Phase Space Research of One Non-autonomous Dynamic System

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Abstract: - The non-autonomous dynamic system considered in this paper represent a system of differential equations of variable mass coaxial bodies (gyrostat) motion around a fixed point. The modification of mass parameters of coaxial bodies (mass, moment of inertia) causes nontrivial changes of system angular motions. The special qualitative research method of a system phase space is developed. The method is based on an evaluation of a phase trajectory curvature. On the basis of this method it is possible to determine the phase trajectory shape. Also it is possible to synthesize conditions of realization of motion special cases (for example, a monotone diminution or magnification of an angle of a nutation). The application value of researches is connected with the analysis of space vehicles angular motion around a center of mass at realization of active inter-orbital maneuvers.

Key-Words: - Non-autonomous Dynamic System, Coaxial Bodies, Space Vehicles, Variable Mass, Angular Motion, Qualitative Method, Phase Trajectory Curvature

1 Introduction

The non-autonomous dynamic system considered in this paper represents differential equations of angular motions of variable mass rigid coaxial bodies system (also called unbalanced gyrostat). Research of angular motions of rigid bodies systems and gyrostats with constant and variable structure still remains one of the important problems of theoretical and applied mechanics. Despite of modern development the indicated problem still is far from completion. In particular it concerns study of dynamics of variable structure systems. The analysis and synthesis of conditions of precession motion with diminished amplitude of nutation oscillations is lead in this paper.

Such motion has the important application value to space flight mechanics tasks. Especially it concerns gyroscopic stabilization of space vehicles (SV) at realization of active maneuvers.

For understanding of an essence of this problem it is important to describe briefly the main considered engineering singularities of realization of SV active maneuvers. For a realization of active maneuvers, for example inter-orbital passage, it is necessary to create thrust of a jet engine for accelerating or braking impulse ($\Delta\mathbf{V}$).

This impulse is produced in a necessary calculated direction. Engine thrust is usually directed along SV direct-axis. Therefore it is necessary to conduct stabilization of a direct-axis for stabilization of an impulse output direction. Stabilization of a direct-axis can be carried out in a gyroscopic mode when SV spins around a direct-axis oriented in the calculated direction.

Shaping of impulse is not instant and demands operation of the jet engine within several seconds (or minutes). During engine operaton SV makes two motions: trajectory motion of a center of mass and angular motion around the center of mass. SV angular motion obviously changes direct-axis direction and, hence, a thrust direction.

The time history of a thrust direction is a reason of magnitude deviation and direction deviation of the inter-orbital passage impulse from the computational values. Consequently, the passage happens to the orbit, which differs from the calculated one. There is stray "sputtering" of thrust (fig. 1). Therefore, it is very important to take into account SV angular motions on the active leg.

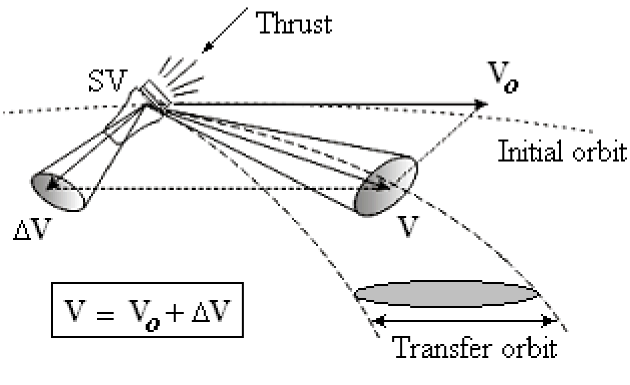

Fig.1

It is necessary to supply such angular motions at which SV direct-axis (and the thrust vector) makes precession motion with monotonically diminished nutation angle. Thus the direct-axis goes inside of an initial cone of a nutation and the thrust vector naturally comes nearer to an axis of a precession which is a calculated direction of output of transitional impulse ("is focused" along a necessary direction).

At an angular motions without a monotone diminution of nutation angle the direct-axis can make rather complicated motion. In this case the thrust vector also makes complicated motion and "sputters" transitional impulse. Sputtering a transfer orbit in this case takes place.

Gyroscopic stabilization can be carried out at the expense of fast rotation not all SV, but only his parts - the stabilizing unit. In this case a SV will in essence consist as a minimum of two coaxial bodies. Such SV can refer to the "SV with double rotation" or "SV-gyrostat" ("Satellite – gyrostat"). One of the important constructional singularities is possibility of using of the stabilizing unit as a power plant [1]. The study of angular motions of such SV will be carried out on the basis of mechanical model of coaxial bodies of a variable mass.

It is necessary to mark that the angular motions dynamic of variable mass coaxial bodies is important not only for application space flight mechanics problems, but also represents large interest within the framework of basic researches of rigid bodies systems dynamic.

2 Problem Formulation

Let's consider motion of coaxial bodies (unbalanced gyrostat) of variable mass under an operation of the dissipative and boosting exterior moments depending on components of angular velocities. Let gyrostat will consist of dynamically symmetrical main body (coaxial body 2) of a constant mass and a rotor (coaxial body 1) of the variable mass remaining dynamically symmetrical during modification of a mass (fig. 2).

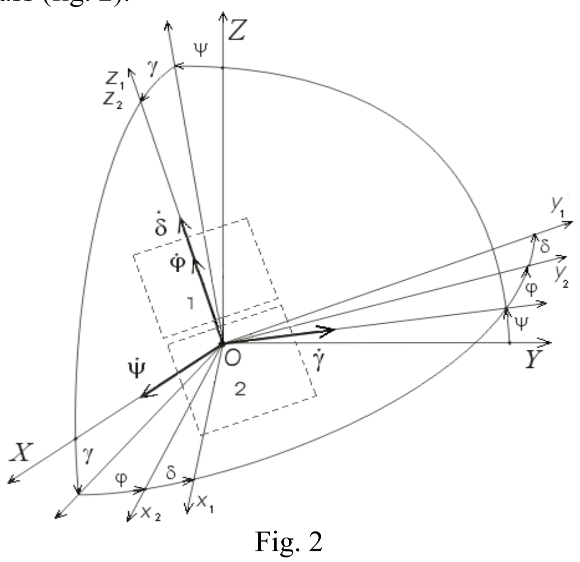

Fig. 2

The fixed point O coincides a primal geometrical position of center of masses of a system. We will use following coordinate systems: $OXYZ$ - fixed, $Ox_2y_2z_2$ - connected to the main body, $Ox_1y_1z_1$ - connected to the rotor. The rotor rotates along direct-axis Oz_1. The unbalance gyrostat has vacillating relative angular velocity of rotor rotation concerning the main body. It is possible in connection with interior moment operation between coaxial bodies. Let there is a moment of jet forces around of direct-axis Oz_1. It is possible to note following equations of system motions [2]:

$$\begin{aligned}
&A(t)\dot{p}+\left(C(t)-A(t)\right)qr+C_r(t)q\sigma=M_x^e\\
&A(t)\dot{q}-\left(C(t)-A(t)\right)pr-C_r(t)p\sigma=M_y^e\\
&C(t)\dot{r}+C_r(t)\dot{\sigma}=M_z^R+M_z^e\\
&C_r(t)\left(\dot{r}+\dot{\sigma}\right)=M_r+M_z^R+M_{z,r}^e
\end{aligned}\qquad(1)$$

Here $A(t)=A_m+A_r(t)-m\rho^2(t)$, $C(t)=C_m+C_r(t)$, A_m, $A_r(t)$, C_m, $C_r(t)$ are equatorial and longitudinal moments of inertia of the coaxial bodies – main body and rotor (indexes "m" and "r", accordingly); $M_i^e=M_{i,m}^e+M_{i,r}^e$ are exterior forces moments $\left(i=x,y,z\right)$; M_z^R is moment of jet forces; M_r is interior moment operating between coaxial bodies (a friction torque or a moment of the untwisting engine); $m\rho^2\left(t\right)$ is a term which describe "geometrical" transition of center of masses concerning a fixed point [1, 2] in connection with a modification of a mass-inertia structure of a system, $\rho\left(t\right)$ is a varying distance between fixed point and system masses center along direct-axis Oz_1; $m=m(t)=m_m+m_r(t)$ is a varying mass of the gyrostat; p, q, r are main body angular velocity projections on axis of system $Ox_2y_2z_2$; σ is relative angular velocity of the rotor. A point above a numeral means operation of derivation on time.

It is necessary to add kinematic relations (fig. 2):

$$\begin{aligned}
&\dot{\gamma}=p\sin\varphi+q\cos\varphi,\quad \dot{\psi}=\frac{1}{\cos\gamma}\left(p\cos\varphi-q\sin\varphi\right)\\
&\dot{\varphi}=r-\frac{\sin\gamma}{\cos\gamma}\left(p\cos\varphi-q\sin\varphi\right),\quad \dot{\delta}=\sigma
\end{aligned}\qquad(2)$$

Angel $\delta=\angle\left(Ox_1,Ox_2\right)$ is angle of relative rotation of the rotor.

Let's present the new variables corresponding to magnitude of vector of transversal angular velocity G and angle F between this vector and axis Oy_2:

$$p(t)=G(t)\sin F(t),\quad q=G(t)\cos F(t)\qquad(3)$$

Equation (1) will be noted in new variables as follows:

$$\dot{F} = -\frac{1}{A(t)}\left[\left(C(t)-A(t)\right)r + C_r(t)\sigma + f_F\left(G,F\right)\right]$$

$$\dot{G} = \frac{f_G\left(G,F\right)}{A(t)},\ \dot{r} = \frac{M_{z,c}^e - M_r}{C_m} \quad (4)$$

$$\dot{\sigma} = \frac{C(t)M_r}{C_r(t)C_m} + \frac{M_z^R + M_{z,r}^e}{C_r(t)} - \frac{M_{z,m}^e}{C_m}$$

In equations (4) the following disturbing functions describing exposures take place:

$$f_G\left(G,F\right) = \left(M_x^e \sin F + M_y^e \cos F\right)$$

$$f_F\left(G,F\right) = \frac{1}{G}\left(M_x^e \cos F - M_y^e \sin F\right)$$

We will consider a case when the module of a transversal angular velocity of main body is small as contrasted to relative longitudinal rotation rate of the rotor:

$$\varepsilon = \sqrt{p^2+q^2}\Big/\left|\sigma\right| << 1 \quad (5)$$

We will assume angles γ and ψ as small quantity $\left(\gamma = O(\varepsilon),\ \psi = O(\varepsilon)\right)$. Then the angle of a nutation θ (an angle between axes OZ and Oz_2) will be defined by the following approximated formula:

$$\theta^2 \cong \gamma^2 + \psi^2 \quad (6)$$

With the help of relations (3) kinematic equations (2) can be noted as (terms of the second order of a smallness are rejected):

$$\dot{\gamma} \cong G\cos\Phi(t),\ \dot{\psi} \cong G\sin\Phi(t),\ \dot{\varphi} \cong r,\ \dot{\delta} = \sigma \quad (7)$$
$$\Phi(t) = F(t) - \varphi(t)$$

Function $\Phi(t)$ is a phase of spatial oscillations.

Precession motion of the gyrostat with small nutation angles is obviously described by a phase space of variables $\{\gamma,\psi\}$. The phase trajectory in this space completely characterizes motion of the direct-axis Oz_2 (an apex of the direct-axis). Therefore our further researches will be connected to the analysis of this phase space and chances of behaviors of phase curves in this space.

3 Problem Solution

We will develop a special qualitative method of the analysis of a phase space. Main idea of the method is the evaluation of a phase trajectory curvature in the phase plane $\{\gamma,\psi\}$.

On the indicated plane the phase point will have following components of a velocity and acceleration:

$$V_\gamma = \dot{\gamma},\ V_\psi = \dot{\psi},\ W_\gamma = \ddot{\gamma},\ W_\psi = \ddot{\psi}$$

With the help of relations (7) the curvature of a phase trajectory (k) is evaluated as follows:

$$k^2 = \left(\ddot{\gamma}\dot{\psi} - \ddot{\psi}\dot{\gamma}\right)^2\Big/\left(\dot{\gamma}^2+\dot{\psi}^2\right)^3 = \dot{\Phi}^2\big/G^2 \quad (8)$$

If curvature magnitude will increase, there is a motion on a twisted spiral trajectory similar to a steady focal point (fig. 3, case "a") and if decreases - on untwisted. On twisted spiral trajectory motion condition can be noted as:

$$\left|k\right|\uparrow \Rightarrow k\dot{k} > 0 \ \Rightarrow\ \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2 > 0 \quad (9)$$

For the analysis of the condition realization it is necessary to study a disposition of zero of a following function:

$$P(t) = \dot{\Phi}\ddot{\Phi}G - \dot{G}\dot{\Phi}^2 \quad (10)$$

Function (10) we will name as function of phase trajectory evolutions.

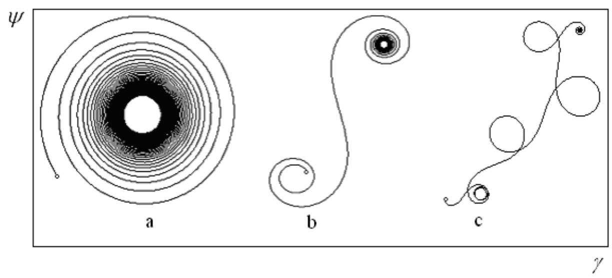

Fig. 3

Different qualitative cases of phase trajectory behaviors are possible depending on zero of function $P(t)$ (fig.3). In the first case (fig. 3, case "a") the function is positive and has no zero on a considered slice of time $t\in\left[0,T\right]$, thus the phase trajectory is spirally twisting. In the second case (fig. 3, case "b") is present one zero and there is one modification in a monotonicity of the trajectory curvature. In the third case (fig. 3, case "c") is present some zero and the trajectory has alternation of untwisted and twisted segments of motion; also there are points of self-intersection.

As an example we will consider motion of coaxial bodies of variable mass at an operation of the constant interior forces moment ($M_r = const$) and constant moment of jet forces ($M_z^R = const$).

The analysis of a phase space we will conduct with the help of developed method of curvature evaluation.

Moments of inertia we will count linear functions of time, and magnitudes $m\rho^2$ we will neglect. In a considered case equations (4) will be noted as follows:

$$\dot{F} = -\frac{\left(C_m + C_r - ct - A_m - A_r + at\right)r + (C_r - ct)\sigma}{A_m + A_r - at}$$

$$\dot{G} = 0, \qquad \dot{\sigma} = \frac{\left(C_m + C_r - ct\right)M_r}{(C_r - ct)C_m} + \frac{M_z^R}{(C_r - ct)} \quad (11)$$

$$\dot{r} = -M_r / C_m$$

Analytical solutions for angular velocity $r(t)$ and $\sigma\left(t\right)$ follow from equations (11):

$$r = r_0 - \frac{M_r}{C_m}t, \quad \sigma = \sigma_0 + s_1 t + s_2 \ln\left(1 - c_1 t\right)$$
$$s_1 = M_r / C_n, \quad s_2 = -\frac{1}{c}\left(M_r + M_z^R\right), \quad c_1 = c / C_r \tag{12}$$

Using solutions (12) it is possible to receive an expansion in a series of a right part of an equation for phase $F(t)$ (11):

$$\dot{F} = F_0 + \sum_{i=1}^{\infty} F_i t^i \tag{13}$$

Following values appear in expansion (13):

$$F_0 = -\frac{D_1}{A_m + A_r}$$
$$F_i = \frac{-1}{A_m + A_r}\left(\sum_{k=1}^{3} D_k a_1^{i-k+1} + \sum_{j=1}^{i-2} E_{j+1,j+2} a_1^{i-j-2}\right)$$
$$D_1 = Br_0 + C_r\sigma_0, \qquad B = C_r + C_m - A_r - A_m$$
$$D_2 = C_r\left(s_1 - s_2 c_1\right) - c\sigma_0 - br_0 - B\frac{M_r}{C_m}$$
$$D_3 = b\frac{M_r}{C_m} - c\left(s_1 - s_2 c_1\right) - C_r c_1^2 s_2 / 2$$
$$a_1 = \frac{a}{A_m + A_r}, \qquad b = c - a$$
$$E_{k,l} = cs_2 c_1^k / k - C_r s_2 c_1^l / l$$

The obtained expansion (13) uniformly converges on the interval $t \in \left[0, \frac{1}{a_1}\right) \bigcap \left[0, \frac{1}{c_1}\right)$. From kinematic equations it is possible to receive the solution for angle φ:

$$\varphi = \varphi_0 + r_0 t - \frac{M_r}{2C_m}t^2 \tag{14}$$

Relations for time derivative from phase of spatial oscillations (Φ) follow from (14):

$$\dot{\Phi} = \dot{F} - \dot{\varphi} = f_0 + \sum_{i=1}^{\infty} f_i t^i, \quad \ddot{\Phi} = f_1 + \sum_{i=2}^{\infty} i f_i t^{i-1}$$
$$f_0 = F_0 - r_0, \qquad f_1 = F_1 + M_r / C_m \tag{15}$$
$$f_j = F_j \quad (j = 2..\infty)$$

On the basis of expansions (15) in a linear approximation we will get a polynomial of the first degree for phase trajectory evolutions function (10):

$$P(t) = f_1\left(f_0 + f_1 t\right) = f_1^2 t + f_1 f_0 \tag{16}$$

There is a unique zero of a polynomial $t_1 = -f_0 / f_1$. For implementation of a condition (9) of twisted spiral motion it is necessary that the polynomial was steady ($t_1 < 0$) and positive at $t \geq 0$. It is possible only in case of following condition fulfilment:

$$f_1 f_0 > 0 \tag{16}$$

We will consider a case when following contingencies are correct:

$$r_0 = 0, \ \sigma_0 < 0, \ M_r > 0, \ M_z^R > 0$$

In this case value f_0 will be positive:

$$f_0 = -C_r\sigma_0 / \left(A_m + A_r\right) > 0$$

For the positiveness of value f_1 is necessary fulfilment of a condition:

$$f_1 = \frac{\sigma_0\left[c\left(A_m + A_r\right) - C_r a\right]}{\left(A_m + A_r\right)^2} - \frac{C_m M_z^R}{C_m\left(A_m + A_r\right)} > 0 \tag{17}$$

From conditions (17) the group of conditions follows:

$$\frac{c}{C_r} < \frac{a}{A_m + A_r}, \quad M_z^R < 0 \tag{18}$$

or two other similar groups of conditions follows:

$$\begin{aligned}
&1) \quad \frac{c}{C_r} < \frac{a}{A_m + A_r}, \quad M_z^R > 0 \\
&\qquad \frac{\sigma_0\left[c\left(A_m + A_r\right) - C_r a\right]}{\left(A_m + A_r\right)} > M_z^R \\
&2) \quad \frac{c}{C_r} > \frac{a}{A_m + A_r}, \quad M_z^R < 0 \\
&\qquad \left|M_z^R\right| < \left|\frac{\sigma_0\left[c\left(A_m + A_r\right) - C_r a\right]}{\left(A_m + A_r\right)}\right|
\end{aligned} \tag{19}$$

In figure (fig. 4) results of numerical calculations of phase trajectory are shown.

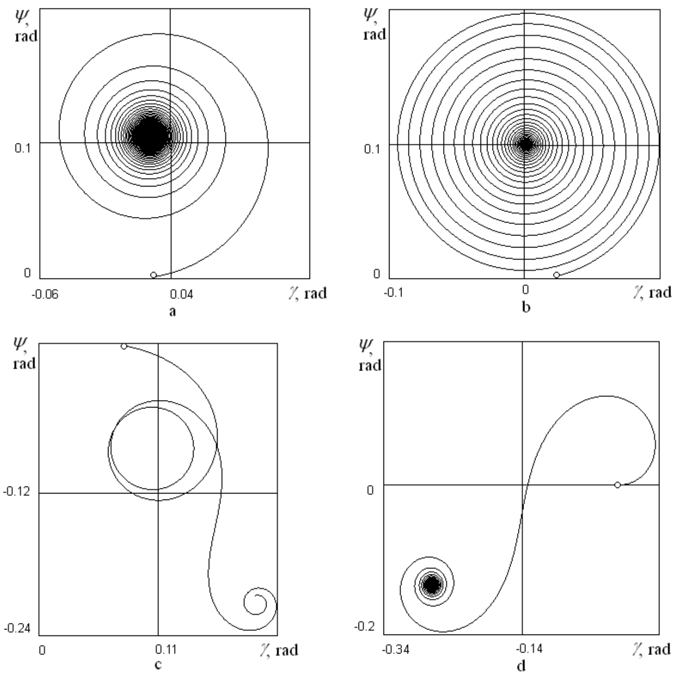

Fig. 4

Cases "a" and "b" (fig. 4) correspond to fulfilment of conditions (18) and (19), cases "c" and "d" (fig. 4) - to their nonfulfilment. Parameters of system and initial conditions for calculations are listed in table 1.

Table 1

Case (fig. 4)	a	b	c	d
A_m, kg·m^2	2.5	2.5	2	2.5
A_r, kg·m^2	2.5	2.5	2	2.5
C_m, kg·m^2	1	1	1	1
C_r, kg·m^2	1	1	1	1
a, kg·m^2/s	0.2	0.2	0.2	0.2
c, kg·m^2/s	0.01	0.01	0.1	0.01
M_r, N·m	1	1	3	1
M^R_z, N·m	-3	0.1	4	2.3
σ_0, rad/s	-10	-10	-1	-10
G_0, rad/s	0.2	0.2	0.1	0.1

Conditions (18) (or (19)) can be used for synthesis of space vehicles parameters. For naturally increase of vehicle direct-axis positioning exactitude it is necessary to realize precession motion with a decreasing nutation angle. This motion will be realized at fulfilment of conditions (18) or (19).

For realization of more exact researches, certainly, it is necessary to take into account of higher degrees polynomials $P(t)$ (10). However it is shown that already linear analysis (with the help of a polynomial of the first degree (16)) allow quite adequately describe of angular motions evolutions of variable mass coaxial bodies.

Examined above case of investigation does not take into account many important aspects of variable mass coaxial bodies motion. However the introduced example has well illustrate the approach to research of non-autonomous dynamic systems of indicated type.

4 Conclusion

We also present results of two hypothetical cases of motion of coaxial bodies of variable mass. Not addressing to any formulas we will show in figures a situation when the polynomial (10) has ten zeros (fig. 5) and numberless quantity of zeros (there is a case when the function contains any harmonic term) (fig. 6). Such interesting cases of motion can quite meet at research of variable mass coaxial bodies and unbalanced gyrostats motion dynamic.

In the article research of a phase space of a non-autonomous dynamic system of motion of coaxial bodies and unbalanced gyrostats of variable mass carried out. New method of investigation behavior of non-autonomous dynamical system is developed. Results of researches have important application value in space flight mechanics problems.

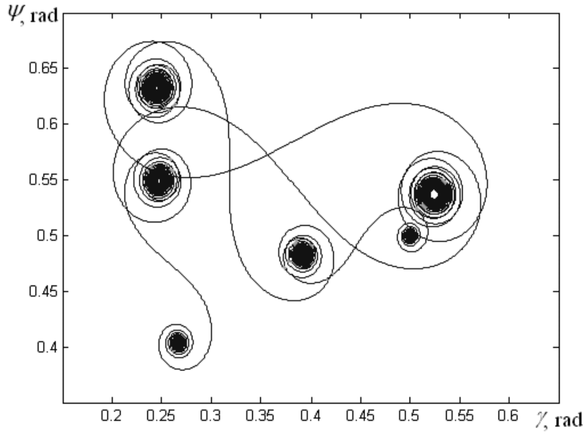

Fig.5

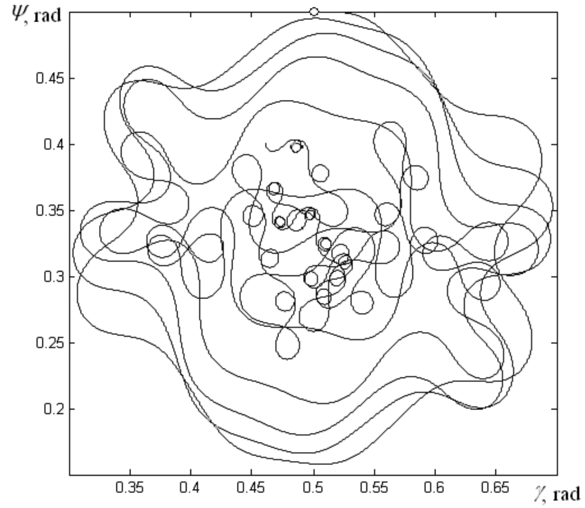

Fig.6

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