# Exact 3-D Solution for System with Rectangular Fins, Part 2

ANDRIS BUIKIS, MARGARITA BUIKE
Institute of Mathematics and Computer Science
University of Latvia
Raina bulv. 29, Riga, LV1459
LATVIA
http://www. lza.lv/scientists/buikis.htm

*Abstract:* - In this paper we construct exact analytical three-dimensional solution for the distribution of the temperature field in the wall with arrays of rectangular fins. We assume that the heat transfer process in the wall and the fin is stationary. This exact solution is obtained by the Green function method. It is obtained in the form of the system of 2nd kind Fredholm integral equations with the numbers of equations equal to the number of the fins.

*Key-Words:* - Steady-state, Three-dimensional, Heat exchange, Rectangular fins, Array, Non-homogeneous environment, Exact analytical solution, Fredholm integral equations.

## 1 Introduction

In part 1 of this paper we have considered mathematical three-dimensional formulation of steady-state problem for one element of system with rectangular fin (as usually in the literature, e.g. [1]-[6]). In this paper we concentrate our attention on the whole system, assembled into array of fins. Such statement essentially generalizes the problems considered earlier in literature, e.g., in paper [7] and doctoral thesis [8]. In this part of we obtain exact analytical solution by the Green function method. The solution has the form of the system of 2nd kind Fredholm integral equations. In other words, we have reduced the original problem for the Laplace equation in domain with extended surfaces (wall with the array of fins) to the system of Fredholm integral equations. The order of this system is equal the number of the fins.

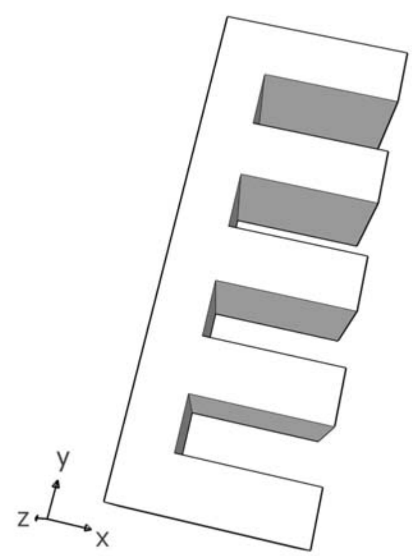
## 2 Mathematical Formulation of 3-D Problem

In this part of our paper we are modeling the whole system with $N(i=\overline{1,N})$ fins, as it is depicted in figure on this page. We will use the same as in part 1 all dimensionless arguments and parameters, adding to them total dimensional length $\overline{Y} = Y\left(B+R\right)$. Here the dimensionless length $Y = 2\left[\left(N-1\right)+\dfrac{B}{B+R}\right]$, where $N-$ the total number of the fins:

$$x=\frac{x'}{B+R},\, y=\frac{y'}{B+R},\, z=\frac{z'}{B+R},\ \delta=\frac{\Delta}{B+R},$$

$$l=\frac{L}{B+R},\ b=\frac{B}{B+R},\ \beta_0^0=\frac{h_0(B+R)}{k_0},$$

$$\beta_0=\frac{h(B+R)}{k_0},\ \beta=\frac{h(B+R)}{k},\ w=\frac{W}{B+R}.$$




The dimensionless temperatures are introduced in the same form as earlier, but in the contradiction to the part 1 in different way will be introduced the averaged environmental temperatures:

$$\bar{V}(x,y,z)=\frac{\tilde{V}(x,y,z)-T_a}{T_b-T_a},$$
$$\bar{V}_0(x,y,z)=\frac{\tilde{V}_0(x,y,z)-T_a}{T_b-T_a},$$
$$\Theta(x,y,z)=\frac{\tilde{\Theta}(x,y,z)-T_a}{T_b-T_a},$$

$$\Theta_0(y,z)=\frac{\tilde{\Theta}_0(y,z)-T_a}{T_b-T_a}.$$

Here $\tilde{\Theta}_0(y,z)$ is the dimensional surrounding (environment) temperature on the left (hot) side (the heat source side) of the wall, $\tilde{\Theta}(x,y,z)$ - the surrounding temperature on the right side of the wall and the fins (on the heat sink side). Further, $\tilde{V}(x,y,z)$ ($\tilde{V}_0(x,y,z)$) are the dimensional temperature in the fin (wall). The wall (base) now occupied bounded serrated domain $\left\{x\in[0,\delta],y\in[0,Y],z\in[0,w]\right\}$. The $N$ rectangular fins occupy the domains ($i=\overline{1,N}$):
$\left\{x\in[\delta,\delta+l],y\in[y_i^-,y_i^+],z\in[0,w]\right\}$.

Here: $y_i^-=2(b+r)(i-1),y_i^+=y_i^-+2b,i=\overline{1,N}.$

Finally, here $T_a(T_b)$ are integral averaged environment temperatures over edges, orthogonal to $x-$direction:

$$T_a=\left(Yw\right)^{-1}\left[\sum_{i=1}^{N-1}\int_{y_i^+}^{y_{i+1}^-}dy\int_0^w\tilde{\Theta}(\delta,y,z)dz\right.$$
$$\left.+\sum_{\bar{i}=1}^{N}\int_0^w dz\int_{y_i^-}^{y_i^+}\tilde{\Theta}(\delta+l,y,z)dy\right],$$
$$T_b=\left(Yw\right)^{-1}\int_0^Y dy\int_0^w\tilde{\Theta}_0(y,z)dz.$$

## 3 Reduction of 3-D Model to 2-D Problem and its Full Mathematical Formulation

We describe the dimensionless temperature field by function $\bar{V}_0(x,y,z)$ $\left(\bar{V}_i(x,y,z),i=\overline{1,N}\right)$ in the wall (fins). They fulfill the equations:

$$\frac{\partial^2\bar{V}_0}{\partial x^2}+\frac{\partial^2\bar{V}_0}{\partial y^2}+\frac{\partial^2\bar{V}_0}{\partial z^2}=0,$$
$$\frac{\partial^2\bar{V}_i}{\partial x^2}+\frac{\partial^2\bar{V}_i}{\partial y^2}+\frac{\partial^2\bar{V}_i}{\partial z^2}=0,\ i=\overline{1,N}.$$

As the model in this part of our paper we consider again the three dimensional statement with given (prescribed) heat fluxes from the flank surfaces (edges):

$$\left.\frac{\partial\bar{V}_0}{\partial z}\right|_{z=0}=Q_{0,2}(x,y),\left.\frac{\partial\bar{V}_0}{\partial z}\right|_{z=w}=Q_{0,3}(x,y),$$
$$\left.\frac{\partial\bar{V}_i}{\partial z}\right|_{z=0}=Q_{2,i}(x,y),\left.\frac{\partial\bar{V}_i}{\partial z}\right|_{z=w}=Q_{3,i}(x,y). \qquad (1)$$

Such type of boundary conditions (BC) allows us in similar to part 1 way make the exact reducing of this three-dimensional problem for Laplace equations to two-dimensional problem for Poisson equations. This can be done by conservative averaging method [12], [13]. For this goal we introduce following integral average values:

$$V_0(x,y)=w^{-1}\int_0^w\bar{V}_0(x,y,z)dz,$$
$$\vartheta_0(y)=w^{-1}\int_0^w\Theta_0(y,z)dz,$$
$$V_i(x,y)=w^{-1}\int_0^w\bar{V}_i(x,y,z)dz, \qquad (2)$$
$$\vartheta(x,y)=w^{-1}\int_0^w\Theta(x,y,z)dz.$$

It remains to realize the integration of main equation by usage of the both BC (corresponding one pair) and we obtain, details see in [13]:

$$\frac{\partial^2V_0}{\partial x^2}+\frac{\partial^2V_0}{\partial y^2}+Q_0(x,y)=0,$$
$$\frac{\partial^2V_i}{\partial x^2}+\frac{\partial^2V_i}{\partial y^2}+Q_i(x,y)=0,\ i=\overline{1,N}. \qquad (3)$$

Here

$$Q_0(x,y)=\frac{1}{w}\left(Q_{0,3}(x,y)-Q_{0,2}(x,y)\right),$$
$$Q_i(x,y)=\frac{1}{w}\left(Q_{3,i}(x,y)-Q_{2,i}(x,y)\right).$$

Again we must add to main partial differential equations (3) needed BC as follow:

$$\left\{\frac{\partial V_0}{\partial x}+\beta_0^0\left[\vartheta_0(y)-V_0\right]\right\}\Bigg|_{x=0}=0,\, y\in(0,Y), \quad (4)$$

$$\left\{\frac{\partial V_0}{\partial x}+\beta_0\left[V_0-\vartheta(x,y)\right]\right\}\Bigg|_{x=\delta}=0, \quad (5)$$

$$y\in(y_i^+,y_{i+1}^-), i=\overline{1,N-1},$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0}=Q_{0,0}(x), \quad (6)$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=Y}=Q_{0,1}(x). \quad (7)$$

We assume the same as in part 1 conjugation conditions on the surface between the wall and the fins (for $y\in[y_i^-,y_i^+], i=\overline{1,N}$). They describe the ideal thermal contact between the wall and the fins:

$$V_0\big|_{x=\delta-0}=V_i\big|_{x=\delta+0}, \quad (8)$$

$$\beta\left.\frac{\partial V_0}{\partial x}\right|_{x=\delta-0}=\beta_0\left.\frac{\partial V_i}{\partial x}\right|_{x=\delta+0}. \quad (9)$$

Remark: we consider here the situation where all fins have the same thermal properties, but physical properties of the fins could be different from the properties of the wall. In most general case (when different fins have different thermal properties) instead of the same dimensionless heat exchange coefficient $\beta$ for all fins we would have had different $\beta=\beta_i$. We underline that the proposed here method works in this general case without principal changes.

We have following BC for the fins:

$$\left\{\frac{\partial V_i}{\partial x}+\beta\left[V_i-\vartheta(x,y)\right]\right\}\Bigg|_{x=\delta+l}=0, \quad (10)$$

$$y\in[y_i^-,y_i^+], i=\overline{1,N},$$

$$\left\{\frac{\partial V_i}{\partial y}+\beta\left[V_i-\vartheta(x,y)\right]\right\}\Bigg|_{y=y_i^+}=0,$$

$$\left\{\frac{\partial V_i}{\partial y}-\beta\left[V_i-\vartheta(x,y)\right]\right\}\Bigg|_{y=y_i^-}=0, \quad (11)$$

$$x\in[\delta,\delta+l], i=\overline{1,N}.$$

As in the part 1 we assume again that all conditions which ensure existence and uniqueness of classic solution of the problem (3)-(11), e.g. continuity of environment temperatures, consistency conditions on the sides of edges etc. are fulfilled.

# 4 Exact Solution of 2-D Problem

The combination of the equations (8), (9) and (5) allow us rewrite them as following BC:

$$\left(\frac{\partial V_0}{\partial x}+\beta_0 V_0\right)\Bigg|_{x=\delta-0}=\beta_0 F_0(\delta,y), \quad (12)$$

where

$$F_0(x,y)=\begin{cases}F_{0,i}(x,y), y_i^-\le y\le y_i^+,\\ i=\overline{1,N};\\ \vartheta(x,y)=\vartheta(\delta,y), y_i^+<y<y_{i+1}^-,\\ i=\overline{1,N-1};\end{cases} \quad (13)$$

$$F_{0,i}(x,y)=\left(\frac{1}{\beta}\frac{\partial V_i}{\partial x}+V_i\right);\quad x\in[\delta,\delta+l].$$

By the given (known) function $F_0(x,y)$ we can represent the solution for the wall in form:

$$V_0(x,y)=\Psi_0(x,y)+ \beta_0\sum_{l=1}^{N}\int_{y_l^-}^{y_l^+}F_{0,l}(\delta,\upsilon)G_0(x,y,\delta,\upsilon)d\upsilon, \quad (14)$$

In formula (14) we have denoted:

$$\Psi_0(x,y)=\int_0^\delta Q_{0,1}(\zeta)G_0(x,y,\zeta,Y)d\zeta -\int_0^\delta Q_{0,0}(\zeta)G_0(x,y,\zeta,0)d\zeta - \beta_0^0\int_0^Y\vartheta_0(\upsilon)G_0(x,y,0,\upsilon)d\upsilon+ \beta_0\sum_{l=1}^{N-1}\int_{y_l^+}^{y_{l+1}^-}\vartheta(\delta,\upsilon)G_0(x,y,\delta,\upsilon)d\upsilon+ \int_0^\delta d\zeta\int_0^Y Q_0(\zeta,\upsilon)G_0(x,y,\zeta,\upsilon)d\upsilon.$$

The Green function in (14) has similar with part 1 form, see, e.g. [11]. Difference is only in the total length of the wall in the $y-$direction:

$$G_0(x,y,\zeta,\upsilon)=\sum_{m,n=1}^{\infty}\frac{G_{0,m}^{(x)}(x,\zeta)\cdot G_{0,n}^{(y)}(y,\upsilon)}{\left[\left(\frac{n\pi}{Y}\right)^2+\mu_m^2\right]}, \quad (15)$$

$$G_{0,m}^{(x)}(x,\zeta)=\frac{\varphi_m(x)\varphi_m(\zeta)}{\|\varphi_m\|^2},$$

$$G_{0,n}^{(y)}(y,\upsilon)=\frac{2}{Y}\times \tag{16}$$

$$\left\{\cos\left[\frac{n\pi}{Y}(y+\upsilon)\right]+\cos\left[\frac{n\pi}{Y}(y-\upsilon)\right]\right\}.$$

The eigenfunctions have following expression for the first one-dimensional Green function:

$$\varphi_m(x)=\cos(\mu_m x)+\frac{\beta_0^0}{\mu_m}\sin(\mu_m x),\|\varphi_m\|^2=$$

$$=\frac{\beta_0}{2\mu_m^2}\frac{\mu_m^2+\left(\beta_0^0\right)^2}{\mu_m^2+\left(\beta_0\right)^2}+\frac{\beta_0^0}{2\mu_m^2}+\frac{\delta}{2}\left(1+\frac{\left(\beta_0^0\right)^2}{\mu_m^2}\right). \tag{17}$$

Here $\mu_m$ are the roots of the transcendental equation:

$$tg(\mu_m\delta)=\frac{\mu_m\left(\beta_0+\beta_0^0\right)}{\mu_m^2-\beta_0\beta_0^0}. \tag{18}$$

The representation (14) for the solution in the wall is under exploitable as solution because of unknown functions $F_{0,i}(x,y)$, i.e. temperature fields $V_i(x,y)$ in the fins. In the same way as for (12) we can rewrite the conjugations conditions in the form of BC on the left side of each rectangular fin:

$$\left.\left(\frac{\partial V_i}{\partial x}-\beta V_i\right)\right|_{x=\delta+0}=\beta F(\delta,y), \tag{19}$$

$$y\in[y_i^-,y_i^+],i=\overline{1,N}.$$

Here the right hand side function of BC (19) has the form:

$$F(x,y)=\left(\frac{1}{\beta_0}\frac{\partial V_0}{\partial x}-V_0\right), \tag{20}$$

$$x\in[0,\delta],y\in[y_i^-,y_i^+],i=\overline{1,N}.$$

Then, similar as in formula (14) for the wall we can represent solution for the $i-th$ fin in following form:

$$V_i(x,y)=\Psi_i(x,y)-$$

$$\beta\int_{y_i^-}^{y_i^+}F(\delta,\eta)G(x,y-y_i^-,\delta,\eta-y_i^-)d\eta \tag{21}$$

The known function $\Psi_i(x,y)$ has the form (on the three boundaries where the traditional boundary conditions are given):

$$\Psi_i(x,y)=$$

$$-\beta\int_{\delta}^{\delta+l}\vartheta(\xi,y_i^-)G(x,y-y_i^-,\xi,0)d\xi+ \tag{22}$$

$$+\beta\int_{\delta}^{\delta+l}\vartheta(\xi,y_i^+)G(x,y-y_i^-,\xi,2b)d\xi+$$

$$\beta\int_{y_i^-}^{y_i^+}\vartheta(\delta+l,\eta)G(x,y-y_i^-,\delta+l,\eta-y_i^-)d\eta$$

$$+\int_{\delta}^{\delta+l}d\xi\int_{y_i^-}^{y_i^+}Q_i(\xi,\eta)G(x,y-y_i^-,\xi,\eta-y_i^-)d\eta.$$

Further, see [11]:

$$G(x,y,\xi,\eta)=$$

$$\sum_{j,k=1}^{\infty}\frac{\phi_j(x)\phi_j(\xi)\psi_k(y)\psi_k(\eta)}{\|\phi_j\|^2\|\psi_k\|^2\left(\lambda_j^2+\kappa_k^2\right)}, \tag{23}$$

$$\phi_j(x)=\cos\left[\lambda_j(x-\delta)\right]+$$

$$\frac{\beta}{\lambda_j}\sin\left[\lambda_j(x-\delta)\right],$$

$$\|\phi_j\|^2=\frac{l}{2}+\frac{\beta}{\lambda_j^2}\left(1+\frac{\beta l}{2}\right),$$

$$\psi_k(y)=\cos\left(\kappa_k y\right)+\frac{\beta}{\kappa_k}\sin\left(\kappa_k y\right),$$

$$\|\psi_k\|^2=b+\frac{\beta}{\kappa_k^2}\left(1+\beta b\right).$$

Here $\lambda_j(\kappa_k)$ are the roots of the transcendental equations:

$$\tan(\lambda_j l)=\frac{2\lambda_j\beta}{\lambda_j^2-\beta^2},\tan(2b\kappa_k)=\frac{2\kappa_k\beta}{\kappa_k^2-\beta^2}. \tag{24}$$

Using notations (13), (22) and representation (21) we obtain easy the following equation:

$$F_{0,i}(x,y)=\tilde{\Psi}_i(x,y)-$$

$$\int_{y_i^-}^{y_i^+}F(\delta,\eta)\Gamma(x,y-y_i^-,\delta,\eta-y_i^-)d\eta, \tag{25}$$

where

$$\Gamma(x,y,\xi,\eta)=\left(\frac{\partial}{\partial x}+\beta\right)G(x,y,\xi,\eta)$$

$$\tilde{\Psi}_i(x,y)=\frac{1}{\beta}\left(\frac{\partial}{\partial x}+\beta\right)\Psi_i(x,y).$$

From (14) we obtain immediately similar representation for the $F(x,y)$:

$$F(x,y)=\tilde{\Psi}_0(x,y)-\sum_{l=1}^{N}\int_{y_l^-}^{y_l^+}F_{0,l}(\delta,\upsilon)\Gamma_0(x,y,\delta,\upsilon)d\upsilon.$$

Here we have introduced following notations:

$$\Gamma_0(x,y,\zeta,\upsilon)=\left(\beta_0-\frac{\partial}{\partial x}\right)G_0(x,y,\zeta,\upsilon),\quad \tilde{\Psi}_0(x,y)=\frac{1}{\beta_0}\left(\frac{\partial}{\partial x}-\beta_0\right)\Psi_0(x,y). \tag{26}$$

On the lines between the wall and fins the function $F(x,y)$ takes the form:

$$F(\delta,y)=\tilde{\Psi}_0(\delta,y)-\sum_{l=1}^{N}\int_{y_l^-}^{y_l^+}F_{0,l}(\delta,\upsilon)\Gamma_0(\delta,y,\delta,\upsilon)d\upsilon. \tag{27}$$

Now we substitute the representation (27) in the right hand side of formula (25):

$$F_{0,i}(\delta,y)=\tilde{\Psi}_i(\delta,y)-\int_{y_i^-}^{y_i^+}\tilde{\Psi}_0(\delta,\eta)\Gamma(\delta,y-y_i^-,\delta,\eta-y_i^-)d\eta+\sum_{l=1}^{N}\int_{y_l^-}^{y_l^+}F_{0,l}(\delta,\upsilon)d\upsilon\times\int_{y_i^-}^{y_i^+}\Gamma_0(\delta,\eta,\delta,\upsilon)\Gamma(\delta,y-y_i^-,\delta,\eta-y_i^-)d\eta.$$

Finally we obtain following system of the second kind Fredholm integral equations regarding the functions $F_{0,i}(\delta,y), i=\overline{1,N}$:

$$F_{0,i}(\delta,y)=-\Phi_i(y)+\sum_{l=1}^{N}\int_{y_l^-}^{y_l^+}K(y,\upsilon)F_{0,l}(\delta,\upsilon)d\upsilon. \tag{28}$$

Here we have introduced following shorter denominations:

$$\Phi_i(y)=\tilde{\Psi}_i(\delta,y)-\int_{y_i^-}^{y_i^+}\tilde{\Psi}_0(\delta,\eta)\Gamma(\delta,y-y_i^-,\delta,\eta-y_i^-)d\eta,$$

$$K(y,\upsilon)=\int_{y_i^-}^{y_i^+}\Gamma_0(\delta,\eta,\delta,\upsilon)\Gamma(\delta,y-y_i^-,\delta,\eta-y_i^-)d\eta.$$

After is solved system of integral equations (28) with continuous kernels from the representation (14) we can obtain immediately the temperature field in the wall and the function $F(\delta,y)$. In its turn the representation (21) gives the temperature fields in all fins.

This problem (with non-homogeneous environment temperatures) and its exact solution allow conjugating temperature field with hydrodynamic (motion of fluid or gas by the side of fins and along the left edge of the wall). Secondly, if we had 3rd type BC instead of the BC (1), we would have had full three-dimensional problem.

## 5 Conclusions

We have constructed exact three dimensional analytical solution for the system with rectangular fins where the wall and the fins consist of materials with different thermal properties. The solution has the form of the system of 2nd kind Fredholm integral equations. The order of this system is equal the number of the fins.

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