The Oscillatory Stable Regime of Nonlinear Systems, with two time constants

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Abstract: - The object of this paper is the nonlinear systems that include a relay with a hysteresis loop characteristic property. The linear parts of systems that are taken into consideration are characterized by an order two transfer function, with two time constants. The paper shows a method to evaluate the period of the limit stabile cycles, which are proper for these nonlinear systems.

Key Words: - nonlinear systems, oscillatory stable regime, relay, hysteresis loop, phase-plane method, calculus algorithm.

1. Introduction

The systems that include linear and nonlinear parts, in assembly, are nonlinear systems. These systems are realized in order to maintain, for any parameters, a specified values or a values in a specified band. The figure 1 presents a typical structure for these nonlinear systems.

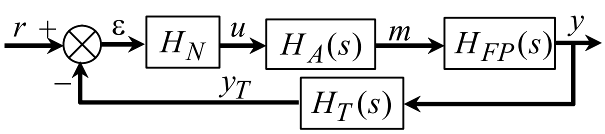

Fig.1. Nonlinear system destined to control of one parameter

The previous schema presents a structure that uses the feedback principle. The linear components of system are: the fixed part of the controlled process, the actuators and the loop's transducer, described form the transfer functions $H_{FP}(s)$, $H_A(s)$ and $H_T(s)$. Usually, the transducer can be considerate that a pure proportional element. The error signal, ε, is the difference between the input, r, and the image y_T (via transducers) of output, y. Based on the error signal, a nonlinear device, described by a nonlinear function, H_N, generates the command u. The command u is applied to actuators. The actuator output m is the input for the fixed part of the controlled process.

The nonlinear component included in the systems can induce a punctual instability. Globally, the response of theses systems to constant inputs bring to an oscillatory stable state. The characteristics of such state, in many situations, can be analytically obtained, and in other situations, only by simulations. Sometimes, the presented nonlinear systems can have a punctually stationary state.

This paper proposes an algorithm to calculus the period of oscillatory stable state, for the nonlinear systems, that include a hysteresis loop relay and the linear components is described that a two order system, without integrator elements. The case of linear parts described by a two order transfer function, with on or two integrators are presented in [1].

2. The study model

In the following, we consider a nonlinear system, which has identically structure that figure 1.

Additionally, is considered that the nonlinear component is a hysteresis loop relay, with $\pm u_0$ the level of output. The hysteresis band is centered in 0 and $2\varepsilon_0$ is the bandwidth. The transducer is pure proportional, and K_T is its gain. The actuators and the plant of process are described by order one transfer function:

$$\begin{cases} H_{FP}(s) = \dfrac{k_F}{s \cdot T_F + 1}, \\ H_A(s) = \dfrac{k_A}{s \cdot T_A + 1}, \end{cases} \quad (1)$$

and the time constants achieved the condition:

$$0 < T_A < T_F . \quad (2)$$

Using the previous hypotheses, the structure presented in the figure 1 can be presented such as the figure 2.

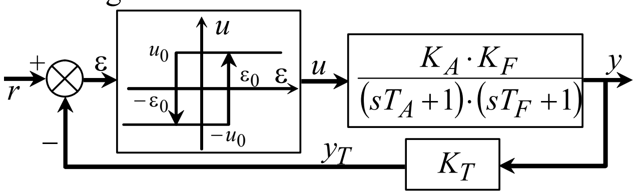

Fig. 2. The nonlinear study model

In order to simplify the next considerations, in the following we'll consider that the system output coincides with the transducer output. The shape of y and y_T evolutions are identically, but the scale differ by the transducers gain.

If the system output is considered the transducer output, the structure can be presented more compact, like in the figure 3.

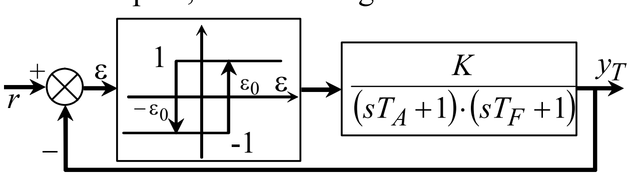

Fig. 3. The compacted study model

The linear parts of previous schema is an order two system, with two negatives poles, and the gain of loop, *K*, is:

$$K = u_0 \cdot K_A \cdot K_F \cdot K_T . \quad (3)$$

In case a constant value of input, the system output y_T tends to one static work points, if the following condition is achieved:

$$\varepsilon_0 > K . \quad (4)$$

In this case, the hysteresis bandwidth is so large, or the gain is so small. The stationary value of system output will be:

$$y_{st} = \pm K . \quad (5)$$

In the previous relation, the sign of y_{st} is identically with the sign of initial value of y_T.

3. The oscillatory stable regime study

In the following, we'll consider that the loop gain is so great that achieved the condition:

$$\varepsilon_0 < K . \quad (6)$$

For the system linear parts, we'll use an equivalent model, which is a parallel connection, showed in the figure 4.

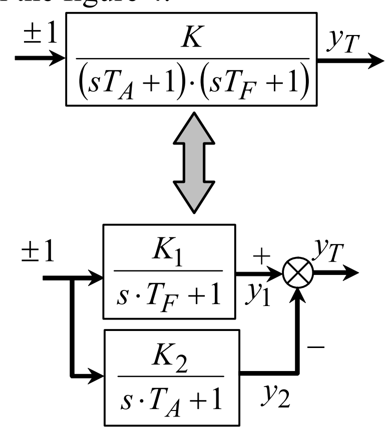

Fig. 4. The equivalent parallel model

The gains of the two order one elements of parallel connection are positives and satisfied the relations:

$$\begin{cases} K_1 = K \cdot \dfrac{T_F}{T_F - T_A}, \\ K_2 = K \cdot \dfrac{T_A}{T_F - T_A}. \end{cases} \quad (7)$$

Observation: if the linear parts has a zero (this case will not treated in this paper):

$$\frac{s \cdot A + K}{(s \cdot T_A + 1)\cdot(s \cdot T_F + 1)}, \quad (8)$$

then it can be represented like a parallel connection of two elements, with the gains:

$$\begin{cases} K_1 = \dfrac{K \cdot T_F - A}{T_F - T_A}, \\ K_2 = \dfrac{K \cdot T_A - A}{T_F - T_A}. \end{cases} \quad (9)$$

For the two elements of the equivalent connection, we'll associate the state variables y_1

and y_2. Using the previous notations, the system output will be:

$$y_T = y_1 - y_2 \tag{10}$$

For a constant relay output, is possible to obtain analytical expressions for y_1 and y_2:

$$\begin{cases} y_1(t) = K_1 + (y_{10} - K_1)\cdot e^{-t/T_F}, \\ y_2(t) = K_2 + (y_{20} - K_2)\cdot e^{-t/T_A}, \end{cases} \tag{11}$$

where y_{10} and y_{20} are the initial values of y_1 and y_2.

If the relay output is constant, is obvious those, indifferently on the initial values, y_1 and y_2 tend to values:

$$\begin{cases} y_1(t \to \infty) = K_1, \\ y_2(t \to \infty) = K_2. \end{cases} \tag{12}$$

In the plane of y_1 and y_2 variables, which is the phase's plane, the state trajectory of the equivalent system tend to the point (K_1, K_2), which is an accumulation point. Using the relation (11), we can obtain an analytical equation of state trajectory, in the phase-plane:

$$\left|\frac{y_2(t) - K_2}{y_{20} - K_2}\right| = \left|\frac{y_1(t) - K_1}{y_{10} - K_1}\right|^{T_F/T_A}. \tag{13}$$

The relation (13) is an equation of a power curve. Indifferently of the initial conditions or of the value of ratio $a = T_F/T_A > 1$, the state trajectory will tend to the accumulation point.

If the parameters of the initial system components are known, is possible to obtain a curve family, who include the state trajectory. The aspect of these curve family is present in the next figure.

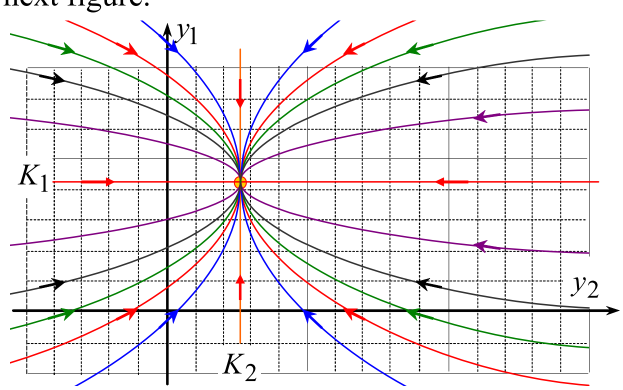

Fig. 5. The state trajectory family in the phase-plane

In case of null value of reference, the commutation lines, in the phase-plane are described trough the equations:

$$|y_1(t) - y_2(t)| = \varepsilon_0. \tag{14}$$

When the relay output are negative, the family of curves that include the trajectory has same graphical shape, but the accumulation point will be other, $(-K_1, -K_2)$. The two curves families and the commutation lines are presented in the next figure.

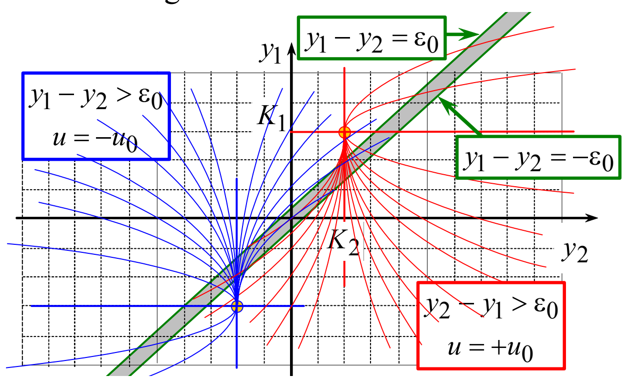

Fig. 6. The two trajectories curves families and the commutation lines

In the previous figure, we can observe:

- if the accumulation points are included in the band delimited by the two commutation lines, according to the initial state, the system output tend to one or other of theses points. In this case the condition (4) is achieved;
- if the accumulation points aren't included in the band delimited by the two commutation lines, the condition (4) isn't achieved. The accumulation points are included in the domains that correspond to a relay output with an opposite sign.

If we consider the typical shape of trajectories, is evident that the system output tends to a limit stable cycle.

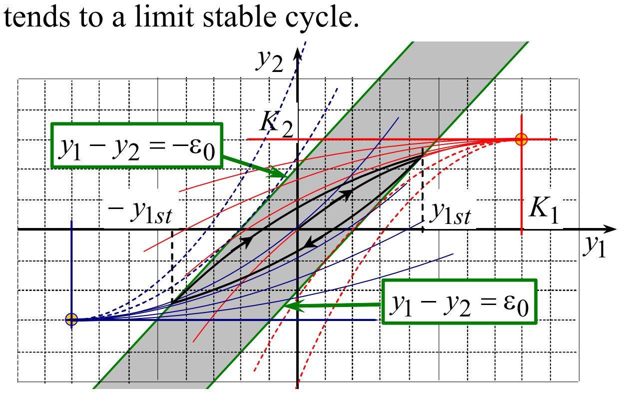

Fig. 7. The limit stable cycle to a null reference

The limit stable cycle is symmetric relative to the coordinate system origin. The extreme points of the cycle are positioned to the two commutation lines and the real output of system will be included in an interval:

$$y(t) \in \left[-\varepsilon_0 / K_T, \varepsilon_0 / K_T\right]. \quad (15)$$

If y_{1st} is the maximum value of state variable y_1, this value satisfied the relation:

$$\left|\frac{K_1 - y_{1st}}{K_1 + y_{1st}}\right| = \left|\frac{K_2 - y_{1st} + \varepsilon_0}{K_2 + y_{1st} - \varepsilon_0}\right|^{T_F / T_E}. \quad (16)$$

The period of limit stable cycle, T, can be obtained using the relation:

$$T_0 = 2 \cdot T_F \cdot \ln \frac{K_1 + y_{1st}}{K_1 - y_{1st}}. \quad (17)$$

In case of one no null constant reference, r_0, we must count that the commutation lines are placed in other position, and its equations will be:

$$\left|y_1 - y_2 - r_0\right| = \varepsilon. \quad (18)$$

Because the trajectories curve families and the accumulation points aren't modified, the limit stable cycle are displaced in the variable y_1 and y_2 plane, like in the next figure.

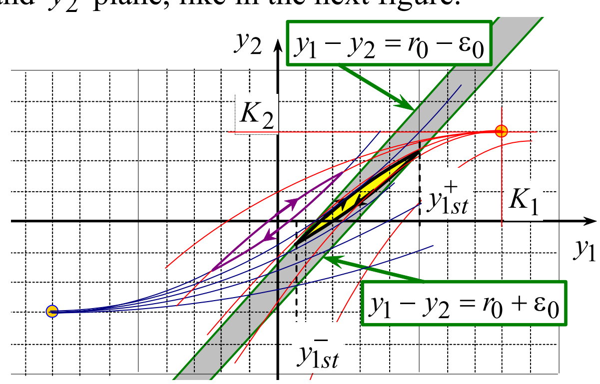

Fig. 8. The limit stable cycle in case a no null reference

The output of system will oscillate in a band, centered to the reference value:

$$y(t) \in \left[r_0 - \varepsilon_0 / K_T, r_0 + \varepsilon_0 / K_T\right]. \quad (19)$$

The limit stable cycle will be delimited in the phase's plane by two extreme points, positioned to the commutations lines:

$$\left(y_{1st}^+, y_{1st}^+ - r_0 - \varepsilon_0\right),$$
$$\left(y_{1st}^-, y_{1st}^- - r_0 + \varepsilon_0\right).$$

The values y_{1st}^+ and y_{1st}^- will satisfied the relations:

$$\left|\frac{y_{1st}^+ - r_0 - \varepsilon_0 - K_2}{y_{1st}^- - r_0 + \varepsilon_0 - K_2}\right| = \left|\frac{y_{1st}^+ - K_1}{y_{1st}^- - K_1}\right|^{T_F / T_A}, \quad (20)$$

and:

$$\left|\frac{y_{1st}^- - r_0 + \varepsilon_0 + K_2}{y_{1st}^+ - r_0 - \varepsilon_0 + K_2}\right| = \left|\frac{y_{1st}^- + K_1}{y_{1st}^+ + K_1}\right|^{T_F / T_A}. \quad (21)$$

The solving of last two equations system permits to obtain the extremes point of limit stable cycle. Using the obtained values y_{1st}^+ and y_{1st}^-, we can calculate the period, such as:

$$T = 2 \cdot T_F \cdot \ln\left(\frac{K_1 - y_{1st}^-}{K_1 - y_{1st}^+}\right). \quad (22)$$

This relations, exactly, is difficult to applied.

The previous considerations are true if the accumulation points are positioned in the phase's plane, relative to the commutation lines like in figure 8. If the reference values r_0 is too high, the state trajectory tends asymptotically to an accumulation point. The limit level of reference module is:

$$r_{0\lim} = \frac{K - \varepsilon_0}{K_T}. \quad (23)$$

If the reference absolute value exceeds the limit value $r_{0\lim}$, the system not tends to an oscillatory stable regime.

4. Example

We'll study the case of a system characterized by the next transfer functions for the linear parts:

$$H_A(s) = \frac{5}{3s+1},$$
$$H_F(s) = \frac{1}{5s+1},$$
$$K_T = 10.$$

We consider for the relay the next values:

$$\varepsilon_0 = 1,$$
$$u_0 = 2.$$

According the presented relations, will be obtained:

$$K = 2 \cdot 5 \cdot 1 \cdot 10 = 100 > \varepsilon_0,$$

and the limit of the reference absolute values is:

$$r_{0\,\lim} = \frac{100-1}{10} = 9.9 .$$

If the reference satisfied the condition:

$$-r_{0\,\lim} < r_0 < r_{0\,\lim}$$

the values of system output will be in the band:

$$\left[r_0 - \frac{\varepsilon_0}{K_T}, r_0 - \frac{\varepsilon_0}{K_T} \right] = \left[r_0 - 0.1, r_0 - 0.1 \right].$$

The gains of two linear components of equivalent connections will be:

$$\begin{cases} K_1 = 100 \cdot \dfrac{5}{5-3} = 250, \\ K_2 = 100 \cdot \dfrac{3}{5-3} = 150. \end{cases}$$

The equation (16) has the next solution:

$$y_{1st} = 47.536$$

and using (17), we obtain the oscillations period for a null reference:

$$T_0 = 2 \cdot 5 \cdot \ln \frac{250 + 47.536}{250 - 47.536} = 3.85(s) .$$

From simulation, it was obtained the values of oscillations period for system output, using non null values for references. These values are presented in the next table.

r_0	0	0.99	1.98	2.97	3.96	4.95
$\dfrac{r_0}{r_{0\,\lim}}$	0	0.1	0.2	0.3	0.4	0.5
T	3.85	3.85	3.85	3.85	3.86	3.86
$100 \cdot \dfrac{T - T_0}{T_0}$	0	0	0	0	0.26	0.26

r_0	5.94	6.93	7.91	8.91	9.5	9.8
$\dfrac{r_0}{r_{0\,\lim}}$	0.6	0.7	0.8	0.9	0.96	0.99
T	3.86	3.86	3.86	3.87	3.88	3.89
$100 \cdot \dfrac{T - T_0}{T_0}$	0.26	0.26	0.26	0.52	0.78	1.04

The results obtained by simulation are practically invariable. The variations of the limit stable cycle periods relative to the result of calculus relation (17) are insignificant.

5. Conclusions

For the nonlinear systems that can be reshaped in an equivalent structure, similarly of those which presented in the figure 2, the paper offers the possibility to calculate the elements of limit stable cycle. The relation (17) can be used to evaluate the period of the limit stable cycle, not only in the null reference case, because, the errors induced in case of non null reference are reduced.

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