Analysis of a Lyapunov Function Behavior for Different Design Strategies

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Abstract: – The problem of analog system design for a minimal computer time has been formulated as the functional minimization problem of the control theory. The design process in this case is formulated as the controllable dynamic system. The optimal sequence of the control vector switch points was determined as a principal characteristic of the minimal-time system design algorithm. The conception of the Lyapunov function was proposed to analyze the behavior of design process. The special function that is a combination of the Lyapunov function and its time derivative was proposed to predict the design time of any strategy by means of the initial time interval analysis.

Key Words: – Minimal-time system design, control theory application, Lyapunov function.

1 Introduction

The problem of the computer time reduction of a large system design is one of the essential problems of the total quality design improvement. Besides the traditionally used ideas of sparse matrix techniques and decomposition techniques [1]-[5] some another ways were proposed to reduce the total computer design time [6]-[7]. The generalized approach for the analog system design on the basis of control theory formulation was elaborated in some previous works, for example [8]. This approach serves for the minimal-time design algorithm definition. On the other hand this approach gives the possibility to analyze with a great clearness the design process while moving along the trajectory curve into the design space. The main conception of this theory is the introduction of the special control functions, which, on the one hand generalize the design process and, on the other hand, they give the possibility to control design process to achieve the optimum of the design cost function for the minimal computer time. This possibility appears because practically an infinite number of the different design strategies that exist within the bounds of the theory. The different design strategies have the different operation number and executed computer time. On the bounds of this conception, the traditional design strategy is only a one representative of the enormous set of different design strategies. As shown in [8] the potential computer time gain that can be obtained by the new design problem formulation increases when the size and complexity of the system increase. However it is realized only in case when the algorithm for the optimal strategy of design is constructed.

2 Problem Formulation

The design process for any analog system design can be defined in discrete form [8] as the problem of the generalized cost function $F(X,U)$ minimization by means of the equation (1) with the constraints (2):

$$X^{s+1} = X^{s} + t_{s} \cdot H^{s} \quad (1)$$

$$\left(1 - u_{j}\right) g_{j}(X) = 0\,, \quad (2)$$

$$j = 1,2,\ldots, M$$

where $X \in R^{N}$, $X=(X',X'')$, $X' \in R^{K}$ is the vector of the independent variables and the vector $X'' \in R^{M}$ is the vector of dependent variables ($N=K+M$), $g_{j}(X)$ for all j presents the system model, s is the iterations number, t_{s} is the iteration parameter, $t_{s} \in R^{1}$, $H \equiv H(X,U)$ is the direction of the generalized cost function $F(X,U)$ decreasing, U is the vector of the special control functions

$U = (u_1, u_2, \ldots, u_m)$, where $u_j \in \Omega$; $\Omega = \{0;1\}$. The generalized cost function $F(X,U)$ is defined as:

$$F(X,U) = C(X) + \psi(X,U) \qquad (3)$$

where $C(X)$ is the non negative cost function of the design process, and $\psi(X,U)$ is the additional penalty function:

$$\psi(X,U) = \frac{1}{\varepsilon}\sum_{j=1}^{M} u_j \cdot g_j^2(X) \qquad (4)$$

This formulation of the problem permits to redistribute the computer time expense between the solution of problem (2) and the optimization procedure (1) for the function $F(X,U)$. The control vector U is the main tool for the redistribution process in this case. Practically an infinite number of the different design strategies are produced because the vector U depends on the optimization procedure current step. The problem of the optimal design strategy search is formulated now as the typical problem for the functional minimization of the control theory. The functional that needs to minimize is the total CPU time T of the design process. This functional depends directly on the operations number and on the design strategy that has been realized. The main difficulty of this definition is unknown optimal dependencies of all control functions u_j.

The continuous form of the problem definition is more adequate for the control theory application. This continuous form replaces Eq. (1) and can be defined by the next formula:

$$\frac{dx_i}{dt} = f_i(X,U), \qquad (5)$$

$$i = 0,1,\ldots,N$$

This system together with equations (2), (3) and (4) composes the continuous form of the design process. The structural basis of different design strategies that correspond to the fixed control vector includes 2^M design strategies. The functions of the right hand part of the system (5) are determined for example for the gradient method as:

$$f_i(X,U) = -\frac{\delta}{\delta x_i} F(X,U), \qquad (6)$$

$$i = 1,2,\ldots,K$$

$$f_i(X,U) = -u_{i-K}\frac{\delta}{\delta x_i}F(X,U) + \frac{(1-u_{i-K})}{t_s}\left\{-x_i^s + \eta_i(X)\right\} \qquad (6')$$

$$i = K+1, K+2, \ldots, N$$

where the operator $\frac{\delta}{\delta x_i}$ hear and below means $\frac{\delta}{\delta x_i}\varphi(X) = \frac{\partial \varphi(X)}{\partial x_i} + \sum_{p=K+1}^{K+M} \frac{\partial \varphi(X)}{\partial x_p}\frac{\partial x_p}{\partial x_i}$, x_i^s is equal to $x_i(t-dt)$; $\eta_i(X)$ is the implicit function ($x_i = \eta_i(X)$) that is determined by (2).

The function $f_0(X,U)$ is determined as the necessary time for one-step integration of the system (5). This function depends on the concrete design strategy. The additional variable x_0 is determined as the total computer time T for the system design. It is necessary to find the optimal behavior of the control functions u_j during the design process to minimize the total design computer time.

The idea of the system design problem formulation as the functional minimization problem of the control theory is not depend of the optimization method and can be embedded into any optimization procedures. In this paper the gradient method is used, nevertheless any optimization method can be used as shown in [8].

Now the analog system design process is formulated as a dynamic controllable system. The time-optimal design process can be defined as the dynamic system with the minimal transition time in this case. So we need to find the special conditions to minimize the transition time for this dynamic system.

3 Lyapunov function of design process

On the basis of the analysis in previous section we can conclude that the minimal-time algorithm has one or some switch points in control vector where the switching is realize among different design strategies. As shown in [9] it is necessary to switch the control vector from like modified traditional design strategy to like traditional design strategy with an additional adjusting. Some principal features of the time-optimal algorithm were determined previously. These are: 1) an additional acceleration effect that appeared under special circumstances [9]; 2) the start point special selection outside the separate hyper-surface to guarantee the acceleration effect, at least one negative component of the start value of the vector X is can be recommended for this; 3) an optimal structure of the control vector with the necessary switch points. The two first problems were discussed in [9-10].

The main problem of the time-optimal algorithm construction is unknown optimal sequence of the switch points during the design process. We need to define a special criterion that permits to realize the optimal or quasi-optimal algorithm by means of the optimal switch points searching. A Lyapunov function of dynamic system serves as a very informative object to any system analysis in the control theory. We suppose that the Lyapunov function can be used for the revelation of the optimal algorithm structure. First of all we can compare the behavior of the different design strategies by means of the Lyapunov function analysis.

There is a freedom of the Lyapunov function choice because of a non-unique form of this function. Let us define the Lyapunov function of the design process (2)-(6) by the following expression:

$$V(X,U) = [F(X,U)]^r \tag{7}$$

$$V(X,U) = \sum_i \left(\frac{\partial F(X,U)}{\partial x_i} \right)^2 \tag{8}$$

where *F(X,U)* is the generalized cost function of the design process. The formula (7) can be used when the general cost function is non-negative and has zero value at the stationary point *a*. Other formula can be used always because all derivatives $\partial F / \partial x_i$ are equal to zero in the stationary point *a*.

We can define now the design process as a transition process for controllable dynamic system that can provide the stationary point (optimal point of the design procedure) during some time. The problem of the time-optimal design algorithm construction can be formulated now as the problem of the transition process searching with the minimal transition time. There is a well-known idea [11]-[12] to minimize the time of transition process by means of the special choice of the right hand part of the principal system of equations, in our case these are the functions $f_i(X,U)$. It is necessary to change the functions $f_i(X,U)$ by means of the control vector *U* election to obtain the maximum speed of the Lyapunov function decreasing (the maximum absolute value of the Lyapunov function time derivative $\dot{V} = dV/dt$). Normally the time derivative of Lyapunov function is non-positive for the stable processes. However we define more informative function as a relatively time derivative of the Lyapunov function: $W = \dot{V}/V$. In this case we can compare the different design strategies by means of the function *W(t)* behavior and we can search the optimal position for the control vector switch points.

4 Numerical results

All examples were analyzed for the continuous form of the optimization procedure (5). Functions *V(t)* and *W(t)* were the main objects of the analysis and its behavior has been analyzed for all strategies that compose the structural basis of the general design methodology. The behavior of the functions *V(t)* and *W(t)* for the network of Fig. 1 is shown in Fig. 2. The nonlinear element has the following dependency: $y_{n1} = y_0 + b(V_1 - V_2)^2$. The vector *X* includes five components: $x_1^2 = y_1$, $x_2^2 = y_2$, $x_3^2 = y_3$, $x_4 = V_1$, $x_5 = V_2$. The model of this network (2) includes two equations (*M*=2) and the optimization procedure (5) includes five equations. The cost function $C(X)$ has been determined as the sum of the squared differences between beforehand-defined values and current values of the nodal voltages for two nodes with additional inequalities for some circuit elements.

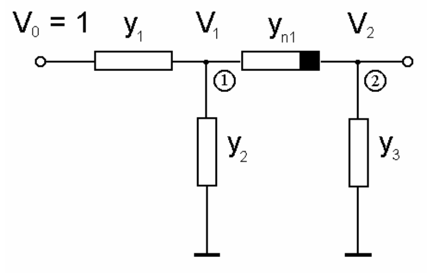

Figure 1. Two-node nonlinear passive network.

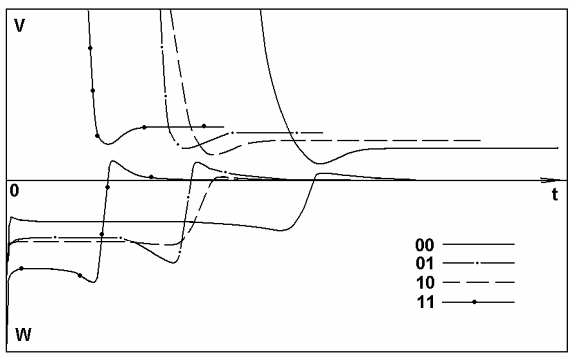

Figure 2. Behavior of the functions *V(t)* and *W(t)* for four design strategies during the design process for network in Fig.1.

The network in Fig. 1 is characterized by two dependent parameters (two nodal voltages) and the control vector includes two control functions: $U=(u_1, u_2)$. The structural basis of design strategies includes four design strategies: 00, 01, 10, and 11. The Lyapunov function was calculated by formula (8) for r=0.5. As we can see from Fig. 2 the functions *V(t)* and *W(t)* can give an exhaustive explanation for the design process characteristics. First of all we can conclude that the speed of decreasing of the Lyapunov function is inversely proportional to the design time. The minimal value of the Lyapunov function that corresponds to the maximum precision is approximately equal for all strategies and exactly is equal to 8.7_{10}-6, 1.7_{10}-5, 1.3_{10}-5, 2.0_{10}-5 for the strategies 00, 01, 10, 11 accordingly. We can see from Fig. 2 that after the minimal value decision the Lyapunov function increases a little. This small increasing corresponds to the small positive value of the Lyapunov function time derivative. Later on this derivative aspire to zero and the Lyapunov function has a permanent value.

The relative design time for four design strategies is equal to 1, 0.44, 0.78 and 0.3 for the strategies 00, 01, 10, 11 accordingly. This time was defined for the time point with the minimal value of the function *V*. As we can see from Fig. 2 a large absolute value of the function *W(t)* corresponds to a more rapid decreasing of the function *V(t)* and a smaller computer design time.

Another passive nonlinear network with three nodes (Fig. 3) was analyzed below.

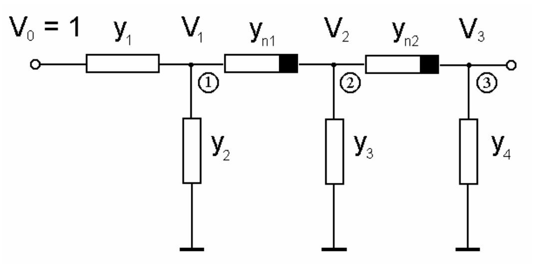

Figure 3. Three-node nonlinear passive network.

The nonlinear elements have been defined by following dependencies: $y_{n1} = a_{n1} + b_{n1} \cdot (V_1 - V_2)^2$, $y_{n2} = a_{n2} + b_{n2} \cdot (V_2 - V_3)^2$. The vector X includes seven components: $x_1^2 = y_1$, $x_2^2 = y_2$, $x_3^2 = y_3$, $x_4^2 = y_4$, $x_5 = V_1$, $x_6 = V_2$, $x_7 = V_3$. The model of this network (2) includes three equations (M=3) and the optimization procedure (5) includes seven equations. This network is characterized by three dependent parameters and the control vector includes three control functions: $U=(u_1, u_2, u_3)$. The behavior of the functions *V(t)* and *W(t)* for this network is shown in Fig. 4.

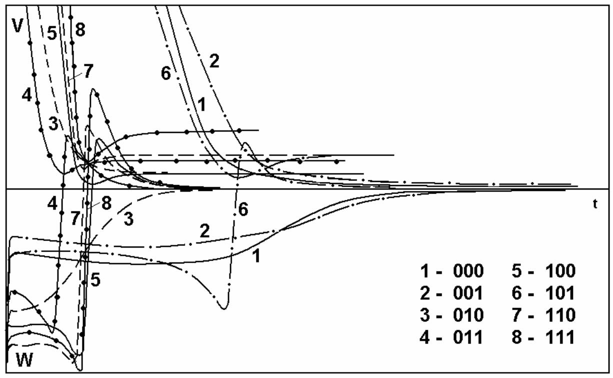

Figure 4. Behavior of the functions *V(t)* and *W(t)* for eight design strategies during the design process for network in Fig.3.

The structural basis of design strategies includes eight design strategies: 000, 001, 010, 011, 100, 101, 110 and 111. As for previous example, for the network in Fig.3 we also can conclude that the speed of decreasing of the Lyapunov function is inversely proportional to the design time. The minimal value of the Lyapunov function that corresponds to the maximum precision is in the limits from 1.2_{10}-5 for strategy 000 to 5.9_{10}-5 for strategy 111. We can see from Fig. 4 that the Lyapunov function increases a little for some strategies after the minimal value decision. The relative design time for all design strategies is equal to 1, 0.886, 0.569, 0.091, 0.129, 0.25, 0.131 and 0.105 for the strategies 000, 001, 010, 011, 100, 101, 110 and 111 accordingly. This time was defined for the time point with the minimal value of the function *V*. We can see from Fig. 4 that a large absolute value of the function *W(t)* corresponds to a more rapid decreasing of the function *V(t)* and a smaller computer design time. The strategies 011, 100, 110 and 111 have a large value of the function *W(t)* during all design process till a small value of the function *V(t)*. That is why these strategies have a relative little computer time.

Next example corresponds to the active network in Fig.5. The vector X includes six components: $x_1^2 = y_1$, $x_2^2 = y_2$, $x_3^2 = y_3$, $x_4 = V_1$, $x_5 = V_2$, $x_6 = V_6$. The model of this network (2) includes three equations (M=3) and the optimization procedure (5) includes six equations. The total structural basis contains eight different strategies. The control vector has three

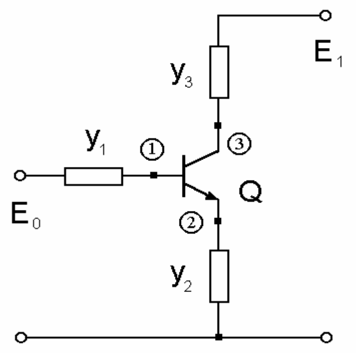

Figure 5. Three-node nonlinear active network.

components in this case and the structural basis consists of eight design strategies. The control vector includes three control functions: $U=(u_1, u_2, u_3)$. The Ebers-Moll static model of the transistor has been used.

As for the previous examples, Fig. 6 shows the behavior of the functions *V(t)* and *W(t)* for a time interval when the majority of the design strategies are finished.

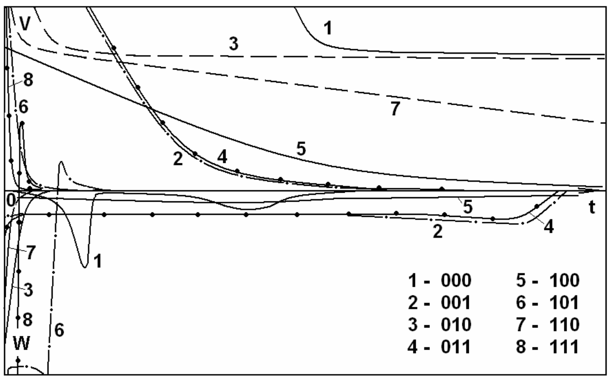

Figure 6. Behavior of the functions *V(t)* and *W(t)* for different design strategies during the design process for network in Fig.5.

The strategies with control vector 101 and 111 have extremely large value of the relative derivative *W(t)* from the beginning of the design process and that is why the Lyapunov function is decreases very rapidly. The relative design time is very small for two these strategies and it is equal to 0.00057 and 0.00018 accordingly. The strategies with the control vector 001, 011 and 100 have the sufficient level of the function *W* during the analyzed interval and the relative design time is equal to 0.0054, 0.0061 and 0.0114 accordingly. Nevertheless three other design strategies with the control vector 000, 010 and 110 are not finished during the presented interval. It occurs because the function *W* for these strategies decreases rapidly while the Lyapunov function had a relatively large value. After this the Lyapunov function decreases very slowly and the relative design time is equal to 1.0, 0.127 and 0.027 accordingly.

Other example corresponds to the network in Fig.7.

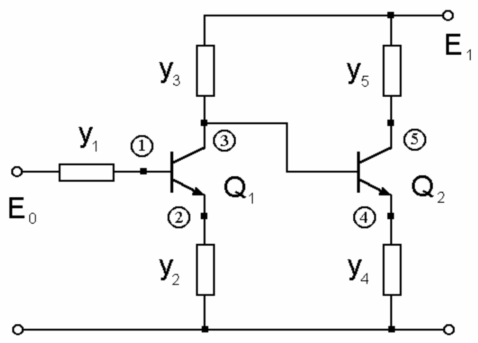

Figure 7. Five-node nonlinear active network.

This network is characterized by five dependent parameters and the control vector includes five control functions: $U=(u_1, u_2, u_3, u_4, u_5)$. The structural basis consists of 32 design strategies. Some graphs of all the structural basis strategies are presented in Fig. 8. These graphs correspond to a time interval when the majority of the design strategies are finished. The strategies 6, 7, 8 and 9 have a large value of the relative derivative *W* from the initial of the design process.

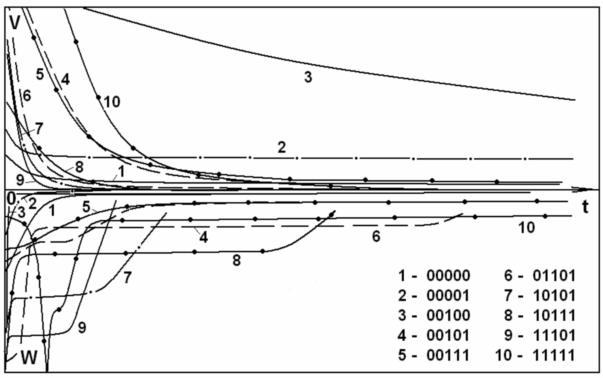

Figure 8. Behavior of the functions *V(t)* and *W(t)* for some design strategies during the design process for circuit in Fig. 7.

This property provides extremely fast decreasing of the Lyapunov function. The design time for these design strategies is presented in Table 1.

Table 1. Some strategies of the structural basis.

N	Control vector	Iterations number	Total design time (sec)
1	(0 0 0 0 0)	165962	299.56
2	(0 0 0 0 1)	337487	737.55
3	(0 0 1 0 0)	44118	68.87
4	(0 0 1 0 1)	14941	19.06
5	(0 0 1 1 1)	21971	22.03
6	(0 1 1 0 1)	4544	4.56
7	(1 0 1 0 1)	2485	1.65
8	(1 0 1 1 1)	7106	3.57
9	(1 1 1 0 1)	2668	1.32
10	(1 1 1 1 1)	79330	10.11

We can see that just these strategies 6, 7, 8 and 9 have the design time lesser than other strategies. The strategies 4, 5 and 10 have an average value of the function W in the initial part of the design process and these strategies have an average value of the design time. At last, the strategies 1, 2 and 3 have a large design time and just these strategies have a very fast decreasing of the function W during initial part of the design process when the Lyapunov function had a relatively large value. After this the Lyapunov function decreases very slowly and the design time for these strategies is large.

So, the main feature of the analyzed examples can be formulated by the next manner: the behavior of the Lyapunov function V and the relative time derivative W surely determine the design time. It means that it is possible be guided by means of these functions to predict the computer design time for any design strategy. We could analyzed the initial time interval of the functions $V(t)$ and $W(t)$ behavior for the different strategies and by this analysis we can predict the strategies that have a minimal computer design time.

5 Conclusion

The problem of the minimal-time design algorithm construction can be solved adequately on the basis of the control theory. The design process in this case is formulated as the controllable dynamic system. The Lyapunov function and its time derivative include the sufficient information to select more perspective design strategies from infinite set of the different design strategies that exist into the general design methodology. The special function $W(t)$ was proposed to compare the different design strategies and to predict the strategy that has a minimal design time. The successful solution of this problem permits to construct the minimal-time system design algorithm.

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