On the Saint-Venant Problem in a Doubly Connected Domain

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Abstract

The Saint-Venant elliptic over-determined problem in a doubly connected domain is considered. The use of Weinberger functional leads us to conclude that the domain is in effect an N-ball. The tool of this investigation are best maximum principles and Rellich's identity.

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1 Introduction

An alternative technique for determining the configuration of domains in the class of elliptic problems when an over-specification on the boundary of the domain is prescribed is to re-formulate the problem in an equivalent integral form, where the most important ingredients in partial differential equations, maximum principles, are not used. Instead, the integral dual is then used in order to deduce that the domain in consideration is an N-ball. For an account on these topics we refer the reader to [1,2,3,5,6,8,10,12].

In their famous paper [6], L. E. Payne and P. W. Schaeffer investigated this new approach without using maximum principles. Among a variety of class of over-determined problems considered in [6] involving Green's functions as well as classical boundary value problems, they showed the following two theorems taking into consideration the following Saint-Venant problem.

Let u be a classical solution of the following Saint-Venant problem

$$\Delta u = -1 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

$$\frac{\partial u}{\partial n} = -c, \qquad c = \text{constant} \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a simply connected regular, bounded domain of $\mathbb{R}^N$, $N \geq 2$ and $\frac{\partial u}{\partial n}$ is the exterior normal derivative of u on the boundary $\partial\Omega$ which is assumed to be sufficiently regular. We cite the two statements investigated in [6] without proof.

Theorem 1.1 *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then the following statements are equivalent*

$$\begin{array}{ll} (i)\ u \text{ satisfies} & (1.1),(1.2), \\ (ii)\ \displaystyle\int_\Omega h\, d\mathbf{x} = c \int_{\partial\Omega} h\, d\mathbf{s} & \end{array} \tag{1.3}$$

for all functions h harmonic in Ω.

Theorem 1.2 *If (1.3) holds, then Ω is an N-ball.*

The aim of this note is to extend this result for a doubly connected domain. In fact, we assume that u is a classical solution of the following Saint-Venant problem and Ω_0 and Ω_1 are two simply connected C^2 domains such that $\Omega := \Omega_0 \setminus \bar{\Omega}_1$. Here and in the following $n(x)$ denotes always the unit inner normal with respect to Ω. We consider u a C^2 solution of

$$\Delta u = -1 \quad \text{in } \Omega, \tag{1.4}$$

$$u = 0 \quad \text{on } \partial\Omega_0, \tag{1.5}$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega_1, \tag{1.6}$$

$$\frac{\partial u}{\partial n} = -c^2 \quad \text{on } \partial\Omega_0, \tag{1.7}$$

$$u = b^2 \quad \text{on } \partial\Omega_1. \tag{1.8}$$

The next theorem establishes an equivalence between two assertions: the original problem and the dual problem expressed as integral identity involving harmonic functions. This auxiliary result is formulated as follows

Theorem 1.3 *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$. Then the following statements are equivalent*

$$\begin{array}{ll} (i)\ u \text{ satisfies} & (1.4)-(1.8), \\ (ii)\ \displaystyle\int_\Omega h\, d\mathbf{x} = c^2 \int_{\partial\Omega_0} h\, d\mathbf{s} - b^2 \int_{\partial\Omega_1} \frac{\partial h}{\partial n}\, d\mathbf{s} & \end{array} \tag{1.9}$$

for all functions h harmonic in Ω, where

$$c^2 := \frac{|\Omega|}{|\partial\Omega_0|}$$

and

$$b^2 := \frac{|\Omega|}{|\partial\Omega_1|}.$$

For the proof of Theorem 1.3, we start with the second part by showing that (ii) implies (i). Then for any harmonic function h, we have

$$\int_{\partial\Omega_0} h\{\frac{\partial u}{\partial n} + \frac{|\Omega|}{|\partial\Omega_0|}\}\, \mathrm{d\ s} + \int_{\partial\Omega_1} \frac{\partial h}{\partial n}\{\frac{|\Omega|}{|\partial\Omega_1|} - u\}\, \mathrm{d\ s} = 0. \tag{1.10}$$

It is worth noting that the differential integral (1.10) is valid for any function h harmonic of class C^2 on $\bar{\Omega}$, and therefore h must be a solution of the following elliptic problem

$$\begin{aligned} \Delta h &= 0 \quad \text{in } \Omega, \\ h &= \frac{\partial u}{\partial n} + \frac{|\Omega|}{|\partial\Omega_0|} \quad \text{on } \partial\Omega_0 \\ \frac{\partial h}{\partial n} &= \frac{|\Omega|}{|\partial\Omega_1|} - u \quad \text{on } \partial\Omega_1. \end{aligned} \tag{1.11}$$

Making substitution of the value h and $\frac{\partial h}{\partial n}$ appearing in (1.11) into (1.10) conduct us to the following boundary conditions

$$\frac{\partial u}{\partial n} = -\frac{|\Omega|}{|\partial\Omega_0|} = -c^2 \quad \text{on } \partial\Omega_0, \tag{1.12}$$

and

$$u = \frac{|\Omega|}{|\partial\Omega_1|} = b^2 \quad \text{on } \partial\Omega_1. \tag{1.13}$$

Thus in view of (1.12) – (1.13), u must satisfy the above elliptic over-determined problem (1.4) – (1.8).

Now for the reverse implication, the first part, we suppose that (i) holds. Applying the second classical formula of Green and taking in account the boundary conditions $(1.5)-(1.8)$ we get the desired integral identity (ii) which complete the proof.

The following theorem states that the only configuration of the doubly connected domain in consideration are N-concentric spheres. This result is very well known in view of other approach see [5,12] where the used technics are moving plane method [9] and maximum principles of E. Hopf [3,4,7,10] jointly with P-functions [11]. Now we are able to show the main result of this paper formulated in the theorem under below, using duality theorems which extends the result (Theorem 1.2) investigated in [6].

Theorem 1.4 *If* (1.9) *holds with* $c^2 := \frac{|\Omega|}{|\partial\Omega_0|}$ *and* $b^2 := \frac{|\Omega|}{|\partial\Omega_1|}$ *then* Ω *is a concentric sphere annulus provided that*

$$0 < u < b^2.$$

In order to prove Theorem 1.4, we want to get some integral identity which leads us able to deduce that the only possibility for the combination Φ (that will be defined later) is to be constant. This is due to the application of several Green'theorem to the Weinberger functional $r\frac{\partial u}{\partial r}$. This investigation relies heavily on the following observation already known in [11] for a single elliptic equation (Saint-Venant problem) in a simply connected domain. The tool of this investigation are Best maximum principles and Rellich'identity.

Let us for completeness sake re-write it as

$$\begin{aligned} \Delta\,(r\frac{\partial u}{\partial r}) &= r\frac{\partial \Delta\, u}{\partial r} + 2\Delta\, u \qquad (1.14)\\ &= -2, \end{aligned}$$

Now multiplying (1.14) by $-u$ and using Green's theorem, we get

$$\int_\Omega (-u\Delta\,(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\Delta\, u)\,\mathrm{d}\mathbf{x} \quad = \quad \int_\Omega (2u - r\frac{\partial u}{\partial r})\,\mathrm{d}\mathbf{x} \qquad (1.15)$$

Next, we express the last term in (1.15) from the right in light of classical formula of Green, we obtain

$$\begin{aligned} \int_\Omega r\frac{\partial u}{\partial r}\,\mathrm{d}\mathbf{x} &= \int_\Omega \nabla\,(\frac{r^2}{2})\nabla\,(u)\,\mathrm{d}\mathbf{x} \qquad (1.16)\\ &= -N\int_\Omega u\,\mathrm{d}\mathbf{x}, \end{aligned}$$

from one hand. From an other hand, in view of second classical formula of Green, the right-hand side of (1.15) takes the form

$$\begin{aligned} &\int_\Omega (-u\Delta\,(r\frac{\partial u}{\partial r}) + r\frac{\partial\, u}{\partial r}\Delta\, u)\,\mathrm{d}\mathbf{x} = \qquad (1.17)\\ &\int_{\partial\Omega} (-u\frac{\partial}{\partial n}(r\frac{\partial u}{\partial r}) + r\frac{\partial u}{\partial r}\frac{\partial u}{\partial n})\,\mathrm{d}\mathbf{s} \\ &= \int_{\partial\Omega_0} c^4 r\frac{\partial r}{\partial n}\,\mathrm{d}\mathbf{s} - \{\int_{\partial\Omega_1} b^2\frac{\partial}{\partial n}(r\frac{\partial u}{\partial r})\,\mathrm{d}\mathbf{s}\} \\ &= \int_{\partial\Omega_0} c^4 r\frac{\partial r}{\partial n}\,\mathrm{d}\mathbf{s} - \{\int_{\partial\Omega_1} b^2[\frac{\partial}{\partial n}(r\frac{\partial r}{\partial n})\frac{\partial u}{\partial n} \\ &+ r\frac{\partial r}{\partial n}(\frac{\partial^2 u}{\partial n^2})]\,\mathrm{d}\mathbf{s}\}. \end{aligned}$$

Now we need to compute explicitly the second directional derivative of $\frac{r^2}{2}$

$$\begin{aligned} \frac{\partial}{\partial n}(r\frac{\partial r}{\partial n}) &= \frac{\partial^2}{\partial n^2}(\frac{r^2}{2}) \qquad (1.18)\\ &= n_i n_j(\frac{r^2}{2})_{,ij} \\ &= n_i n_i = 1, \end{aligned}$$

where n_i denotes a unit vector normal to the boundary.

Inserting (1.18) into (1.17) we obtain

$$\begin{aligned} \int_\Omega (-u\Delta\,(r\frac{\partial u}{\partial r}) + r\frac{\partial\, u}{\partial r}\Delta\, u)\,\mathrm{d}\mathbf{x} &= \\ \int_{\partial\Omega_0} c^4 r\frac{\partial r}{\partial n}\,\mathrm{d}\mathbf{s} - & \qquad (1.19)\\ \{\int_{\partial\Omega_1} b^2[\frac{\partial}{\partial n}(r\frac{\partial r}{\partial n})\frac{\partial u}{\partial n} + r\frac{\partial r}{\partial n}(\frac{\partial^2 u}{\partial n^2})]\,\mathrm{d}\mathbf{s}\}. \end{aligned}$$

Therefore combining together (1.15) − (1.16), we obtain

$$(N+2)\int_{\Omega} u\,\mathrm{d}\mathbf{x} = Nc^4|\Omega_0| + Nb^2|\Omega_1|. \quad (1.20)$$

From which we deduce that

$$Nc^4|\Omega_0| + Nb^2|\Omega_1| - (N+2)\int_{\Omega} u\,\mathrm{d}\mathbf{x} = 0. \quad (1.21)$$

Now with a straightforward calculation one sees that the following combination

$$\Phi := |\nabla u|^2 + \frac{2}{N}u, \quad (1.22)$$

where $F(u) := \int_0^u f(s)\,\mathrm{ds}$, satisfies $\Delta\Phi \geq 0$, and therefore takes its maximum value on the boundary $\partial\Omega$ unless Φ is constant. The next aim is to show that Φ is harmonic in Ω. Since Φ is sub-harmonic in Ω, it is sufficient to prove that $\Delta\Phi \leq 0$. To do so, we use a new argument, namely dual integrals investigated in [6]. Indeed, let u be a solution of $(1.4)-(1.8)$ and use classical formula of Green, one get

$$\begin{aligned} N\int_{\Omega} u\Delta\Phi\,\mathrm{d}\mathbf{x} &= -N\int_{\Omega}\Phi\,\mathrm{d}\mathbf{x} + Nc^6|\Gamma_0| \quad (1.23)\\ &= -(N+2)\int_{\Omega} u\,\mathrm{d}\mathbf{x} + Nc^6|\Gamma_0| \\ &= -(N+2)\int_{\Omega} u\,\mathrm{d}\mathbf{x} + Nc^4|\Omega_0|. \end{aligned}$$

Combining (1.20) and (1.24), we are then led to

$$\int_{\Omega} u\Delta\Phi\,\mathrm{d}\mathbf{x} = -Nb^2|\Omega_1| < 0. \quad (1.24)$$

Since u is positive we get $\Delta\Phi < 0$ in Ω. So Φ is harmonic and following the final part of Weinberger we conclude that Ω is an N-ball and u is radially symmetric.

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