Use of Functional Programming in the Research on Dynamical Systems and the Deterministic Chaos with the Example of Feigenbaum and Non-Feigenbaum Function Iterations

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Abstract: The paper shows the results of computer simulations which were performed with the use of OCaml functional language. The simulations show Feigenbaum trees for a broad spectrum of functions. The possibility to magnify selected areas of the generated fractals permits viewing a whole diversity of structures which are invisible on a normal scale. What is most important, however, is that for non-Feigenbaum functions, the Author discovered the structures which differ completely in terms of quality from those of classic Feigenbaum trees.

Key-Words: - Feigenbaum tree, Feigenbaum functions, non-Feigenbaum functions, orbit, bifurcation point

1 Introduction

The present paper aims at presenting the results which were obtained while studying Feigenbaum trees for numerous functions with the use of functional programming methods. The Author chose functional programming due to the opportunity it provided to naturally construct complex mathematical objects - higher order procedures, etc. The results are the images of the fractals which were generated. It turns out that if the class of the functions under study is enlarged by non-Feigenbaum functions, a standard Feigenbaum tree becomes extremely diverse. Additional interesting structures are generated. In addition, thanks to the computer techniques which were used, the Author discovered fractal structures which were beyond the critical point and which did not qualitatively resemble any fractal images occurring in Feigenbaum trees the Author had seen so far.

2 Problem Formulation

The Author thoroughly studies fractals for non-Feigenbaum functions and compares them with the ones generated for classic Feigenbaum functions. He is interested in: the behavior of the orbits in those two cases, the frequency of occurrence of subsequent bifurcation points and the possibility to determine it by providing a constant which is analogous to the Feigenbaum constant, and the determination of its numerical value. The behavior of Feigenbaum trees for the non-Feigenbaum functions beyond the critical point is analyzed in particular. Before the solution of the problem is presented, essential definitions are given.

2.1 Definitions

Definitions which are used in the paper are introduced below:

Attraction point of the function f:[a,b]->[c,d] – point which is the limit of the sequence defined recursively: a(0)=x0; a(n+1)=f(a(n)) for n>=0; a<x0<b)

Sequence {t(n)} converges to the set of points W only if with n -> ∞: $\mathrm{Inf}\left|t(n)-w\right|_{w\in W} \rightarrow 0$

S-orbit of the function f:[a,b]->[c,d] – set of points to which the above sequence defined recursively converges

Function f with parameter a – function g defined as follows: g(x)=a*f(x), where a is an additional parameter from a determined range, and f(x) is a determined function

Bifurcation point of the function f with parameter a – value of parameter a for which the number of the orbit of the function g(x)=a*f(x) increases

Feigenbaum tree – diagram showing dependence of the orbits of the function f with parameter on this parameter's value, and more precisely: value of the parameter a is marked on the x-axis, and the points from the orbit for the function f with parameter a – on the y-axis; the Feigenbaum tree is an example of a fractal

Critical point – value of the coefficient **a** for which the dynamics of the iteration process defined above is changed in terms of quality; there occurs a finite increase of the orbit cardinality in the bifurcation points below this point; chaotic behavior and orbits of infinite cardinality appear above this point, which is represented by an infinite number of branches in the Feigenbaum tree diagram

Cardinality function for a given function f with parameter – function which assigns the cardinality of the orbit of the function g(x)=a*f(x) to a given parameter a; according to the definition of critical point, this function is determined only on the segment [0,cp] where cp constitutes the critical point's value

Schwarz derivative of the three times differentiable function f – function Sf(x) defined with the formula: Sf(x)=f3(x)/f1(x)-3/2*(f2(x)/f1(x))^2, where f1, f2, f3 are respectively: the first, the second and the third derivative of the function f

Feigenbaum constant – number which is the limit of the sequence p(n)=(b(n+1)-b(n))/(b(n+2)-b(n+1)), where b(n) is the value of the parameter a for which an n-fold bifurcation occurs for the function f with parameter a

Feigenbaum functions – functions f for which the p(n) sequence defined above has a limit (the Feigenbaum constant). In addition, these functions are determined on a certain segment [0,b], they have one local extreme (maximum) on this segment, they are non-negative, they are zero at the points: 0 and b, and they have negative Schwarz derivative on the segment [0,b]

Non-Feigenbaum functions - smooth functions which are determined on a certain segment [0,b] and which do not meet one of the two criteria of the above definition – the condition of extremes or the sign of the Schwarz derivative

3 Problem Solution

3.1 Analysis before the critical point

In the beginning, images of generalized Feigenbaum trees (i.e. also of those which occur for non-Feigenbaum functions) are presented before the critical point. It is possible to observe here additional knots in the case of non-Feigenbaum functions which are invisible on a normal scale:

3.1.1 Function x*(1-x)^3

The part of the fractal marked red is magnified in the next figure.

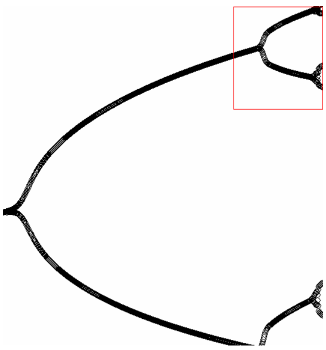


Fig.1

the figure below shows the magnified part, additional knots are clearly visible on the right of the image

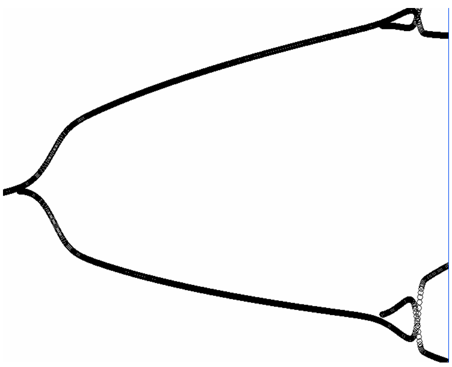


Fig.2

3.1.2 Function x*(1-x)^3*(1+x) – another example of a non-Feigenbaum function for which the structures (humps of knots) which do not occur in Feigenbaum functions are generated before the critical point; it is

worth to point out that they are a lot more complex than in the case of the function which was studied above:

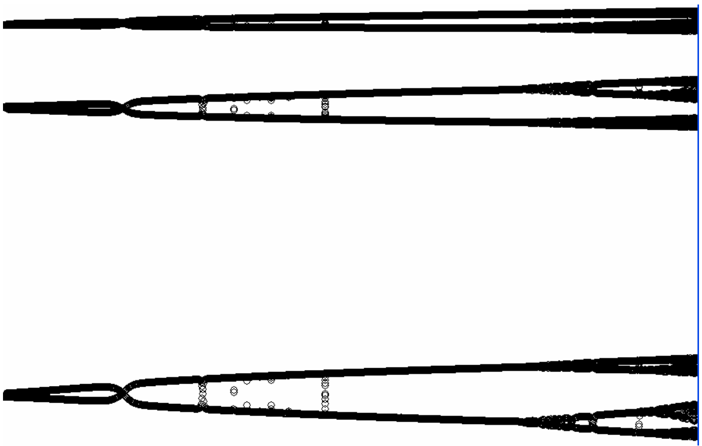


Fig.3

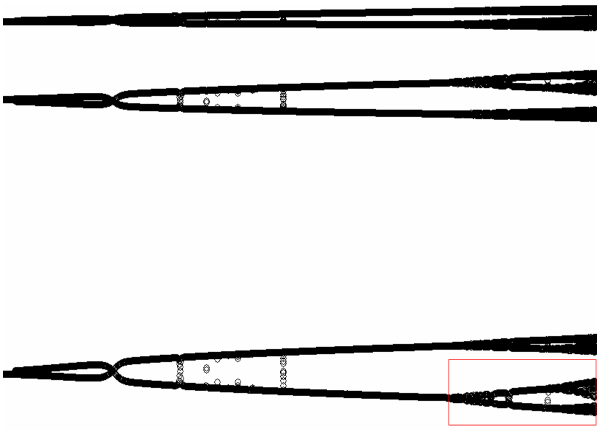


Fig.4

The part marked red on the right of the above image is magnified below; such structures did not occur before the critical point for Feigenbaum functions:

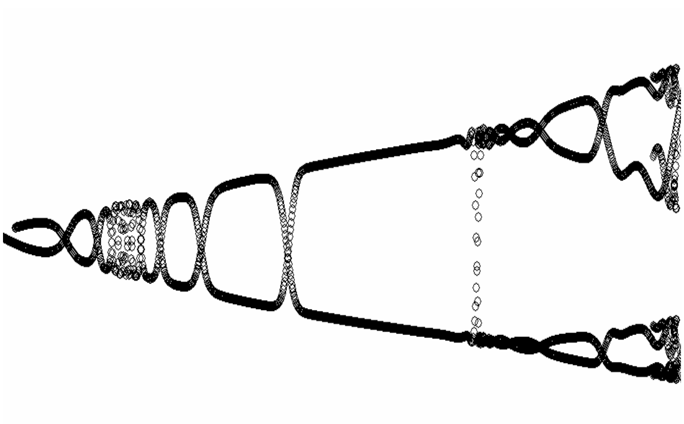


Fig.5

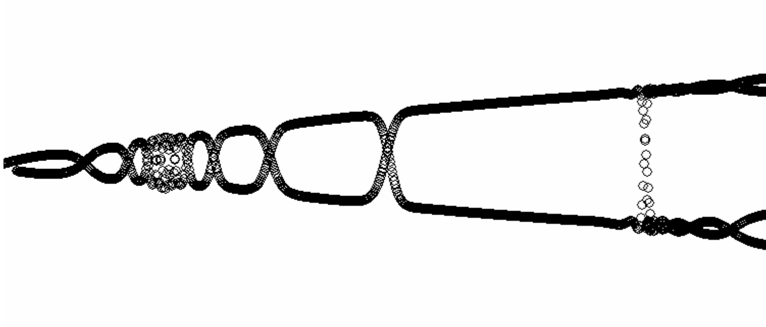


Fig.6

3.2 Analysis beyond the critical point

The quality analysis of Feigenbaum trees for non-Feigenbaum functions beyond the critical point is the most important part of the present paper. The images are more complex even for classic Feigenbaum trees in this area, because they show orbits of infinite cardinality. As it turns out, in the case of non-Feigenbaum functions, there are structures generated locally which do not resemble those which occur before the critical point

3.2.1 (Classic) Function x*(1-x) – in order to compare qualitative features of Feigenbaum trees beyond the critical point for non-Feigenbaum and Feigenbaum functions, the results obtained for one of the simplest Feigenbaum functions are shown first

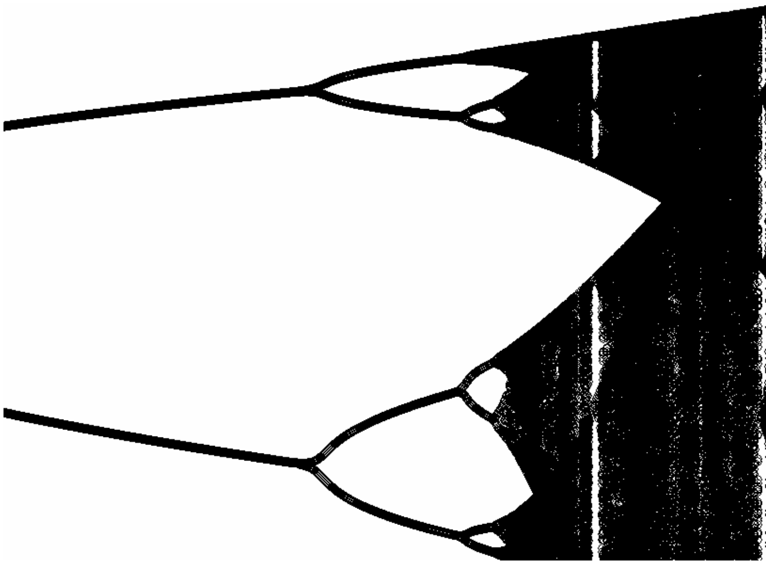


Fig. 7

Below is shown a “window” - a structure which resembles a Feigenbaum tree before the critical point:

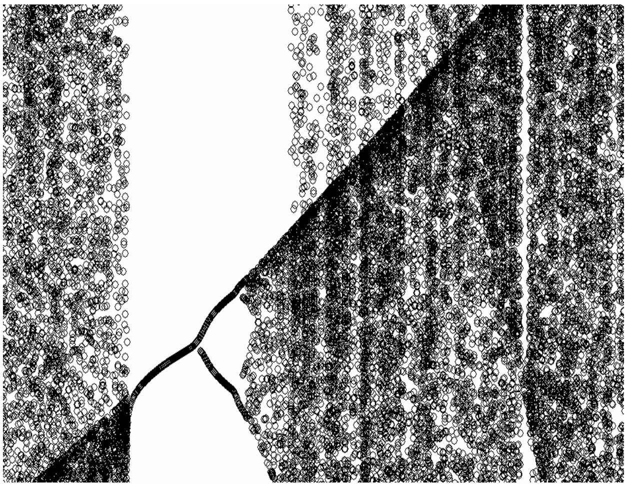


Fig. 8

Below is shown another area beyond the critical point. The orbits in this area behave completely chaotically:

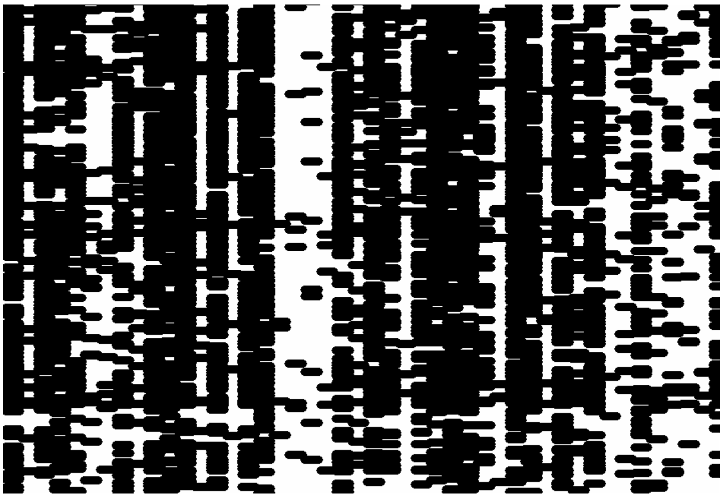


Fig. 9

3.2.2 Function x*(1-x)^3*(1+x) – first non-Feigenbaum function for which a Feigenbaum tree beyond the critical point is shown

On the next pictures, the part occurring near the critical point is visible:

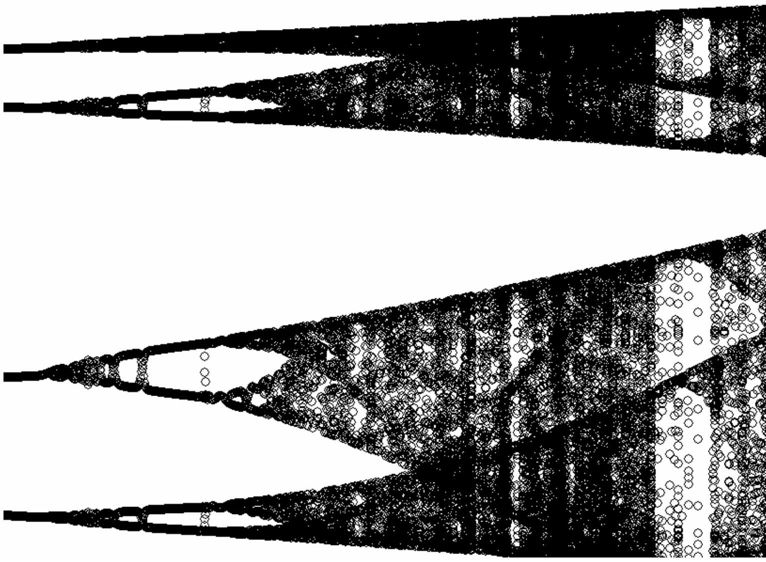


Fig. 10

The previous figure shows a "solar plexus" – dark area to which the lines of the orbits converge. Such plexus also occurs in the case of Feigenbaum functions, but it is less distinct. The plexus is magnified in the figure below:

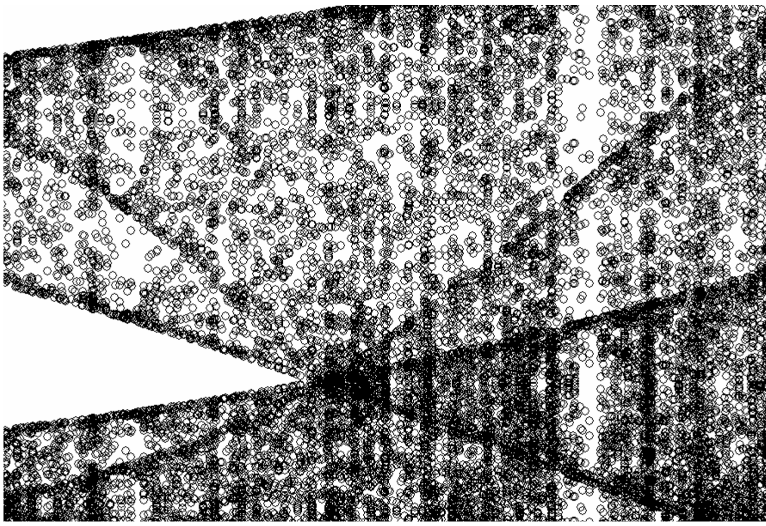


Fig. 11

3.2.3 Function x*(1-x)^4 (after the critical point is crossed), it is worth to point out the new structures.

(a) image of a tree beyond the critical point before magnification, characteristic "cups" are clearly visible on the right of the figure:

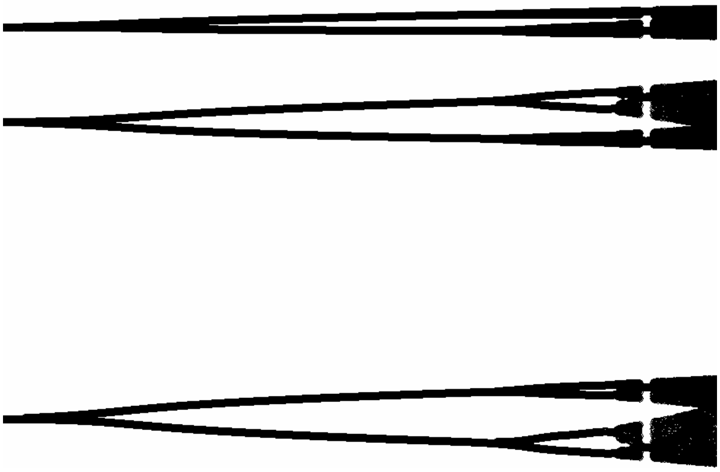


Fig. 12

(b) "cups" after magnification (it is possible to observe that several types of orbits overlap each other – some irregular ones and the ones seen before in the function x*(1-x)^3*(1+x) before the critical point):

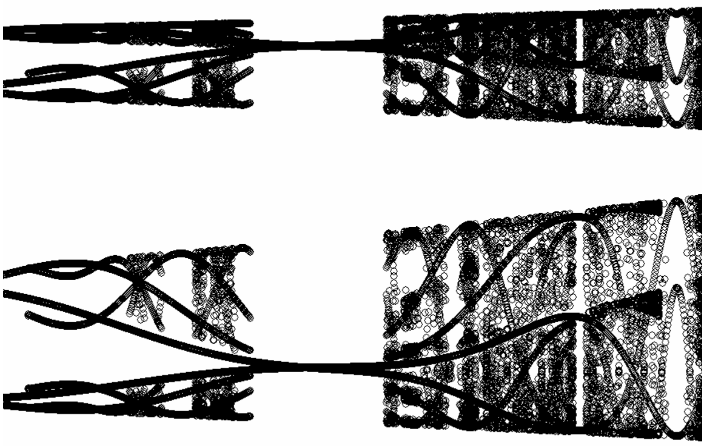
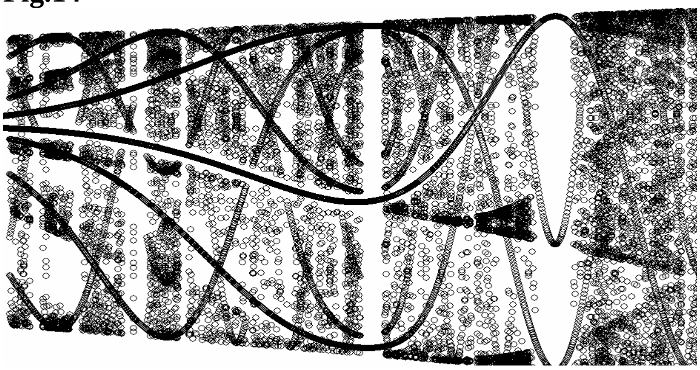


Fig. 13

(c) rectangular "block" from the previous image, magnified:

Fig.14



3.2.4
Function x*(1-x)*tan(x/2) – other fractal patterns are visible here which are different in terms of quality from the ones described previously; they can be clearly seen only after they are precisely magnified; it was possible to observe them after the techniques used by the Author of the paper to produce images of fractals were improved, before these structures were hidden:

(a) no new structures are visible on a regular scale:

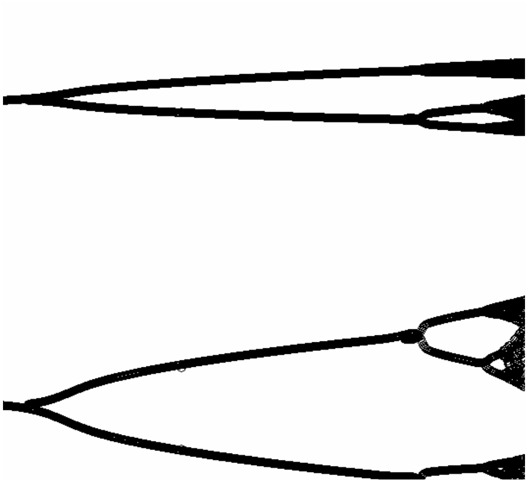


Fig. 15

(b) Images which were produced after subsequent magnifications:

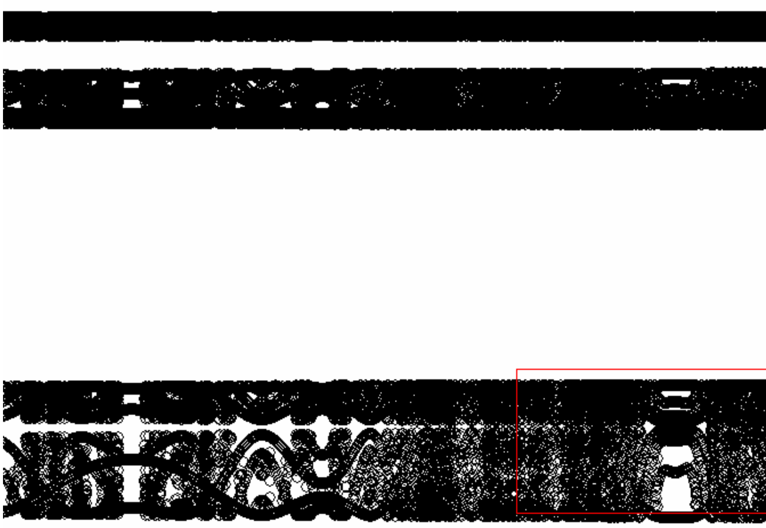


Fig. 16

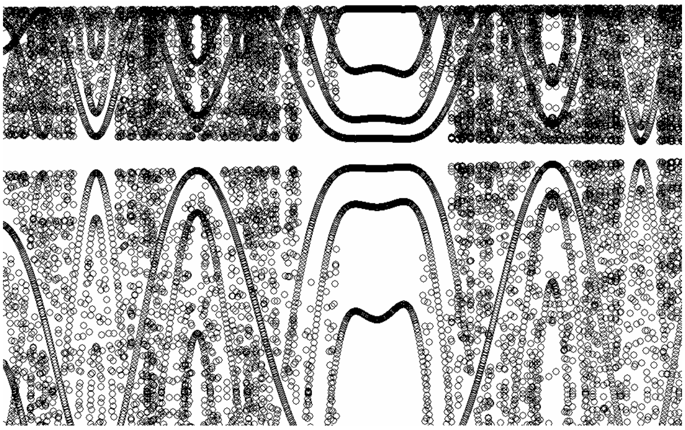


Fig. 17 – image obtained after the area marked red in the previous figure was magnified

When one of the darker areas from the previous figure is magnified again, a characteristic structure of a "tarantula" appears. Darker areas on one axis turn out to be the copies of the image presented below, which is shown in the next figures:

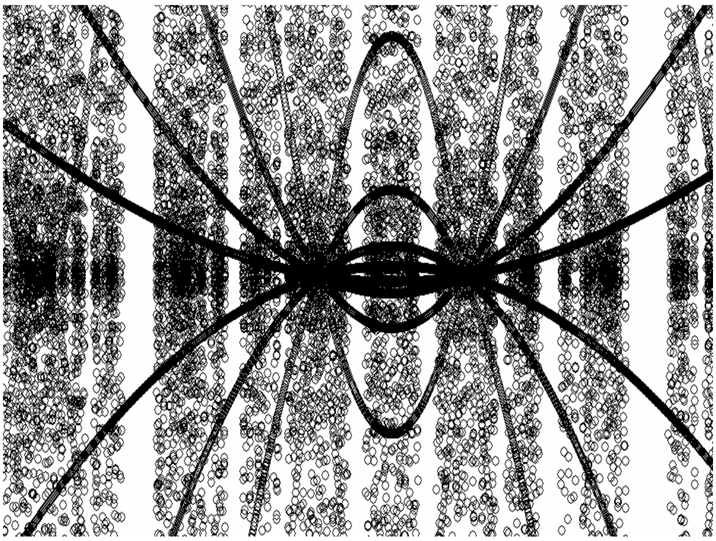


Fig. 18

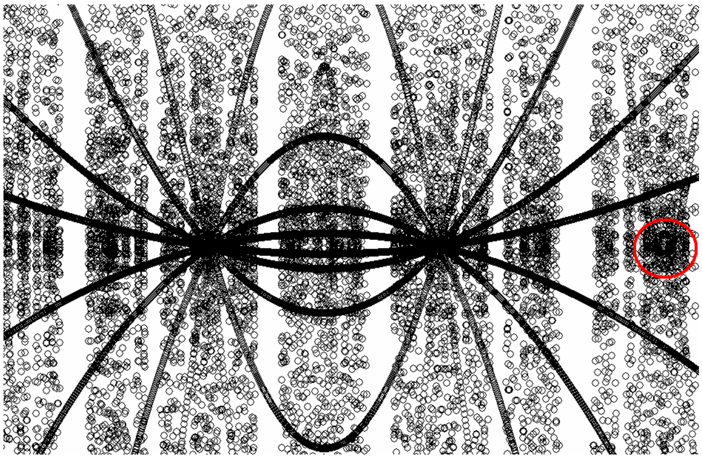


Fig. 19 – the area marked red is shown magnified in the next figure

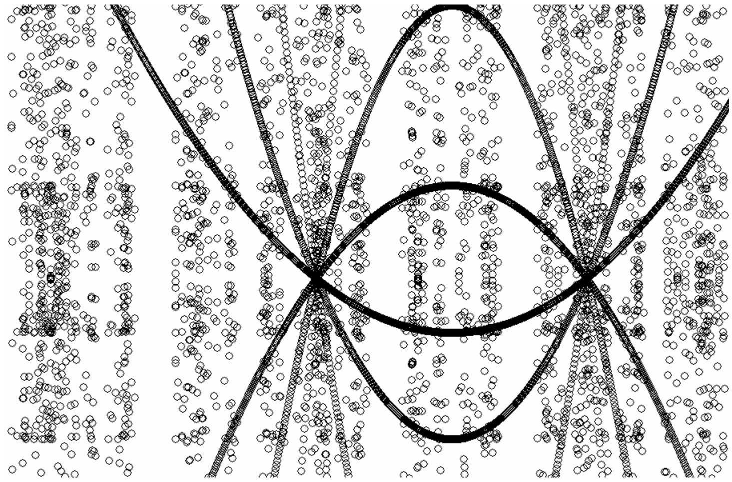


Fig. 20 – the axis seen in the previous figures is almost invisible due to the limited accuracy of the software (this image was produced after a small area of a Feigenbaum tree beyond the critical point was magnified several times)

Below is presented another structure which was found while studying the function. This structure is something in between the structure of "tarantula" and the image of orbits which behave in a regular way:

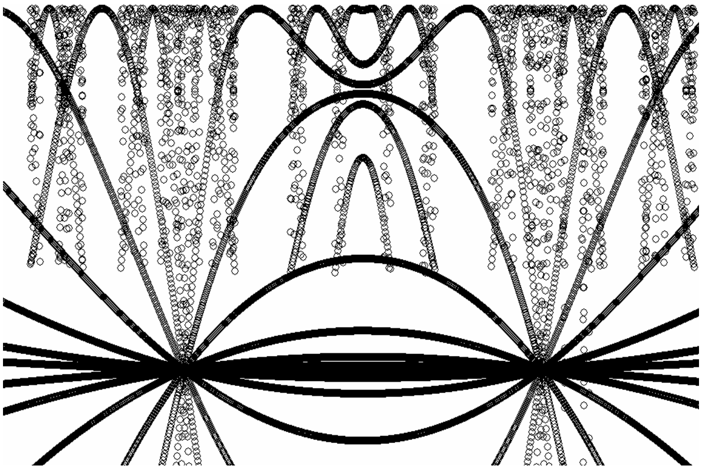


Fig. 21

When another area of the tree in Figure 17 is magnified, the following complex structure is generated:

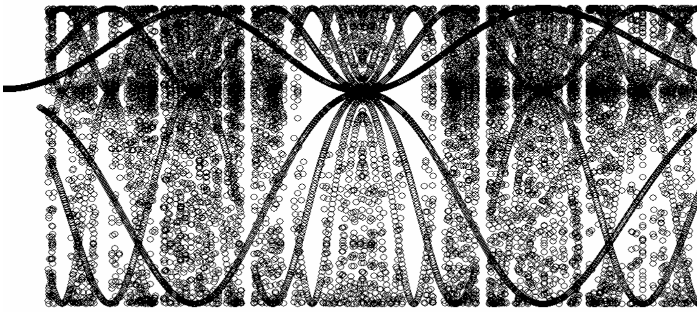


Fig.22

Below are analyzed "rectangular blocks" which are produced by magnifying selected fragments of the tree under study:

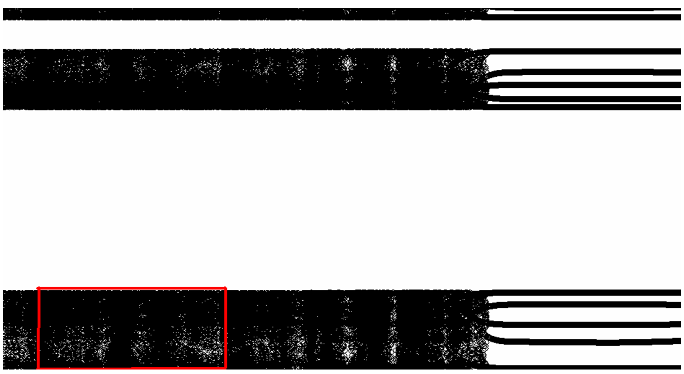


Fig. 23 – characteristic black blocks containing extremely complicated structures which will be seen when the marked area is magnified

The rectangular blocks are in fact very rich fractal structures; in order to ascertain it, the figure below shows the marked area of the above block after it is magnified; a kind of a "tarantula structure" is visible in the center of the figure; in addition, horizontal stripes occur:

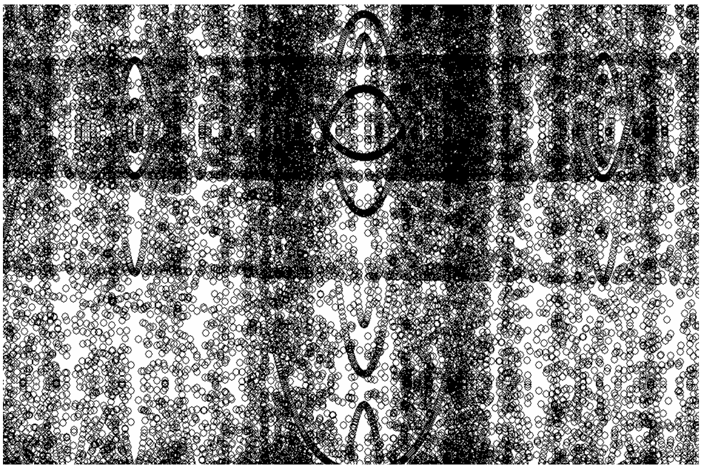


Fig. 24

Below, a magnification of this interesting middle stripe from the above figure is shown; the small "bumps" are probably a copy of the whole figure; due to the limited accuracy of the methods used by the Author, he was not able to magnify these fine "granularities"

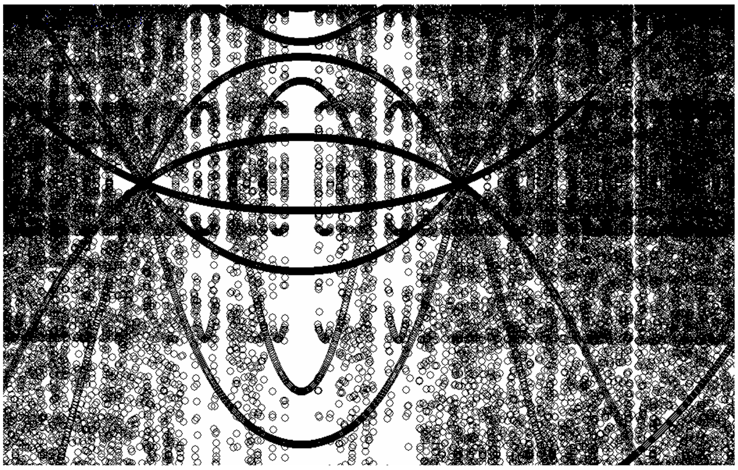


Fig. 25

Below is shown the area bordering the area presented in the figure above; the marked area is magnified in the next figure:

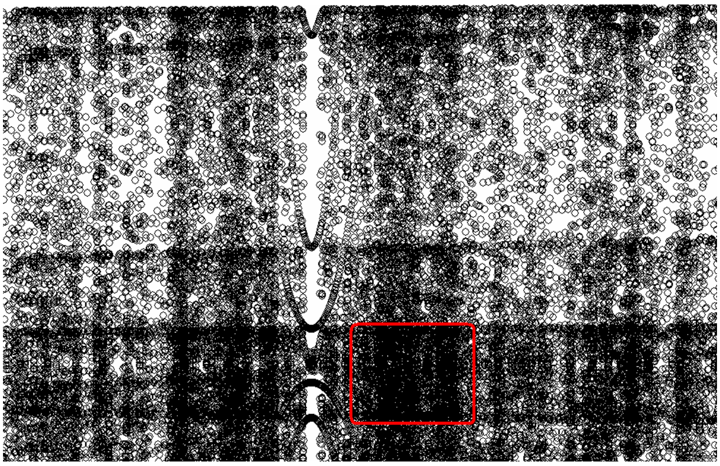


Fig. 26

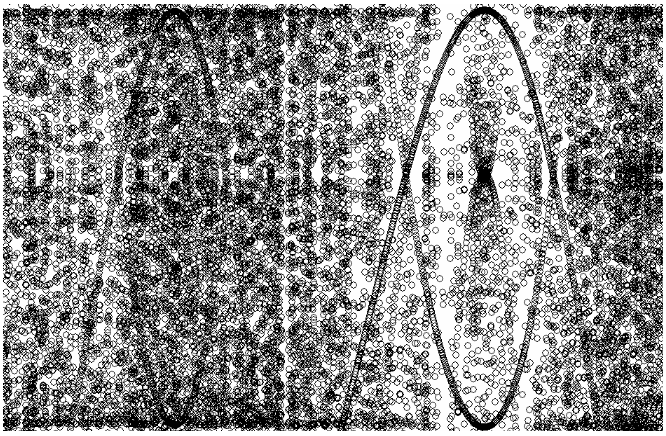


Fig. 27 – rectangular block marked in Figure 26; the lines which resemble fragments of sinusoids overlap the areas which behave chaotically; the dark bumps which lie on one axis are probably "tarantula" structures

4 Conclusion

The results of the simulations clearly indicate that while Feigenbaum trees for non-Feigenbaum functions seem to be isomorphic with classic Feigenbaum trees on a small scale, these functions' different character is proved first of all by the local structures which the Author did not observe in classic trees. What is interesting, the structures which indicated non-Feigenbaum functions occurred even before the critical point. These were the additional knots. They occurred individually or in larger conglomerations. Therefore, the cardinality function defined at the beginning of the paper is not in general monotonic, and this non-monotonicity manifests itself before the critical point, indicating the type of the function under study so to speak. The Author studied Feigenbaum trees for non-Feigenbaum functions, paying special attention to quantitative results at the same time. For the majority of the non-Feigenbaum functions under study, a constant determining the bifurcation frequency which was extremely similar to the Feigenbaum constant was generated (despite qualitative differences in the trees). It would suggest that the conditions imposed on the functions, i.e. that their iteration tree (Feigenbaum tree) should be distinguished by this constant are too restrictive and they can certainly be lessened.

It is worth to underline that it was the use of functional programming that permitted a thorough examination of the generated fractals while retaining a high level of abstraction, and identification of characteristic local areas.

The character of the newly-discovered structures in Feigenbaum trees for non-Feigenbaum functions seems to be a purely mathematical challenge and will probably constitute the subject of the Author's next paper.

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