmatrix} - \begin{pmatrix} k_{q,r} B_{1,1}^r \mathbf{q}_5^{(e_2)} \\ k_{q,r} B_{4,1}^r \mathbf{q}_5^{(e_2)} \\ B_{4,1}^s \mathbf{w}_0^{(0)} \\ B_{1,1}^s \mathbf{w}_0^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{ff}_{r0} \\ \mathbf{ff}_{r5} \\ \mathbf{ff}_{s0} \\ \mathbf{ff}_{s5} \end{pmatrix} \quad (2-33)$$

By expanding the right hand side of each of the above equations and merging them, we now obtain the following equations:

$$\begin{pmatrix} I & k_{q,r}(B_{2,4}^{pq} B_{4,1}^q + B_{1,1}^q) \\ k_{q,r}(B_{1,1}^{rs} B_{4,1}^r + B_{1,1}^r) & I \end{pmatrix} \begin{pmatrix} \mathbf{q}_5^{(e_2)} \\ \mathbf{r}_0^{(e_3)} \end{pmatrix} = - \begin{pmatrix} B_{1,4}^{pq} B_{4,1}^p \mathbf{z}_0^{(0)} \\ B_{2,1}^{rs} B_{4,1}^s \mathbf{w}_0^{(0)} \end{pmatrix} + \begin{pmatrix} \mathbf{ff}_{pq} \\ \mathbf{ff}_{rs} \end{pmatrix} \quad (2-34)$$

where

$$\left. \begin{aligned} \mathbf{ff}_{pq} &= B_{1,4}^{pq} \mathbf{ff}_{p5} + B_{2,4}^{pq} \mathbf{ff}_{q0} + \mathbf{ff}_{q5} \\ \mathbf{ff}_{rs} &= \mathbf{ff}_{r0} + B_{1,1}^{rs} \mathbf{ff}_{r5} + B_{2,1}^{rs} \mathbf{ff}_{s0} \end{aligned} \right\} \quad (2-35)$$

Using the inverse matrix of the coefficient matrix of Eq. (2-34), we finally find solutions for variables $\mathbf{q}_5^{(e_2)}$ and $\mathbf{r}_0^{(e_3)}$ as follows (see [14]).

$$\left. \begin{aligned} \mathbf{q}_5^{(e_2)} &= -D_1(B_{1,4}^{pq}(B_{4,1}^p \mathbf{z}_0^{(0)})) - D_2(B_{2,1}^{rs}(B_{4,1}^s \mathbf{w}_0^{(0)})) + D_1 \mathbf{ff}_{pq} + D_2 \mathbf{ff}_{rs} \\ \mathbf{r}_0^{(e_3)} &= -D_3(B_{1,4}^{pq}(B_{4,1}^p \mathbf{z}_0^{(0)})) - D_4(B_{2,1}^{rs}(B_{4,1}^s \mathbf{w}_0^{(0)})) + D_3 \mathbf{ff}_{pq} + D_4 \mathbf{ff}_{rs} \end{aligned} \right\} \quad (2-36)$$

where

$$\left. \begin{aligned} D_1 &= [I - k_{q,r} k_{q,r} (B_{2,4}^{pq} B_{4,1}^q + B_{1,1}^q) (B_{1,1}^{rs} B_{4,1}^r + B_{1,1}^r)]^{-1} \\ D_2 &= -k_{q,r} D_1 (B_{2,4}^{pq} B_{4,1}^q + B_{1,1}^q) \\ D_4 &= [I - k_{q,r} k_{q,r} (B_{1,1}^{rs} B_{4,1}^r + B_{1,1}^r) (B_{2,4}^{pq} B_{4,1}^q + B_{1,1}^q)]^{-1} \\ D_3 &= -k_{q,r} D_4 (B_{1,1}^{rs} B_{4,1}^r + B_{1,1}^r) \end{aligned} \right\} \quad (2-37)$$

The above solutions are now substituted into Eqs. (2-32) and (2-33) to find solutions for $\mathbf{p}_0^{(e_1)}$, $\mathbf{p}_5^{(e_1)}$, $\mathbf{q}_0^{(e_2)}$, $\mathbf{r}_5^{(e_3)}$, $\mathbf{s}_0^{(e_4)}$, and $\mathbf{s}_5^{(e_4)}$. These solutions are then back substituted into Eqs. (2-25), (2-26), (2-27), and (2-28) and the solutions for the remaining variables are obtained. (Finally, we need to find solutions for the variables associated with each material. This can be done by way of repeated substitutions of the values for relevant variables into certain equations. (See [14] for more details.)

3 EXPERIMENTS

We applied the above mentioned technique to two-dimensional, steady-state heat conduction analysis of multi-layer VLSI chips. We used a

simplified model of four layer materials of different thermal conductivities. As depicted in Fig. 3.1, the model is a basic structural pattern of a chip consisting of four layers of SiO₂, Cu, ILD (interlayer dielectric), and passivation. Note that the goal of this paper is to demonstrate the applicability of our technique to heat conduction analysis for VLSI chips. Therefore, we assume for simplicity that each material layer is of the same depth.

The value of thermal conductivity of each material is taken from Im, *et al.* [9]. Also following Im, *et al.* [9], we used the values of junction temperature ($T_j = 85^\circ\text{C}$) and ambient temperature ($T_{\text{amb}} = 45^\circ\text{C}$) as the boundary conditions at the bottom and top edges, respectively, of the region analyzed. These values were taken from ITRS (International Technology Roadmap for Semiconductors) [6]. As for the left and right boundary conditions, we set $T_{\text{right}} = T_{\text{left}} = 20^\circ\text{C}$ as the reference temperatures for our analysis [18]. Furthermore, we calculated the heat generation per unit volume (W/m^3) by Joule heating from metal 1 (M1) of material Cu using the parameters of the 65nm process which were also used in Im, *et al.* [9].

In order to evaluate the performance of our technique, we wrote a C program for solving the single Poisson and three Laplace equations described above. We ran it on a Dell Precision 470 (Linux, CPU Xeon 2.4GHz, memory 2GB). Fig. 3.2 depicts the temperature distribution. As seen from the figure, the temperature increased to $T_{\text{max}} = 125^\circ\text{C}$ from $T_j = 85^\circ\text{C}$ due to the Joule heating from M1 and the difference of 40°C was the same value as reported in Im, *et al.* [9]. This supports the correct temperature distribution produced by our technique.

We also developed a C program for a block LUD method. We ran this LUD program as well as the ICCG method borrowed from the LASPACK packages [7] in place of S-PSM. Table I shows the CPU time, residual, and memory usage for each of the three methods of solving the system of linear equations derived by finite difference method. The results demonstrate that our S-PSM ran 70 and 3 times faster and used up to 3 times and 1.8 times less memory than the LUD and ICCG methods, respectively, and that its residual was smaller by one order of magnitude than that by the LUD. Note that recent work on thermal analysis by Multigrid method reported very fast operations but suffered from high degree of inaccuracy of the solutions. [10][25].

4 Conclusions

We have presented a new technique of thermal analysis for multi-layer VLSI chips. After modeling multiple layers of materials of different heat conductivities by a combination of Laplace and Poisson equations, it discretizes the equations and applies S-PSM to the resulting systems of linear equations.

We have applied our technique to two-dimensional, steady-state heat conduction analysis for a simplified four layer model. The experimental results showed that our S-PSM achieved 70 times speed-up, 3 times memory usage reduction, and a residual improvement

by an order of magnitude as compared to the LUD method.

When compared with ICCG for the case of $EPS=1.0E-10$, our S-SPM (1) ran faster by a factor of 3, (2) obtained a solution with a smaller residual by an order of magnitude, and (3) required a smaller amount of memory. It should be noted that when transient analysis is concerned, the direct method of LUD becomes much more efficient than the iterative methods due to an effective backward substitution [10]. Research on the extension of our technique to transient heat conduction analysis for multi-layer materials of more complex shapes is under way.

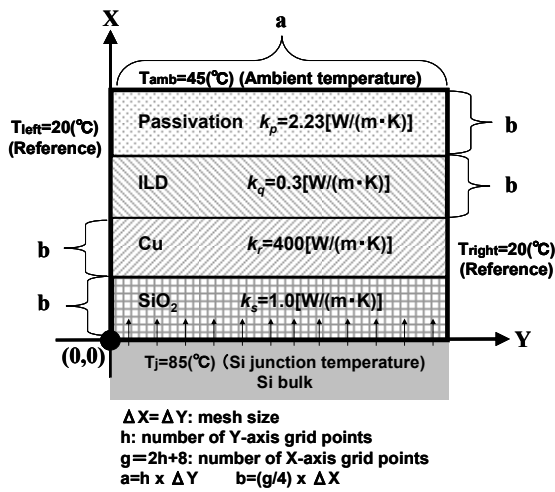


Fig. 3.1: Two dimensional analysis for steady-state heat conduction of four materials of thermal conductivities k_p , k_q , k_r and k_s .

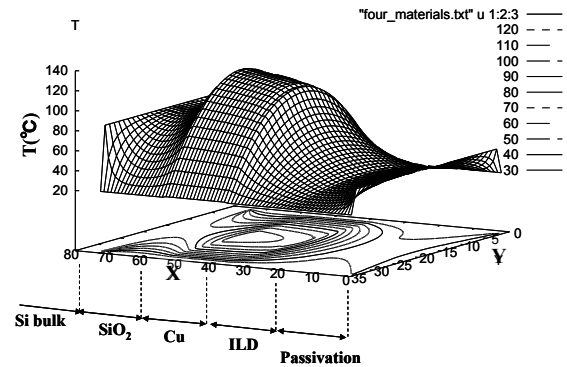


Fig. 3.2: Temperature distribution of steady-state heat conduction for four layers of materials.

Table I: CPU times / Memory usage of S-PSM, LUD, and ICCG for different arrangements of grid points.

| Number of grid points | | S-PSM | | | LUD | | | ICCG (EPS=1.0E-6) | | ICCG (EPS=1.0E-10) | | |
|-----------------------|-----|------------|----------|-------------|---------------|----------|-------------|-------------------|----------|--------------------|----------|-------------|
| g | h | CPU (sec) | Residual | Memory (MB) | CPU (sec) | Residual | Memory (MB) | CPU (sec) | Residual | CPU (sec) | Residual | Memory (MB) |
| 72 | 32 | 0.01 (1.0) | 2.9E-14 | 5.8 (1.0) | 0.06 (6.0) | 7.3E-13 | 6.4 (1.1) | 0.06 (6.0) | 9.2E-07 | 0.07 (7.0) | 8.9E-11 | 6.1 (1.1) |
| 136 | 64 | 0.07 (1.0) | 1.4E-13 | 6.4 (1.0) | 0.94 (13.4) | 9.1E-13 | 8.8 (1.4) | 0.34 (4.9) | 9.2E-07 | 0.45 (6.4) | 9.2E-11 | 7.4 (1.2) |
| 264 | 128 | 0.62 (1.0) | 5.7E-13 | 9.0 (1.0) | 14.6 (23.6) | 1.0E-12 | 16.4 (1.8) | 2.65 (4.3) | 9.3E-07 | 3.90 (6.3) | 9.5E-11 | 11.8 (1.3) |
| 520 | 256 | 5.8 (1.0) | 2.8E-12 | 19.4 (1.0) | 232.9 (40.2) | 9.2E-12 | 48.7 (2.5) | 16.6 (2.9) | 9.6E-07 | 24.3 (4.2) | 8.9E-11 | 30.9 (1.6) |
| 1032 | 512 | 53.7 (1.0) | 1.1E-12 | 60.5 (1.0) | 3761.9 (70.1) | 1.0E-11 | 182.0 (3.0) | 111.8 (2.1) | 9.6E-07 | 164.0 (3.1) | 1.0E-10 | 106.7 (1.8) |

Notes: The values in parentheses are the ratios of increase in time and memory, respectively, as measured relative to the time and memory used by S-PSM.

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