Energy-Efficient Train Control and Speed Constraints

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Abstract: This paper deals with the description of the nature of the optimal driving strategy for an electric-powered train with speed constraints as well as the calculation of the switching times of optimal driving regimes for special types of resistance functions. We apply the Pontryagin principle and some related tools of optimal control theory to develop the optimal strategy and to derive equations for computation of switching times and the corresponding speed profile in the case of global speed constraints. The problem of varying constraints is briefly discussed as well. The emphasize is put on exact forms of solutions with a minimal use of numerical mathematics.

Key–Words: Critical time, energy-efficient train control, optimal driving strategy, Pontryagin principle, resistance function, speed constraint, speed profile, switching times

1 Introduction

The problem of energy-efficient train control has become a typical problem that can be solved with use of the Pontryagin principle and related tools. It was formulated and discussed in some particular cases in [4] in 1971. In [5] it was proved that the optimal strategy consists of at most four successive control levels (full power, speed holding, coasting and full braking) in a general case. There appeared several papers dealing with various modifications of the basic problem especially during the nineties. Among all we can mention e.g. [11] considering a vehicle with discrete control settings and speed limits or [7] describing a track with a non-zero gradient. A summary of the main results was presented in [6]. Under consideration of these theoretical results there were introduced systems (Metromiser and later FreightMiser) for calculating efficient driving advice during the journey. They were developed as on-board systems which displayed efficient driving advice to the driver and were used with positive results to timetabled suburban and long-haul trains (e.g. in Adelaide, Brisbane or Toronto). Some further alternative approaches to these and similar problems can be found in [8] and [9].

This paper recalls the basic problem of the energy-efficient train control and discusses an extended version of the problem where global and varying speed constraints are involved. We describe the optimal strategy and derive equations for switching times of optimal driving regimes under assumption of the most common types of the resistance function. We also illustrate our ideas with sample speed profiles. We are going to use analytical forms of solution where it is possible. In the above cited papers the problem was discussed with slightly different assumptions and with use of numerical mathematics.

The paper is organised as follows: In Section 2 we describe the formulation of the basic problem and some of its extensions. Section 3 deals with the optimal strategy in case of non-active speed constraints. The main result of this section is the speed profile of the journey and especially the calculation of the maximal speed of the train within the whole track. We also briefly discuss the problem of the critical time. Section 4 is devoted to the problem with active speed constraints and the main accent is put on global speed limits. The last Section 5 summarizes the derived results.

2 Formulation of the problem

We are going to study the following problem of the energy efficient train control:

$$J = \int_0^T u^+ (t)\, x_2 (t)\, \mathrm{d}t \;\; \to \;\; \min \tag{1}$$

with respect to the system of differential equations

$$\dot{x}_1 = x_2 \quad (2)$$
$$\dot{x}_2 = u(t) - r(x_2) \quad (3)$$

and boundary conditions

$$x_1(0) = 0\,,\ x_2(0) = 0\,, \quad (4)$$

$$x_1(T) = L\,,\ x_2(T) = 0\,, \quad (5)$$

where function u^+ is defined as

$$u^+(t) := \begin{cases} u(t) & \text{for } u(t) > 0 \\ 0 & \text{for } u(t) \le 0\,. \end{cases}$$

Further, we assume that u is a piecewise continuous function mapping $\langle 0, T\rangle$ into $\langle -\alpha, \beta\rangle$, where $\alpha, \beta > 0$ are given constants. Function $r = r(x_2)$ (which represents the frictional resistance) is a differentiable function (with respect to x_2) with the properties $r, r' \ge 0$ and $r'(x_2)\,x_2$ is nondecreasing for $x_2 \ge 0$. The most usual type of resistance function r (which satisfies all these conditions) is a polynomial function

$$r(x_2) = a + bx_2 + c\,(x_2)^2\ .$$

To simplify the computations, we consider the linear resistance function $r(x_2) = bx_2$ and the quadratic resistance function $r(x_2) = c\,(x_2)^2$.

The problem (1)-(5) describes the motion of a train along a straight level track of length $L > 0$ with minimal consumption of electric energy J. We assume that the mass of the train $m = 1$. Phase coordinates x_1 and x_2 correspond to position and speed of the train. Given parameter T represents the time that is available according to the timetable for the train to complete the track.

The basic problem of energy-efficient train control (1)-(5) can be extended in several ways. We are going to study the problem involving the speed limits. The most comprehensive approach to this issue is represented by the following conditions

$$x_2 \le M_{j+1} \quad \text{for} \quad x_1 \in (X_j, X_{j+1})\ , \quad (6)$$

where $0 = X_0 < X_1 < \ldots < X_p = L$. We shall further concentrate especially on the global speed constraint in the form

$$x_2(t) \le v_{max}\,, \quad t \in \langle 0; T\rangle\ . \quad (7)$$

3 Non-active speed constraints

In this section we are going to summarize the previous results and extend them in a certain way in order to obtain the value of the maximal speed x_{2max} of the train within the whole track under assumption of the basic problem (1)-(5) without any further constraints. We have to calculate the value of x_{2max} so that we determine whether the global speed constraint (7) is active ($x_{2max} \ge v_{max}$) or not. In the latter case, we may easily apply the results of this section (optimal strategy and the values of switching times) also for the case of the global speed constraint.

First of all, it is necessary to determine the value of the minimal time T_{min}, that it is possible to complete the journey within. It can be found by solving the corresponding minimum time problem (we arrive at the standard "bang-bang" control). In what follows, we assume that the given time of the journey T satisfies relation $T > T_{min}$.

The solution of energy-efficient control problem (1)-(5) is specified by the following theorem (for further details see e.g. [5]).

Theorem 1. *Let $(\hat{x}_1(t), \hat{x}_2(t); \hat{u}(t))$, $t \in \langle 0, T\rangle$ be the energy optimal solution of (1)-(5). Then there exist t_1, t_2, t_3, where $0 < t_1 \le t_2 < t_3 < T$, such that*

$$\hat{u}(t) = \begin{cases} \beta & \text{for } 0 \le t < t_1 \\ r(\hat{x}_2(t)) \equiv const. & \text{for } t_1 < t < t_2 \\ 0 & \text{for } t_2 < t < t_3 \\ -\alpha & \text{for } t_3 < t \le T. \end{cases}$$

If $t_1 = t_2$, then the values of the switching times can be easily calculated by integration of (2) and (3) on separate intervals, comparing values of position and speed in boundary points of these time intervals (i.e. in $t = t_1 = t_2$ and $t = t_3$) and involving conditions (4) and (5). The second phase (speed-holding) is omitted in this consideration.

Under consideration of the linear resistance function r we obtain equation for t_3

$$Lb^2 + \alpha bT - \alpha bt_3 = \beta \ln\left(\frac{\alpha}{\beta}\,\mathrm{e}^{bT} - \frac{\alpha}{\beta}\,\mathrm{e}^{bt_3} + 1\right)\ .$$

Consequently, we can determine the value of $t_1 = t_2$ via relation

$$t_1 = \frac{1}{b} \ln\left(\frac{\alpha}{\beta}\,\mathrm{e}^{bT} - \frac{\alpha}{\beta}\,\mathrm{e}^{bt_3} + 1\right)$$

and the value of the maximal speed x_{2max} as

$$x_{2max} = -\frac{\beta}{b}\left(\frac{\alpha}{\beta}\,\mathrm{e}^{bT} - \frac{\alpha}{\beta}\,\mathrm{e}^{bt_3} + 1\right)^{-1} + \frac{\beta}{b}\,.$$

For quadratic type of resistance function r we can derive analogical equations.

Fig. 1 represents a typical speed profile for the case of the linear resistance function.

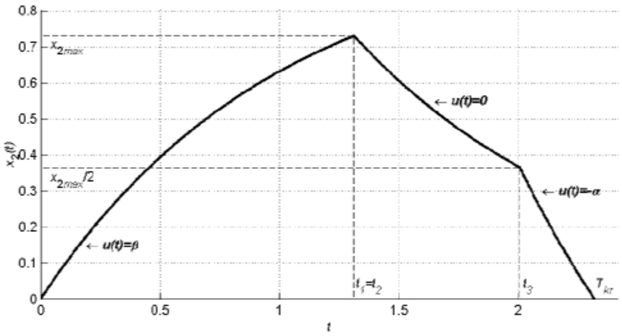

Fig. 1: A typical speed profile for $t_1 = t_2$ and parameters $\alpha = 1$, $\beta = 1$, $b = 1$, $L = 1$, $T = 2,17$

If we assume the relation $t_1 < t_2$, we have to determine the values of three unknown variables t_1, t_2 and t_3. Therefore, it is necessary to compare the values of the corresponding Hamilton function under suitable choices of the independent variable t and utilize the property $H \equiv$ const on $\langle 0, T\rangle$. Further, we make use of continuity of Lagrange multiplicators on $\langle 0, T\rangle$ and obtain equation for calculation of the time t_2 for linear resistance function r in the form

$$\left(\alpha\, \mathrm{e}^{b(T-t_2)} - 2\alpha - \beta\right) \ln\left[-\frac{\alpha}{\beta}\, \mathrm{e}^{b(T-t_2)} + \frac{2\alpha}{\beta} + 1\right]$$
$$= Lb^2 + \alpha bT + \alpha b t_2 - \alpha \ln 2 - \alpha b t_2\, \mathrm{e}^{b(T-t_2)}$$

and relations for the remaining switching times t_1 and t_3 in the form

$$t_1 = -\frac{1}{b} \ln\left(-\frac{\alpha}{\beta}\, \mathrm{e}^{b(T-t_2)} + \frac{2\alpha}{\beta} + 1\right),$$

$$t_3 = t_2 + \frac{1}{b} \ln 2 .$$

The value of the maximal speed x_{2max} for this case can be determined via relation

$$x_{2max} = \frac{\alpha}{b}\, \mathrm{e}^{b(T-t_2)} - \frac{2\alpha}{b} .$$

Analogically as in the case of linear resistance function r it si possible to solve the problem with $t_1 < t_2$ under assumption of quadratic resistance function.

A typical speed profile for the case $t_1 < t_2$ and linear resistance function is shown in Fig. 2.

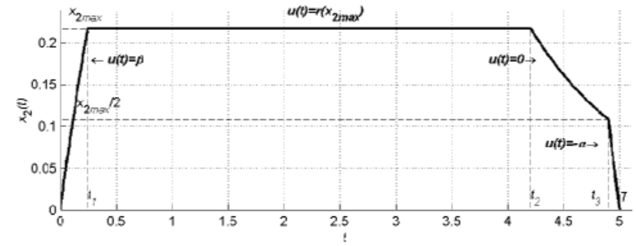

Fig. 2: A typical speed profile for $t_1 < t_2$ and parameters $\alpha = 1$, $\beta = 1$, $b = 1$, $L = 1$, $T = 5$

We have derived the values of the switching times for both possible strategies, i.e. involving the speed-holding phase or not. However, we have not specified yet which of these strategies is optimal for given values of entry parameters of the problem. The optimal solution can be found by evaluation of the cost functional J for both cases and by comparing the obtained values. With use of expression (1) we can easily arrive at relations for computation of the values of J. In case $t_1 = t_2$ and linear resistance function r we obtain

$$J = -\frac{\beta^2}{b^2} + \frac{\beta^2}{b} t_1 + \frac{\beta^2}{b^2}\, \mathrm{e}^{-bt_1}$$

and for quadratic resistance function r

$$J = \frac{\beta}{c} \ln\cosh\left(\sqrt{\beta c}\, t_1\right) .$$

In case $t_1 < t_2$ we derive relation

$$J = -\frac{\beta^2}{b^2} + \frac{\beta^2}{b} t_1 + \frac{\beta^2}{b^2}\, \mathrm{e}^{-bt_1} + b\left(x_{2max}\right)^2 (t_2 - t_1)$$

for linear type of resistance function and

$$J = \frac{\beta}{c} \ln\cosh\left(\sqrt{\beta c}\, t_1\right) + c\left(x_{2max}\right)^3 (t_2 - t_1)$$

for quadratic type.

However, numerical calculations (based on algorithms in [2]) show that the choice of the optimal strategy depends on the value of the given entry parameter T. Therefore, it is convenient to apply the mathematical theory of parametric programming and related tools to analyse the behaviour of the solution of problem (1)-(5) with respect to parameter T. We cannot present all important concepts and theorems of the theory of nonlinear parametric programming. The reader can find them in [1]. Further details and precise proofs of the following claims can be found in [10].

With use of Theorem 1 we can easily arrive at the formulation of the problem (1)-(5) in the form of nonlinear programming problem (for linear resistance function r, the quadratic case can be solved analogically)

$$J = \frac{\beta^2}{b^2}\left(bt_1 + \mathrm{e}^{-bt_1} - 1\right) + \frac{\beta^2}{b}\left(t_2 - t_1\right)\left(1 - \mathrm{e}^{-bt_1}\right)^2 \quad \to \ \min \tag{8}$$

$$\alpha\left(\mathrm{e}^{b(T-t_3)} - 1\right) = \beta\left(1 - \mathrm{e}^{-bt_1}\right)\mathrm{e}^{b(t_2-t_3)} \tag{9}$$

$$\alpha\left(t_3 - T\right) + \beta\left(t_2 - t_2\,\mathrm{e}^{-bt_1} + t_1\,\mathrm{e}^{-bt_1}\right) = bL \tag{10}$$

$$0 \le t_1 \le t_2 \le t_3 \le T\,. \tag{11}$$

We shall denote with $M(T)$ the set of all feasible solutions of the given problem, i.e. the set of all (t_1, t_2, t_3) satisfying (9)-(11) for a given T. Further, we introduce the following assumption, which is in general complicated to be verified formally (it can be verified for specified values of entry parameters α, β, b and L).

Hypothesis 2. *The point-to-set mapping $M(T)$ is continuous in T for all $T \ge T_{min}$.*

Now, we introduce the notion of the critical time T_{cr} and describe its calculation.

Definition 3. *A parameter T is said to be the critical time of the problem (8)-(11) (and we shall further denote it as T_{cr}) if there exists an $\epsilon > 0$ such that for $T = T_{cr}$ the nonlinear programming problem (8)-(11) has an optimal solution with property $t_1 = t_2$ and for $T \in (T_{cr}, T_{cr} + \epsilon)$ the corresponding optimal solution satisfies $t_1 < t_2$.*

Lemma 4. *Let T_{cr} be the critical time of the problem (8)-(11) and let the Hypothesis 2 be fulfilled. Then T_{cr} is the unique positive solution of the equation*

$$\alpha b T_{cr} + Lb^2 + (\alpha + \beta)\ln\left(\frac{2\alpha + \beta}{\beta + \alpha\,\mathrm{e}^{bT_{cr}}}\right) = \alpha \ln 2\,.$$

Summarizing the considerations we arrive at the following theorem.

Theorem 5. *Let (t_1, t_2, t_3) be the optimal solution of the problem (8)-(11) and let the Hypothesis 2 be fulfilled. Then either $t_1 = t_2$ for every $T \ge T_{min}$ or there exists a unique value of T_{cr} with the property that for $T \in \langle T_{min}, T_{cr}\rangle$ the optimal solution satisfies $t_1 = t_2$ and for $T > T_{cr}$ the property $t_1 < t_2$ is fulfilled (moreover, this value T_{cr} can be found as the unique positive solution of the previous equation).*

The numerical calculations show that considering parameter T large enough the optimal solution (t_1, t_2, t_3) of the problem (8)-(11) satisfies $t_1 < t_2$ for fixed parameters α, β, L and b. E.g., under the choice $\alpha = \beta = b = L = 1$ we get $T_{cr} \approx 2,17$ (thus, Fig. 1 and Fig. 2 illustrate the case $T = T_{cr}$ and $T > T_{cr}$ respectively). We may therefore introduce a conjecture that the first variant described in Theorem 5 (i.e. $t_1 = t_2$ for every $T \ge T_{min}$) does not actually occur. However, the proof of this claim for arbitrary (unspecified) values of α, β, L and b remains open.

For quadratic resistance function we show at least the necessary condition for T_{cr}. First, we calculate the value of t_{cr} according to equation

$$\frac{2}{3}\,\mathrm{e}^{cL}\left|\cos\arctan\left(\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}\,t_{cr}\right)\right)\right| - \cosh\left(\sqrt{\beta c}\,t_{cr}\right) = 0$$

and then obtain the value of T_{cr} via relation

$$T_{cr} = \frac{1}{\sqrt{\alpha c}}\arctan\left(\sqrt{\frac{\beta}{\alpha}}\frac{2}{3}\tanh\left(\sqrt{\beta c}\,t_{cr}\right)\right) + t_{cr} + \frac{1}{2\sqrt{\beta c}\tanh\left(\sqrt{\beta c}\,t_{cr}\right)}\,.$$

4 Active speed constraints

Let us assume the problem (1)-(5) again with additional assumption (7). First, we have to determine the value of the minimal time T^*_{min} that it is possible to complete the track within (involving the speed constraint (7)). Let $x_{2max} \ge v_{max}$ (where x_{2max} denotes the maximal speed achieved by the train within the whole track without any speed constraints by time optimal strategy). With use of Pontryagin principle and some further tools concerning the path constraints (for further details see e.g. [3]) we can easily arrive at the following equation for calculation of T^*_{min}

$$T^*_{min} = \frac{1}{b^2 v_{max}}\ln\left[1 + \frac{b}{\alpha}v_{max}\right]^{\alpha}\left[1 - \frac{b}{\beta}v_{max}\right]^{\beta} + \frac{L}{v_{max}} + \frac{1}{b}\ln\frac{1 + \frac{b}{\alpha}v_{max}}{1 - \frac{b}{\beta}v_{max}}$$

in case of linear resistance function and similarly for quadratic resistance function.

In what follows, we assume that $T > T^*_{min}$ and $x_{2max} > v_{max}$. Let us denote

$$S(x_1, x_2, t) := x_2(t) - v_{max}.$$

Then it holds for the first total time derivative of S that

$$S^{(1)}(x_1, x_2, t) = \dot{x}_2(t) = u(t) - r(x_2).$$

Thus, (7) is a first order state variable inequality constraint. Hamilton function is in the form

$$H = \lambda_0 u^+ x_2 + \lambda_1 x_2 + (\lambda_2 + \mu)(u - r(x_2)) ,$$

where λ_0, λ_1, λ_2 and μ denote the corresponding Lagrange multiplicators (without loss of generality we put $\lambda_0 \equiv -1$, the case $\lambda_0 \equiv 0$ corresponds to time optimization). The variables λ_1 and λ_2 have to satisfy the adjoint system

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0\\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = u^+ - \lambda_1 + \lambda_2 r'(x_2) + \mu r'(x_2).\end{aligned}$$

Further, $\mu \leq 0$ on the constraint boundary ($S = 0$) and $\mu = 0$ off the constraint boundary. The path entering onto the constraint boundary has to meet the tangency constraint $S = 0$ and if we denote t_1 as the entry point onto the boundary constraint, then the following jump conditions have to be satisfied:

$$\begin{aligned}\lambda_1(t_1^-) &= \lambda_1(t_1^+)\\ \lambda_2(t_1^-) &= \lambda_2(t_1^+) + \pi \quad (\pi \in \mathbf{R})\\ H(t_1^-) &= H(t_1^+) ,\end{aligned}$$

where t_1^- and t_1^+ denote the corresponding one-sided limits. Thus, $\lambda_1(t) \equiv C_1 =$ const for $t \in \langle 0, T\rangle$ and λ_2 might be discontinuous at time t_1. Off the constraint boundary we may use the Pontryagin principle and derive the same four possible driving strategies as in the case without the speed constraints, i.e. full power, speed holding, coasting and full braking. Let us denote t_2 the time when the path is leaving the speed boundary. On the constraint boundary (if $t_1 < t_2$) it holds $u(t) = r(v_{max})$ and $\frac{\partial H}{\partial u} = 0$. Thus, for $t \in \langle t_1, t_2)$ (with use of relation $x_2(t) \equiv v_{max}$) it holds

$$\lambda_2(t) = v_{max} - \mu(t) \geq v_{max} .$$

As $\mu(t) \leq 0$ on the constraint boundary the relation $\lambda_2(t) \geq v_{max}$ must hold for $t \in \langle t_1, t_2)$. Further, let us assume the linear case $r(x_2) = bx_2$ (for quadratic resistance function r we can use analogical approach). With use of jump condition for Hamilton function in time t_1 it can be shown that $\lambda_2(t_1^-) = v_{max}$. Further, according to the continuity of λ_2 in t_2 it holds $\lambda_2(t_2^+) > 0$, thus $u(t_2^+) = 0$ and further $\lambda_2(t_2) = v_{max}$. Therefore,

$$H = C_1 v_{max} - k v_{max}^2 > 0 , \quad \rightarrow \quad C_1 > k v_{max} .$$

Summarizing the previous ideas and analysing the properties of function $\lambda_2(t)$ based on previous results it is possible to prove the following theorem.

Theorem 6. *Let $(\hat{x}_1(t), \hat{x}_2(t); \hat{u}(t))$, $t \in \langle 0, T\rangle$ be the energy optimal solution of (1)-(5) and (7). Let $r(x_2) = bx_2$ ($r(x_2) = c(x_2)^2$). Then there exist t_1, t_2, t_3 such that*

$$\hat{u}(t) = \begin{cases} \beta & \text{for } t \in \langle 0, t_1)\\ b v_{max} \quad (c(v_{max})^2) & \text{for } t \in \langle t_1, t_2)\\ 0 & \text{for } t \in \langle t_2, t_3)\\ -\alpha & \text{for } t \in \langle t_3, T\rangle \end{cases} ,$$

where $0 < t_1 \leq t_2 < t_3 < T$.

The case $t_1 = t_2$ corresponds to the relation $x_{2max} = v_{max}$. By integration of equations (2) and (3) on separate time intervals and involving the boundary conditions (4) and (5) it is easy to find the equations for calculation of the switching times t_1, t_2 and t_3 for both linear and quadratic resistance functions. If $r(x_2) = bx_2$ then

$$t_1 = -\frac{1}{b} \ln\left(1 - \frac{b v_{max}}{\beta}\right) .$$

Further, we can derive equation for unknown t_3

$$\left(\frac{v_{max}}{b} - \frac{\beta}{b^2}\right) \ln\left(1 - \frac{b v_{max}}{\beta}\right) - \frac{\alpha}{b}(T - t_3) = L - \frac{v_{max}}{b} \ln\left(\frac{\alpha}{b v_{max}}\left(\mathrm{e}^{bT} - e^{bt_3}\right)\right)$$

and consequently calculate the value of t_2 via relation

$$t_2 = \frac{1}{b} \ln\left(\frac{\alpha}{b v_{max}}\left(\mathrm{e}^{bT} - \mathrm{e}^{bt_3}\right)\right) .$$

For $r(x_2) = c(x_2)^2$ we obtain relation

$$t_1 = \frac{1}{\sqrt{\beta c}} \operatorname{arctanh}\left(\sqrt{\frac{c}{\beta}} v_{max}\right) .$$

Thereafter we calculate the value of t_3 via equation

$$\begin{aligned}&\sqrt{\frac{c}{\alpha}} v_{max} \operatorname{cotan}\left(\sqrt{\alpha c}(T - t_3)\right) - \ln v_{max} +\\ &\ln \sqrt{\frac{\alpha}{c}} \frac{|\cos(\sqrt{\alpha c}(T - t_3))|}{\operatorname{cotan}(\sqrt{\alpha c}(T - t_3))} + cL\\ &+ \sqrt{\frac{c}{\beta}} v_{max} \operatorname{arctanh}\left(\sqrt{\frac{c}{\beta}} v_{max}\right) =\\ &c v_{max} t_3 + 1 + \ln\cosh\operatorname{arctanh}\left(\sqrt{\frac{c}{\beta}} v_{max}\right)\end{aligned}$$

and the value of t_2 from relation

$$t_2 = t_3 + \frac{1}{cv_{max}} - \frac{1}{\sqrt{\alpha c}} \operatorname{cotan}\left(\sqrt{\alpha c}\,(T - t_3)\right) .$$

The equations for computation of the switching time t_3 usually yield two different possible values of t_3. However, only one of them satisfies relations $0 < t_1 \leq t_2 < t_3 < T$.

Fig. 3 shows a typical speed profile for energy-efficient strategy with global speed constraint compared with the case without any constraints.

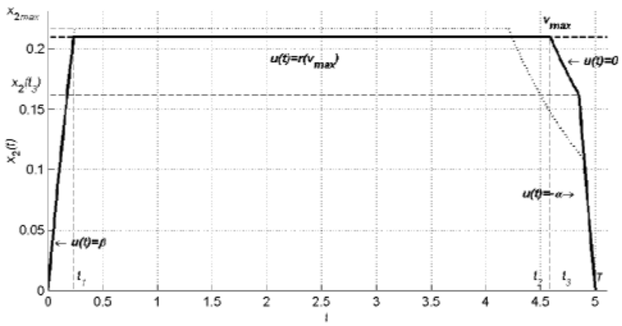

Fig. 3: A typical speed profile for constrained optimization and parameters $\alpha = 1$, $\beta = 1$, $b = 1$, $L = 1$, $T = 5$ and $v_{max} = 0.21$ (the dotted line represents the case without speed constraint)

The complex problem of speed constraints in the form (6) is much more complicated and it is going to be an object of author's further investigations. One way of solving this problem could be partitioning of the time interval $\langle 0, T\rangle$ on subintervals $\left\langle t_j^*, t_{j+1}^*\right\rangle$, $\quad j = 0, \ldots p-1$ with respect to the speed constraints (6), solving the corresponding energy-efficient train control problems on the separate intervals with global speed constraints (6) and with unknown values of the speed at the boundary points, comparing these values and solving the resulting nonlinear programming problem of minimization J according to the values of t_j^*. However, this leads to application of some numerical algorithms or methods of artificial intelligence and exceeds the aim of this paper.

5 Conclusion

We have shown the optimal strategies for energy-efficient train control in the basic case and derived equations for calculation of the switching times. Further, we discussed the question of the critical time with use of parametric optimization. We solved the problem of the global speed constraint for both mentioned types of resistance functions and discussed the problem with varying speed constraints.

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