Dynamics of non-local systems handled by fractional calculus

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Abstract: - Mechanical vibrations of non-local systems with long-range, cohesive, interactions between material particles have been studied in this paper by means of fractional calculus. Long-range cohesive forces between material particles have been included in equilibrium equations assuming interaction distance decay with order α. This approach yields as limiting case a partial fractional differential equation of order α involving space-time variables. It has been shown that the proposed model may be obtained by a discrete, mass-spring model that includes non-local interactions by non-adjacent particles and the mechanical vibrations of the particles have been obtained by an approximation fractional finite difference scheme already used for static analysis. Modal shapes and natural frequency of the non-local systems may then be obtained from the proposed model with boundary conditions coalescing with classical mechanics boundary conditions and solution obtained with the proposed model is capable to capture local characters as particular case of the real coefficient α. Numerical applications reported show a remarkable non-local feature of the state variables of the analyzed system.

Key-Words: - Non-local interactions, Long-range forces, Fractional calculus, Non-local dynamics, Eigenproperties.

1 Introduction

Mathematical theories representing some mechanical behavior may be efficiently applied within some bounds dependent of the studied problem. If the mathematical description of the problem at hand is no longer representative of the observed phenomenon then some remarkable differences between theoretical and experimental results may be experienced. This consideration is also applicable to solid state mechanics in which the obtained results depend of an opportune, internal, scale of material considered. As in fact any engineering material possess an internal substructure which may be observed at micro or nano-level. Internal constitutive substructure at molecular or crystalline level may be considered by means of molecular dynamic as shown in some studies conducted by the mid-fifties.

On one hand engineering studies by molecular dynamics do not allow for efficient descriptions of structural systems due to the large number of particles and consequently large number of motion equations involved in analysis. On the other hand the well-established continuum elastic mechanics, widely used in engineering applications, do not involve any information about the internal scale of material.

As a consequence, the results provided by continuum mechanics theory show excellent agreement only if the observation scale of the phenomenon is much larger than internal scale of material.

First paradoxes of the continuum mechanics theory have been observed about infinitely large stresses at crack tip and about independence of the phase velocity of mechanical waves across an elastic medium on the wave length. Despite paradoxes of continuum mechanics for some class of problem, the powerful and easy approach of continuum mathematical theory of elasticity is extremely attractive to model and solve engineering problems. Some attempts to include the accuracy of atomic theory in the simplicity of continuum mechanics has regarded the introduction in the material constitutive equations of some terms involving long-range interactions between non-adjacent particles. In this context two wide class of theories have been developed: The gradient elasticity theory (weak non-locality) and the integral non-local theory (strong non-locality). The first approach consists in the consideration of opportune terms including gradient of strains in the constitutive equations of the considered material (Mindlin, et al. 1968; Aifantis, 1994) with opportune coefficients dependent of material microstructure. The main drawback of gradient elasticity model regards fulfillment of the boundary conditions associated to the problem considered. In this context several strategies, which make use of variational formulations, have been recently proposed (Polizzotto 2001, 2003). In other studies the problem has been framed in thermodynamic setting (Polizzotto, Borino, 1998; Borino et al.,2003) . The approach yields results in good agreement with experiments but the mechanical aspects of the boundary conditions and selections of parameters involved in the analysis is still an open problem. For a review of recent developments in gradient theories see Aifantis (2003) and references cited therein.

As an alternative the non-local integral model of elasticity has been introduced as intuitive change of scale in molecular dynamic (Kroner, 1967; Krumshals 1968; Eringen, Edelen 1972) allowing to overcome several paradoxes in classical theory of elasticity in presence of damage and fatigue For recent advances on the non-local integral theory the readers are referred to several papers (Bazant et al., 2002; Benvenuti et al., 2002; Fuschi, Pisano, 2003; Polizzotto et al., 2004) to cite but a few. Non-local approach involves the definition of an attenuation function (usually gaussian) describing the behavior of long-range interactions. The internal stress in the integral theory is related to the correspondent strains by a convolution integral with kernel provided by the selected attenuation function.

Very recently an attempt to model the non-local behavior by fractional calculus has been made by Lazopoulos (2006) postulating that strain energy functional involves fractional derivative of the strain but no explicit formulation in terms of equilibrium or kinematics equations have been derived. Fractional calculus has been extensively used by Carpinteri and associates (2001, 2004) to describe damage and fracture mechanics in heterogeneous materials.
In this paper non-local mechanics formulation will be considered as extension of a discrete non-local model. It will be shown that properly setting the attenuation function of long-range interactions between particles as $\left|x_j - x_k\right|^{-(1+\alpha)}$, with $0<\alpha<1$, as molecular distance approaches zero a fractional differential equation involving Marchaud derivatives (Miller et al. 1993; Samko et al. 1988) of the displacement function has been obtained. As it is widely known fractional derivatives are convolution integrals enjoying some properties (Leibinitz rule, integration by parts and others) analogous to classical derivatives. The proposed approach is then an intermediate approach between gradient elasticity and integral theory of non-local interactions. The proposed approach involves two main attractive: i) The mathematical model reproduces a well-defined mechanical model; ii) The boundary conditions may be written simply taking into account fractional differential equation theory overcoming the main problem of non-local theory that relies in the proper definition of boundary conditions. It will be shown that solution of governing fractional differential equation of the problem by the fractional finite differences (Shkhanukov, 1996) yields the same results of the discrete model constituted by linear springs connecting non-adjacent elements with decaying stiffness. Dynamic analysis will be reported in sec.3 including inertial forces in the equilibrium of discrete volume elements. This approach led us, as limiting case toward a partial, fractional differential equation in the state variables of the system. The equation has been solved, for the equivalent discrete model, in free vibrations providing natural frequencies and eigenvectors of the discrete model.
A numerical application contrasting local and non-local system dynamics have been reported at the bottom of sec.3 providing local and non-local mode of vibration and natural frequencies of the analyzed system.

2 Non-local Fractional Model

Consider an elastic bar with infinite length, as depicted in Fig.(1) loaded with external self-equilibrated volume forces denoted $f(x)$ and boundary loads F and let us discretize the bar in volume elements $V_j = A\Delta x$ $(j=-\infty,...,\infty)$ with A the cross-section and Δx the length of the element. Volume element V_j is located at abscissa x_j and it is in equilibrium under external loads, contact forces provided by adjacent volume elements, V_{j-1} and V_{j+1}, namely $N_j^{(l)}$ and $N_{j+1}^{(l)}$, and long-range actions, denoted $N_j^{(nl)}$ applied on V_j by the surrounding non-adjacent elements of the bar (Fig.1):

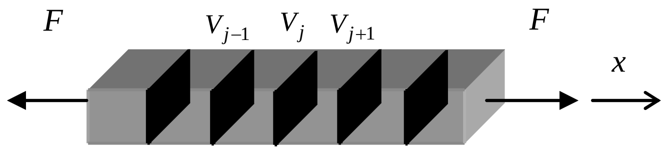

Fig. 1: Discrete elastic bar

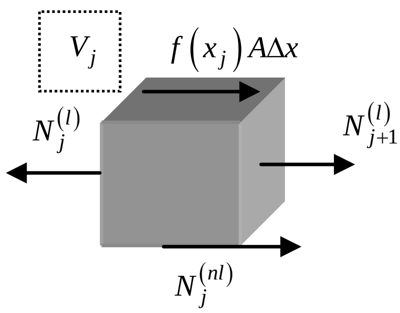

Fig. 2: Equilibrium of volume V_j

Under these circumstances equilibrium equation of volume V_j is provided by the equation:

$$\Delta N_j^{(l)} + N_j^{(nl)} = -f_j A\,\Delta x \tag{1}$$

where: $j=-m,...,-2,-1,\ 0,\ 1,2,...,m$, with $(m\to\infty)$, $f_j = f\left(x_j\right)$, $\Delta N_j^{(l)} = N_{j+1}^{(l)} - N_j^{(l)}$ is the difference between the contact forces N_j^l and N_{j+1}^l provided by volume elements V_{j+1}, V_{j-1}. The resultant $N_j^{(nl)}$ acting on the elementary volume V_j is the algebraic sum of the long-range forces surrounding volume elements

$V_h \quad (h=-m,...,-2,-1,\ 0,\ 1,2,...,h \neq j,...,m \ \ (m \to \infty))$ applied on element V_j and they will be denoted $Q^{(h,j)}$ and reported in Fig.3

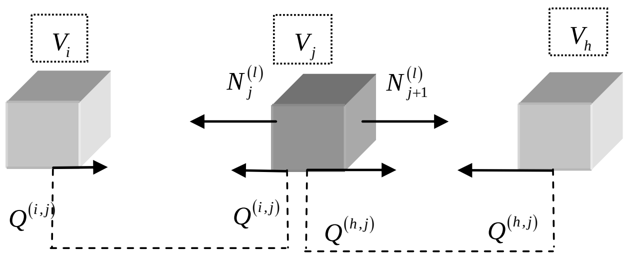

Fig. 3: Long-range terms in equilibrium of volume V_j

In this setting the resultant of long-range interactions $N_j^{(nl)}$ reads:

$$N_j^{(nl)} = -Q^{(-\infty,j)} - ... - Q^{(j-1,j)} + Q^{(j+1,j)} + ... + Q^{(\infty,j)} \tag{2}$$

Long range forces $Q^{(h,j)}$ $(h=-\infty,...,0,...,h \neq j,...,\infty)$ represent molecular interactions between non-adjacent volume elements and hence they depend of volume size of both interacting elementary volumes V_j and V_h as in applied mechanics problems with interacting forces. In the following long-distance interactions $Q^{(h,j)}$ will be modelled as forces decaying with the relative distance between volume elements V_j and V_h as $\left|x_h - x_j\right|^{-(\alpha+1)}$ with α a real, material dependent parameter $(0<\alpha<1)$, and they are expressed by:

$$\begin{aligned} Q^{(h,j)} &= \left(u(x_h) - u(x_j)\right) g\left(\left|x_j - x_h\right|\right)\ V_j\ V_h, \quad \text{if } j<h \\ Q^{(h,j)} &= \left(u(x_j) - u(x_h)\right) g\left(\left|x_j - x_h\right|\right)\ V_j\ V_h, \quad \text{if } j>h \end{aligned} \tag{3a, b}$$

where $u(x_j)$ and $u(x_h)$ are axial displacements of volume elements V_j and V_h, respectively and the decaying function $g\left(\left|x_j - x_h\right|\right)$ is a real-valued, monotonically decreasing function expressed as:

$$g\left(\left|x_j - x_h\right|\right) = \frac{c_\alpha \alpha}{\Gamma(1-\alpha)} \ \left(\left|x_j - x_h\right|\right)^{-(1+\alpha)} \tag{3 c}$$

where c_α is a dimensional coefficient $\left[c_\alpha\right] = \left[F\ L^{\alpha-6}\right]$ and function $\Gamma(\alpha) = \int_0^\infty exp[-t]\ t^{\alpha-1} dt$ is the Euler gamma function. Introduction of the material internal scale α allows for a large variety of non-local behaviour of long range interactions and the classical local case, without cohesive forces, may be recovered as $\alpha \to 0$. Substitution of eq.(3) in the equilibrium equation reported in eq.(1), accounting for eq.(2), the equilibrium equation of volume V_j is written as:

$$\Delta N_j^{(l)} - \frac{c_\alpha \alpha}{\Gamma(1-\alpha)} \left[\sum_{h=-\infty}^{j-1} \frac{u(x_j) - u(x_h)}{\left(x_j - x_h\right)^{1+\alpha}} V_h V_j + \sum_{h=j+1}^{\infty} \frac{u(x_h) - u(x_j)}{\left(x_h - x_j\right)^{1+\alpha}} \ V_h V_j \right] = -f_j V_j \tag{4 a}$$

that may be cast, accounting for volume sizes $V_j = V_h = A\Delta x$, in the form:

$$\Delta N_j^{(l)} - \frac{c_\alpha \alpha\ A^2 \Delta x}{\Gamma(1-\alpha)} \left[\sum_{h=-\infty}^{j-1} \frac{u(x_j) - u(x_h)}{\left(x_j - x_h\right)^{1+\alpha}} \Delta x + \sum_{h=j+1}^{\infty} \frac{u(x_h) - u(x_j)}{\left(x_h - x_j\right)^{1+\alpha}} \ \Delta x \right] = -f_j A \Delta x \tag{4 b}$$

Dividing eq.(4) by Δx and taking limit for $\Delta x \to 0$ the differential equilibrium equation is obtained as:

$$\frac{dN^{(l)}(x)}{dx} - C_\alpha \left[\mathbf{D}_+^\alpha u(x) + \mathbf{D}_-^\alpha u(x) \right] = -f(x) A \tag{5}$$

where $C_\alpha = c_\alpha A^2$ with dimensions $\left[C_\alpha\right] = \left[F\ L^{\alpha-2}\right]$ and the left-hand and right-hand Marchaud derivative has been introduced (Samko et al., 1987; Miller, Ross, 1993) defined by:

$$\begin{aligned} \mathbf{D}_+^\alpha s(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{s(x) - s(\xi)}{(x-\xi)^{(1+\alpha)}} d\xi \\ \mathbf{D}_-^\alpha s(x) &= \frac{\alpha}{\Gamma(1-\alpha)} \int_{x}^{+\infty} \frac{s(x) - s(\xi)}{(\xi - x)^{(1+\alpha)}} d\xi \end{aligned} \tag{6 a, b}$$

Eq.(5) may be rewritten, in terms of Riesz fractional derivative as:

$$\frac{dN^{l}(x)}{dx} - \bar{C}_\alpha D^\alpha u(x) = -f(x) A \tag{7}$$

where Riesz fractional derivative, indicated with $D^\alpha u(x)$ is related to the Marchaud fractional derivative as:

$$D^\alpha s(x) = \frac{1}{2 cos(\alpha\pi/2)} \left[\mathbf{D}_+^\alpha s(x) + \mathbf{D}_-^\alpha s(x) \right] \tag{8}$$

and coefficient $\bar{C}_\alpha = 2 C_\alpha\, cos(\alpha\pi/2)$. Eq.(7) may be recast in terms of conventional stress $\sigma^{(l)}(x) = N^{(l)}(x)/A$ as:

$$\frac{d\sigma^{(l)}(x)}{dx} - \bar{C}_{\alpha}^{\sigma} D^{\alpha} u(x) = -f(x) \tag{9}$$

with $\bar{C}_{\alpha}^{\sigma} = \bar{C}^{\alpha}/A$, with dimensions $\left[F\,L^{\alpha-4}\right]$. Eq.(9) is the equilibrium equations of the volume $dV = Adx$ located at abscissa x in which long-range interactions between surrounding non-adjacent volumes have been taken into account.

Assuming elastic-state material, the conventional stress strains relation may be used:

$$\sigma^{(l)}(x) = E\varepsilon(x) = E\,du/dx \tag{10}$$

being E the Young modulus and yielding the equilibrium equation in terms of the displacement field for the infinitesimal volume as:

$$E\frac{d^2u(x)}{dx^2} - \bar{C}_{\alpha}^{\sigma} D^{\alpha} u(x) = -f(x) \tag{11}$$

that is an ordinary fractional differential equation. Relevant boundary conditions associated to eq.(11) now born in natural way since they may be framed in the context of the fractional Dirichlet problem (Kilbas et al., 2006). Eq.(7) may be converted into a fractional differential equation involving local and non-local axial stresses introducing a fractional differential equation in the form:

$$\frac{d\sigma^{(l)}(x)}{dx} - D^{\alpha/2}\sigma^{(nl)}(x) = -f(x) \tag{12}$$

As in fact by using the virtual work principle we get the kinematics variables associated to the $\sigma^{(l)}(x)$ and $\sigma^{(nl)}(x)$, as:

$$\begin{aligned}
L_{est} &= \int_{-\infty}^{\infty} u(x)Af(x)dx + u(x)F\big|_{x=\infty} - u(x)F\big|_{x=-\infty} = \\
&= -\int_{-\infty}^{\infty} u(x)\left(\frac{d\sigma^{(l)}}{dx} - D^{\frac{\alpha}{2}}\sigma^{(nl)}\right)dx + u(x)F\big|_{x=\infty} - u(x)F\big|_{x=-\infty} = \\
&= \int_{-\infty}^{\infty} \sigma^{(l)}(x)\frac{du}{dx}\,dx + \int_{-\infty}^{\infty} \sigma^{(nl)}(x)D^{\alpha/2}u(x)\,dx = \\
&= \int_{-\infty}^{\infty} \sigma^{(l)}(x)\varepsilon^{(l)}(x) + \sigma^{(nl)}(x)\varepsilon^{(nl)}(x)dx = L_{int}
\end{aligned} \tag{13}$$

in which kinematics variables in the latter term of the equality in eq.(13) have been defined as:

$$\varepsilon^{(l)}(x) = du/dx = \varepsilon(x)\ ;\ \varepsilon^{(nl)}(x) = D^{\alpha/2}u(x) \tag{14 a, b}$$

related to the stresses, by relations:

$$\sigma^{(l)}(x) = E\ \varepsilon^{(l)}(x)\ \ ;\ \ \sigma^{(nl)}(x) = \bar{C}_{\alpha}^{\sigma}\varepsilon^{(nl)}(x) \tag{15 a, b}$$

Combining eqs.(14-15) with eq.(12) the fractional differential equation reported in eq.(7) is recovered. At this stage we may remark that the mechanical representation of non-local axial stress $\sigma^{(nl)}(x)$ in eq.(12) is obtained assuming that the long-range stresses $\sigma^{(h,j)} = Q^{(h,j)}/A$ depends of the non-local resultant axial force $\sigma_h^{(nl)} = N_h^{(nl)}/A$ as:

$$\sigma^{(j,h)} = \frac{1}{\Gamma(1-\alpha)}\sigma_h^{(nl)}\left|x_h - x_j\right|^{-\alpha/2} \tag{16}$$

Assuming that the elastic problem, with long-range interactions, is defined in a finite domain of length L, as in engineering problems, we may discretize the bar in m elements of length $\Delta x = L/m$. In this context the sums in eq.(3) may be rewritten, for the equilibrium equation of the volume element V_j as:

$$\begin{aligned}
\Delta N_j^{(l)} - \frac{c_\alpha \alpha\ A^2\Delta x}{\Gamma(1-\alpha)}\Bigg[\sum_{h=1}^{j-1}\frac{u(x_j)-u(x_h)}{(x_j - x_h)^{1+\alpha}}\Delta x + \\
+\sum_{h=j+1}^{m}\frac{u(x_h)-u(x_j)}{(x_h - x_j)^{1+\alpha}}\Delta x\Bigg] = -f_j A\Delta x
\end{aligned} \tag{17}$$

and at the limit $\Delta x \to 0$ the fractional differential equilibrium equation may be written, with the same considerations leading to eq.(11) as:

$$E\frac{d^2u(x)}{dx^2} - \bar{C}_{\alpha}^{\sigma}\hat{D}^{\alpha}u(x) = -f(x) \tag{18}$$

where we define the *truncated* Riesz fractional derivative as

$$\hat{D}^{\alpha}s(x) = \frac{1}{2cos(\alpha\pi/2)}\left[\hat{\mathbf{D}}_{0^+}{}^{\alpha}s(x) + \hat{\mathbf{D}}_{L^-}{}^{\alpha}s(x)\right] \tag{19}$$

and the *truncated* Marchaud fractional derivatives as

$$\hat{\mathbf{D}}_{0^+}^{\alpha}s(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^x \frac{s(x)-s(\xi)}{(x-\xi)^{(1+\alpha)}}d\xi\ ; \tag{20 a}$$

$$\hat{\mathbf{D}}_{L^-}^{\alpha}s(x) = \frac{\alpha}{\Gamma(1-\alpha)}\int_x^L \frac{s(x)-s(\xi)}{(\xi-x)^{(1+\alpha)}}d\xi \tag{20,b}$$

respectively. This Marchaud truncated operator are related to the Marchaud derivatives in eq.(6 a, b) by the relations:

$$\hat{\mathbf{D}}^{\alpha}_{0^+} s(x) = \mathbf{D}^{\alpha}_{+} \hat{s}(x) - \frac{s(x)}{\Gamma(1-\alpha)x^{\alpha}}; \tag{21 a}$$

$$\hat{\mathbf{D}}^{\alpha}_{L^-} s(x) = \mathbf{D}^{\alpha}_{-} \hat{s}(x) - \frac{s(x)}{\Gamma(1-\alpha)(L-x)^{\alpha}} \tag{21 b}$$

where $\hat{s}(x) = \{s(x), x \in [0,L]; 0, x \notin [0,L]\}$.

In the next section the proposed fractional model for the finite domain will be used to describe boundary effects in a bar in tension, contrasting results with a non-local spring-mass model.

2.1 Solution of the fractional differential equation

In this section, the solution to the fractional differential equation (18) will be found by means of a numerical scheme. As like ordinary differential equations can be solved by finite difference schemes, also fractional differential equations can be tackled by appropriate finite difference forms of fractional operators involved.

In particular, fractional differential equation (18) involving Riemann-Liouville fractional derivatives may be studied resorting to the fractional finite differences scheme proposed by Shkhanukov. Following his original idea, the method is easily extended to Riesz fractional operator.

To this aim, introduce a proper discretization of the bar in *m* intervals of amplitude $\Delta x = L/m$ and represent the fractional differential operator D^{α} at the material point $x_i = (i-1)\Delta x$ by the difference operator $\Delta^{\alpha}_{x_i}$ given by:

$$D^{\alpha} s(x_i) = \Delta^{\alpha}_{x_i} s(x) + O(\Delta x) \tag{22}$$

where $O(\Delta x)$ means a quantity of order Δx and the fractional difference operator $\Delta^{\alpha}_{x_i}$ is represented as:

$$\Delta^{\alpha}_{x_i} s(x) = \frac{\alpha^{-1}}{\Gamma(1-\alpha)} \left\{ \sum_{h=1}^{j} \left[\left(x_{j-h+1}\right)^{-\alpha} - \left(x_{j-h}\right)^{-\alpha} \right] s(x_h) + \right. \\ \left. + \sum_{h=j+1}^{m} \left[\left(x_{j-h}\right)^{-\alpha} - \left(x_{j+1-h}\right)^{-\alpha} \right] s(x_h) \right\} \tag{23}$$

The finite difference scheme of eq.(18), by substitution (22) and neglecting terms of order Δx, is rewritten as:

$$\frac{EA}{\Delta x}\Delta^2 u(x_j) - \frac{\bar{C}_{\alpha}\Delta x}{\Gamma(1-\alpha)} \cdot \\ \cdot \left\{ \sum_{h=1}^{j-1} \left(u(x_j) - u(x_h)\right)\left(\left(x_{j-h}\right)^{-\alpha} - \left(x_{j+1-h}\right)^{-\alpha}\right) + \right. \\ \left. + \sum_{h=j+1}^{m} \left(u(x_j) - u(x_h)\right)\left(\left(x_{h-j}\right)^{-\alpha} - \left(x_{h+1-j}\right)^{-\alpha}\right) \right\} = \\ = -F(x_j)\Delta x \tag{24}$$

holding for $j = 1,2,...,m$. In the former, $F(x_j) = f(x_j)A$ and $\Delta^2 u(x_j) = u(x_{j+1}) - 2u(x_j) + u(x_{j-1})$.

Then, eqs.(24) is a system of m algebraic equations in the unknown displacement field $u(x_j)$. Easily one can recast eq.(24) in compact form as:

$$\mathbf{K}\ \mathbf{u} = \mathbf{f} \tag{25}$$

with displacement and force vectors $\mathbf{u}$ and $\mathbf{f}$, collecting nodal displacements and nodal external forces:

$$\mathbf{u}^T = [u_1 \quad u_2 \quad ... \quad u_m] \ ; \ \mathbf{f}^T = [f_1 \quad ... \quad ... \quad f_m]\Delta x \tag{26 a ,b}$$

and $\mathbf{K}$ is a coefficient matrix, that is sum of the local stiffness matrix $\mathbf{K}^{(l)}$ and a non-local stiffness matrix $\mathbf{K}^{(nl)}$. Explicit expression of the element (r,s) of the matrix $\mathbf{K}^{(nl)}$ is:

$$\mathbf{K}^{(nl)}_{rs} = A^2 \Delta x^2 \left(\delta_{rs} \sum_{h=1, h\neq j}^{m} g|x_r - x_h| + (\delta_{rs} - 1)\, g|x_s - x_r| \right) \tag{27}$$

where δ_{jk} is the Kronecker delta. Further details on the stiffness matrix can be found in the paper (Di Paola et al., 2007). The solution in terms of displacements is thus easily derived from eq.(25) and it represents the non-local displacement field of a continuous media with cohesive long range interaction. It is worth to stress that: (i) the assumption of a power decaying attenuation function, leads in a natural way to the solution of a well-posed fractional differential equation of order $\alpha \in]0,1[$; (ii) two mechanical boundary conditions are sufficient to find the solution in terms of displacements; (iii) the numerical solution to eq.(24) and of the discrete model coincide as the number of nodes *m* increases. Physical grounds yielding eq.(1) may be now further explained considering a discretized, spring-mass model of the bar described in fig.(3) Local forces between adjacent particles have been considered by springs with elastic stiffness $K^l = EA/\Delta x$. Non-local interactions have been considered connecting non-adjacent particles by linear springs with distance-decaying stiffness as $K^{nl}_{jh} = g\left(|x_h - x_j|\right)$ where function $g\left(|x_h - x_j|\right)$ is a monotonically decreasing function of the argument. Under these circumstances model of fig.3 may be studied, by the classical displacement approach, observing that equilibrium equation, for the generic node located at abscissa x_j may be written as:

$$K^l u_1 - K^l u_2 - \sum_{h=2}^{m} g\left(\left|x_1 - x_h\right|\right)\left(u_h - u_1\right) = -F_1 \;;$$

$$-K^l u_{j-1} + 2K^l u_j - K^l u_{j+1} - \sum_{h=1}^{m} g\left(\left|x_j - x_h\right|\right)\left(u_h - u_j\right) = -F_j \;; j = 2,...,m-1$$

$$K^l u_m - K^l u_{m-1} - \sum_{h=2}^{m} g\left(\left|x_1 - x_h\right|\right)\left(u_m - u_h\right) = -F_m$$

(28 a-c)

in which first terms correspond to contact forces and the summations represent non-local forces applied at material particle at abscissa and right-hand side of eqs.(28 a, c) are related to the forces $F_1 = -F, F_j = 0, F_m = F; \quad j = 2,3,...,m-1$.

Equilibrium equations reported in eq.(29) may be rewritten in matrix form as in eq.(25) introducing the non-local stiffness matrix in which we denoted the local stiffness matrix coalescing with eq.(27) and the non-local interactions have been incorporated into the symmetric, fully populated, non-local stiffness matrix which reads:

$$\mathbf{K}^{nl} = \begin{bmatrix} K_1^{nl} & -g\left(\left|x_2 - x_1\right|\right) & \cdots & \cdots & -g\left(\left|x_m - x_1\right|\right) \\ -g\left(\left|x_2 - x_1\right|\right) & K_2^{nl} & \cdots & \cdots & -g\left(\left|x_m - x_2\right|\right) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -g\left(\left|x_m - x_1\right|\right) & \cdots & \cdots & -g\left(\left|x_2 - x_1\right|\right) & K_m^{nl} \end{bmatrix}$$

(29)

where we denoted $K_j^{nl} = \sum_{\substack{h=1 \\ h \neq j}}^{m} g\left(\left|x_j - x_h\right|\right)$. A close observation of eq.(29) shows that the stiffness matrix contrasted with the coefficient matrix in eq.(28) at limit when $\Delta x \to 0$ the matrix in eq.(29) yields exactly the non-local matrix (eq.28) Boundary conditions imposed to the displacement vector in eq.(25) involves reduction of the stiffness matrix in eq.(29) and displacement vector of material particles may be obtained by inversion of the associated reduced matrix.

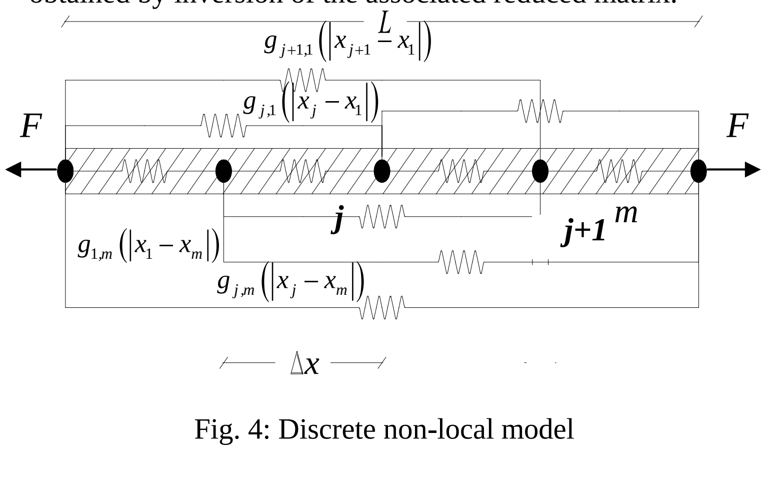

Fig. 4: Discrete non-local model

2.1.1 Numerical Application,

Let us considered an elastic bar restrained as in fig.(5) with length $L = 200\ mm$, cross-section $A = 100\ mm^2$ and modulus of elasticity $E = 72\, GPa$. The bar is loaded by an axial force $F_1 = 10KN$ and it has been studied in local setting and with the non-local fractional description of the previous section. The resulting strains have been contrasted with the non-local discrete model introduced in eq.(4b) and the obtained results have been reported in Fig.4. It can be noted that the strain field $\varepsilon(x)$ has the typical non-local character, as shown either by the discrete model (dotted line) and by the continuous one.

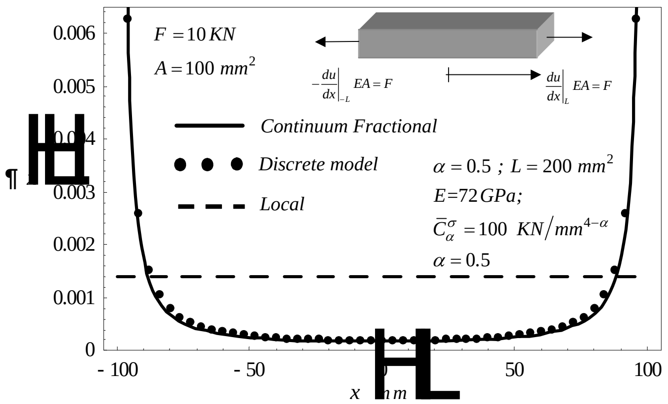

Fig. 5: Axial strain of fractional continuum contrasted with discrete and local elastic model

3 Dynamics of the non-local fractional model

Dynamic behaviour of the non-local model described in sec.(2) may be formulated accounting for the inertial forces arising during vibrations. To this aim let us consider first the discrete spring mass model in fig.(1) under the assumption of homogeneous elastic bar with ρ the mass density of the material, the equilibrium equation of volume V_j, as shown in Fig.6, reads:

$$-(\rho A \Delta x)\ddot{u}_j(t) + \Delta N_j^{(l)}(t) + N_j^{(nl)}(t) = 0 \qquad (30)$$

in which the explicit time-dependence of the state variables terms with $u_j(t) = u(x_j, t)$, M_j the displacement and the mass of the volume V_j, respectively. Inertial forces reported in eq.(30) have been expressed by the time-derivative of the axial displacements denoted $\ddot{u}_j(t) = d^2 u_j(t)/dt^2$

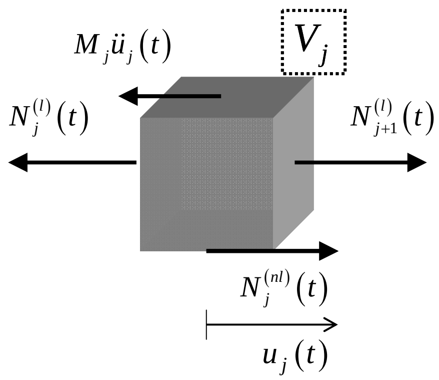

Fig. 6: Equilibrium of the volume V_j with inertial forces

Substitution of the local and non-local forces, as done in the previous section, the equation of equilibrium reads:

$$M_j\ddot{u}_j(t)+\frac{EA}{\Delta x}\Delta^2 u_j(t)+$$
$$-\frac{c_\alpha \alpha A^2\Delta x}{\Gamma(1-\alpha)}\left[\sum_{h=-m}^{j-1}\frac{u_j(t)-u_h(t)}{(x_j-x_h)^{1+\alpha}}\Delta x+\sum_{h=j+1}^{m}\frac{u_h(t)-u_j(t)}{(x_h-x_j)^{1+\alpha}}\Delta x\right]=0 \tag{31}$$

Yielding in the limit a partial fractional differential equation in the axial displacement field $u(t,x)$ as:

$$\rho A\frac{\partial^2 u(t,x)}{\partial t^2}+EA\frac{\partial^2 u(t,x)}{\partial x^2}-\bar{C}_\alpha^\sigma \hat{D}_x^\alpha u(t,x)=0 \tag{32}$$

with $\hat{D}_x^\alpha[\bullet]$ denoting the truncated Marchaud partial fractional derivative with respect to *x*-variable.
Solution of the differential equation reported in eq.(32) is provided by similar technique used in sec.(2.1) and by Shkhanukov approximation of fractional derivatives a linear system of ordinary differential equations in the time variable, involving local and non-local stiffness matrices is obtained as:

$$\mathbf{M}\ddot{\mathbf{u}}(t)+\left(\mathbf{K}^{(l)}+\mathbf{K}^{(nl)}\right)\mathbf{u}(t)=\mathbf{0} \tag{33}$$

where $\mathbf{M}=\delta_{jk}m_j$ is the mass matrix of order *m*.

Classical modal analysis can be now easily applied in order to find the frequencies and the vibrations mode shapes for the system represented by eq.(32). Assuming the solution in the form:

$$\mathbf{u}(t)=\boldsymbol{\phi}_j\exp\left[i\omega_j t\right] \tag{34}$$

with $\boldsymbol{\phi}_j$ and ω_j, respectively, the j-th eigenmode and the natural frequency and $i=\sqrt{-1}$. Substitution of eq.(34) into eq.(33) yields an homogeneous system of algebraic equation involving symmetric and positive definite matrices. In this setting a non-degenerate solution may be obtained only for values of the parameters ω_j satisfying the secular equation:

$$det\left[-\omega_j^2\mathbf{M}+\left(\mathbf{K}^{(l)}+\mathbf{K}^{(nl)}\right)\right]=0 \tag{35}$$

with $\det[\bullet]$ denoting determinant of the argument.
Mode shapes of the non-local system described in eq.(32) have been reported in figs.(7-10) contrasting local and non-local eigenmodes.

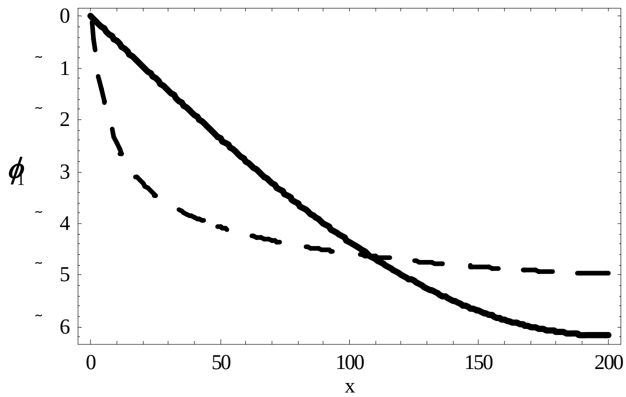

Fig. 7: First modal shape: non-local (dashed), local (continuous)

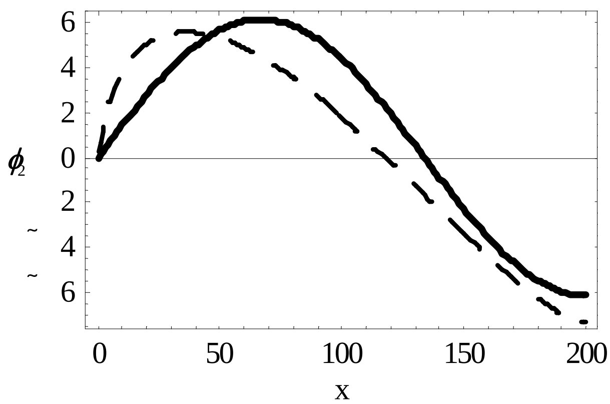

Fig. 8: Second modal shape: non-local (dashed), local (continuous)

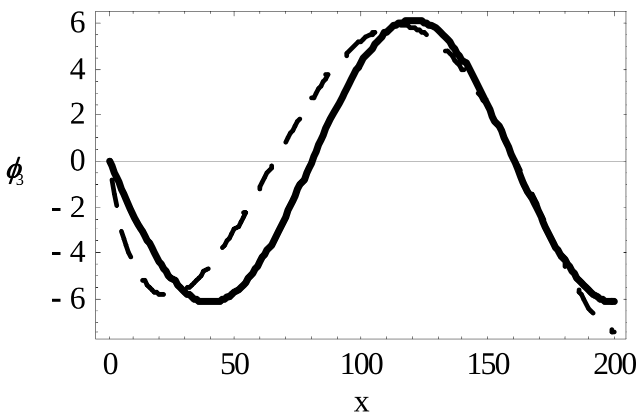

Fig. 9: Third modal shape: non-local (dashed), local (continuous)

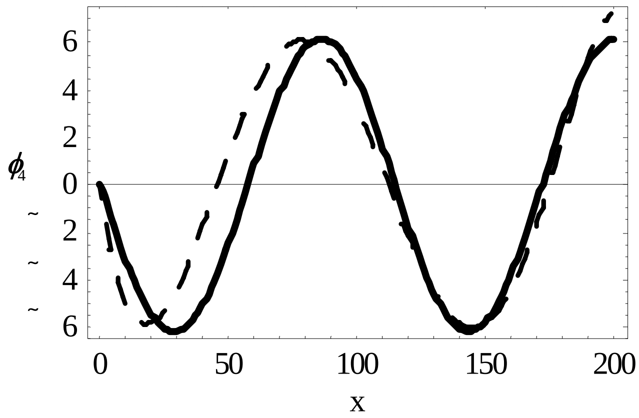

Fig. 10: Fourth modal shape: non-local (dashed), local (continuous)

	D	C	L	ex	e%(L-D)	e%(ex-L)	e%(ex-D)
ω_1	1229.95	1214.33	1298.73	1301.98	5.29633	0.249632	5.53274
ω_2	4405.95	4367.97	3896.12	3905.95	13.0856	0.251672	12.801
ω_3	5996.63	5928.71	6493.27	6509.92	7.64857	0.255754	7.88477
ω_4	7173.78	7076.86	9090.02	9113.88	21.0807	0.261876	21.2874
ω_5	8163.12	8038.56	11686.2	11717.9	30.1474	0.270039	30.336
ω_6	9049.04	8898.42	14281.7	14321.8	36.6389	0.280241	36.8164

Tab. 1: Natural circular frequencies in (rad/s)

Numerical analysis has been conducted with values reported in sec.2.1 and $\rho = 2.5\ 10^{-6}\ Kg\ mm/\sec^2$ and in tab.1 six natural frequencies have been described for the discrete non-local representation (D), for the continuous non-local (C) and for the discrete local (L) model. Column (ex) reports the exact values of the natural frequencies as $\omega_j = \pi(2j-1)/(2L)\sqrt{E/\rho}$. In tab.1 the percentage errors between the investigated models have been reported in the latter columns showing that the errors in the evaluation of the natural frequencies between the local and non local model (e%(L-D)) increase for higher natural frequencies.

4 Conclusion

In this paper the mechanical vibrations of an elastic system with cohesive, long-range interactions have been investigated. The long-range, non-local interactions between the material particles have been included in the investigated model . It has been shown that proper selection of the attenuation function of cohesive interactions as $\left|x_j - x_k\right|^{-(1+\alpha)}$, with $0 < \alpha < 1$, yields a fractional differential equation of order α. Moreover it has been shown that the fractional model of long-distance interactions coincides with a discrete, particle-spring model that includes connections between non-adjacent particles by linear springs with distance-decaying stiffness. The two models of non-local interactions provide the same mechanical response. Dynamics of the non-local system has been subsequently investigated including inertial forces in the equilibrium equations leading to a fractional, partial differential equation of order α in the axial displacement of the model. The solution has been obtained “in vacuo” providing the vibration mode shapes and the natural frequencies of the system. Critical comparison between the proposed non-local model and a local system has been reported showing that the mode shapes are almost unchanged for the local and non-local models. However the natural frequencies of the two models are almost the same for the first two or three frequencies and they show significant differences at the higher eigenfrequencies.

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