Multi-Time Stochastic Control Theory

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Abstract: The object of this paper is (i) to formulate the multi-time stochastic control theory, (ii) to propose a multi-time Itô chain rule, (iii) to find stochastic representations formulas for suitable functions, and (iv) to provide a connection between the multi-time stochastic control theory and multi-time dynamic programming. Section 1 defines and studies the multi-time random partial differential equations, the multi-time stochastic control theory, the multi-time Brownian flow and the multi-time Itô chain rule. Section 2 describes a stochastic representation formulas for harmonic functions, and for solutions of terminal-value problems associated to a heat type PDEs system. Section 3 studies the stochastic multi-time Hamilton-Jacobi-Bellman PDEs system. Section 4 formulates same conclusions regarding the original results of the paper.

Key–Words: Multi-time stochastic control theory, multi-time Itô-Pfaff stochastic system, multi-time dynamic programming, multi-time Hamilton-Jacobi-Bellman PDEs.

1 Multi-time stochastic control theory

In order to extend the theory of single-time stochastic control theory to the multi-time case, when the evolution in R^n is m-dimensional, we can formulate a path-integral stochastic optimal control or a multiple integral stochastic optimal control.

1.1 Multi-time stochastic partial differential equations

Let us formulate the theory of multi-time random PDE (partial differential equations). Consider the vector fields $F_\alpha = (F^i_\alpha),\ \alpha = 1,...,m;\ i = 1,...,n$, on R^n and the Cauchy problem

$$(PDE)\quad \frac{\partial x^i}{\partial t^\alpha}(t) = F^i_\alpha(x(t)),\ x(0) = x_0,\ t \in R^m_+,$$

associated to a completely integrable PDE system. By similarity, some evolution physical phenomena are described by a Cauchy problem

$$\frac{\partial X^i}{\partial t^\alpha}(t) = F^i_\alpha(X(t)) + \sigma\xi^i_\alpha(t),\ X(0) = x_0,\ t \in R^m_+,$$

attached to a *stochastic completely integrable augmented PDE system*, where $X(\cdot) = (X^i(\cdot))$ are *random variables* representing the *stochastic process*, and $\xi_\alpha(\cdot) = (\xi^i_\alpha(\cdot))$ denote *white noises* terms causing random fluctuations.

We extend the previous point of view in two ways:

- the smooth nonholonomic case, re-writing as a stochastic Cauchy-Pfaff problem

$$dX^i(t) = F^i_\alpha(X(t))dt^\alpha + \sigma\xi^i_\alpha(t)dt^\alpha$$

$$X(0) = x_0,\ t \in R^m_+;$$

- the eventually non-smooth case, as a Cauchy problem involving path-dependent stochastic integrals

$$X^i(t) = x^i_0 + \int_{\gamma_{0,t}} (F^i_\alpha(X(s)) + \sigma\xi^i_\alpha(t))ds^\alpha,$$

where $\gamma_{0,t}$ is a piecewise C^1 curve joining the points 0 and t in R^m_+.

A solution $X(\cdot)$ is a collection of sample m-sheets of a stochastic process, plus probabilistic information as to the likelihood of the various multi-time evolutions.

1.2 Multi-time stochastic control theory

On R^m_+ we use the product order. Then, a multi-time interval $[t_0, t] \subset R^m_+$ is identified to the hyper-parallelepiped $\Omega_{t_0,t}$, where t_0, t are diagonal opposite points. The functions $F_\alpha : R^n \times U \to R^n,\ U \subset R^k$

determine the *multi-time controlled stochastic Cauchy problem*

$$(SPDE) \quad \frac{\partial X}{\partial s^\alpha}(s) = F_\alpha(X(s), U(s)) + \sigma\xi_\alpha(s)$$
$$X(t) = x_0,\ s \in \Omega_{t,t_0},$$

under the hypothesis of complete integrability.

A mapping $U(\cdot)$ of Ω_{t,t_0} into $U \subset R^k$, such that for each multi-time $t \leq s \leq t_0$ the value $U(s)$ depends only on s and observations of $X(\tau)$ for $t \leq \tau \leq s$ is called *control*. The corresponding *cost functional* is the expected values over all sample m-sheets for the solution of $(SPDE)$.

There are two different approaches when dealing with cost functionals in the multi-time context. One variant is to use the *curvilinear integral functional*:

$$P_{x,t}(U(\cdot)) =$$
$$= E\left\{\int_{\gamma_{t,t_0}} F^0_\beta(X(s), U(s))ds^\beta + g(X(t_0))\right\},$$

where the *runing cost* $F^0_\beta(X(s), U(s))ds^\beta$ is an integrable 1-form and $g(X(t_0))$ is *the terminal cost*. Another variant is to use a *multiple integral functional*:

$$P_{x,t}(U(\cdot)) =$$
$$= E\left\{\int_{\Omega_{t,t_0}} F^0(X(s), U(s))ds^1...ds^m + g(X(t_0))\right\}.$$

The main goal is to find an optimal control $U^*(\cdot)$ such that

$$P_{x,t}(U^*(\cdot)) = \max_{U(\cdot)} P_{x,t}(U(\cdot)).$$

To do that we adapt the multi-time dynamic programming methods, introducing the *sup value function*

$$v(x,t) = \sup_{U(\cdot)} P_{x,t}(U(\cdot)).$$

For finding an optimal control $U^*(\cdot)$ we follow two steps:

- we look for a multi-time Hamilton-Jacobi-Bellman type PDEs system satisfied by the function $v(x,t)$;

- we use the solution $v(x,t)$ in designing the optimal control $U^*(\cdot)$.

Of course, the stochastic effects modify the structure of the multi-time Hamilton-Jacobi-Bellman type PDEs system $(mtHJB)$, as compared with deterministic case.

In the next explanations we use only the curvilinear integral cost functional.

1.3 Multi-time Brownian flow

A multi-time stochastic process $W(t)$ is called a *Wiener process* or *Brownian m-flow (motion)* if

1) $W(0) = 0$;

2) $W(t),\ t = (t^1, ..., t^m)$ is Gaussian with $\mu = 0,\ \sigma^2_t = \mathrm{vol}\,\Omega_{0,t}$, i.e., $W(t)$ is $N(0, \sigma^2_t)$.

These imply:

3) each sample m-sheet is at least continuous;

4) for all choices of multi-times, $0 < t_1 < \cdots < t_k$, the random variables

$$W(t_1),\ W(t_2) - W(t_1),\ \cdots,\ W(t_k) - W(t_{k-1})$$

are independent (independent increments).

In this context,

$$E(W(t)) = 0$$

$$E(W^2(t)) = \mathrm{vol}\,\Omega_{0,t} = t^1...t^m = \int_{\gamma_{0,t}} d(\tau^1...\tau^m).$$

Define a Brownian m-flow (motion) on R^n as $W(t) = (W^1(t), ..., W^n(t))$, where each $W^i(t)$ is an independent Brownian m-flow (m-motion) on R. Let us accept that $\xi_\alpha(t) = \frac{\partial W}{\partial t^\alpha}(t)$, though the function $t \to W(t,\omega)$ is usually nowhere differentiable. We introduce the *multi-time stochastic PDE*

$$\frac{\partial X}{\partial t^\alpha}(t) = F_\alpha(X(t)) + \sigma\xi_\alpha(t),\ t \in R^m_+,$$

where we informally think of $\xi_\alpha(\cdot) = \frac{\partial W}{\partial t^\alpha}(\cdot)$. Also we accept that the conditions of complete integrability are satisfied, though in many problems this fact is not necessary (see the nonholonomic case and the existence conditions for an integral). Adding the initial point $X(0) = x_0$, we obtain a stochastic Cauchy problem. A multi-time stochastic process is solution of this Cauchy problem if and only if it solves the path-integral equation

$$X(t) = x_0 + \int_{\gamma_{0,t}} F_\alpha(X(s))ds^\alpha + \sigma W(t),$$

where $\gamma_{0,t}$ is an arbitrary C^1 curve joining the points 0 and t in R^m_+. On the other hand, this integral equation can be solved by the method of successive approximation:

$$X_0(t) = x_0$$
$$X_{k+1}(t) = x_0 + \int_{\gamma_{0,t}} F_\alpha(X_k(s))ds^\alpha + \sigma W(t)$$
$$lim_{k\to\infty} X_k(t) = X(t).$$

Now, let us consider a more general completely integrable multi-time stochastic PDE system

$$\frac{\partial X^i}{\partial t^\alpha}(t) = F^i_\alpha(X(t)) + H^i_j(X(t))\xi^j_\alpha(t),\ t \in R^m_+,$$

which can be written

$$\frac{\partial X^i}{\partial t^\alpha}(t) = F^i_\alpha(X(t)) + H^i_j(X(t))\frac{\partial W^j}{\partial t^\alpha}(t)$$

or as an *Itô-Pfaff stochastic system*

$$dX^i(t) = F^i_\alpha(X(t))dt^\alpha + H^i_j(X(t))dW^j(t).$$

By analogy with the foregoing, we say $X(\cdot)$ is a solution, with the initial condition $X(0) = x_0$, if

$$X(t) = x_0 + \int_{\gamma_{0,t}} F_\alpha(X(s))ds^\alpha + H_j(X(s))dW^j(s),$$

where $\gamma_{0,t}$ is a piecewise C^1 curve joining the points 0 and t in R^m_+. The parts $\int_{\gamma_{0,t}} H_j(X(s))dW^j(s)$ are called *Itô stochastic curvilinear integrals*.

Remark. Let $W(\cdot)$ be a multi-time Brownian flow. A process $Y(\cdot)$ with the property that $Y(s),\ 0 \le s \le t$ depends on $W(\tau)$ for $0 \le \tau \le t$, but not on $W(\tau)$ for $s \le \tau$, is called *nonanticipating*. These elements determine the *Itô stochastic curvilinear integrals* $\int_{\gamma_{0,t}} H_j(X(s))dW^j(s)$, where $\gamma_{0,t}$ is an arbitrary C^1 curve joining the points 0 and t in R^m_+. The most important property of such integrals is $E\left(\int_{\gamma_{0,t}} H_j(X(s))dW^j(s)\right) = 0$.

1.4 Multi-time Itô chain rule

As is well-known the *chain rule* in a single-time *Itô calculus* contains additional terms as compared with the usual chain rule from differential calculus. To pass to the multi-time case and to justify how appear new additional terms, we shall consider a suitable multi-time *Cauchy-Itô-Pfaff stochastic problem*

$$dX^i(t) = F^i_\alpha(X(t))dt^\alpha + \sigma dW^i(t),\ X^i(0) = x^i_0,$$

and we define the composed function $Y(t) = u(X(t), t),\ u : R^n \times R^m_+ \to R$. Let us accept the approximative formula

$$dY(t) \approx \frac{\partial u}{\partial t^\alpha}(X(t), t)dt^\alpha + \frac{\partial u}{\partial x^i}(X(t), t)dX^i(t)$$

$$+\frac{1}{2}\frac{\partial^2 u}{\partial x^i \partial x^j}(X(t), t)dX^i(t)dX^j(t).$$

At least from geometrical point of view, the generalization of the single-time Itô chain rule to the multi-time case must bifurcates as follows.

First Itô chain rule. Firstly, we use the euristic rules

$$dW^i dW^j = \delta^{ij}ds,\ ds^2 = \delta_{\alpha\beta}dt^\alpha dt^\beta$$

(arclength element, degree one in dt^α), plug these identities into the previous formula and keep only terms of degree one in dt^α. We obtain the *first Itô chain rule*:

$$dY(t) = \frac{\partial u}{\partial t^\alpha}(X(t), t)dt^\alpha + \frac{\partial u}{\partial x^i}(X(t), t)(F^i_\alpha(X(t))dt^\alpha$$

$$+\sigma dW^i(t)) + \frac{\sigma^2}{2}\Delta u(X(t), t)ds.$$

Second Itô chain rule. Secondly, we use the euristic rules

$$dW^i dW^j = \delta^{ij} c_\alpha(W)dt^\alpha$$

(linear in dt^α), plug these identities into the previous formula and keep only terms which are linear in dt^α. We obtain the *second Itô chain rule*:

$$dY(t) = \frac{\partial u}{\partial t^\alpha}(X(t), t)dt^\alpha + \frac{\partial u}{\partial x^i}(X(t), t)(F^i_\alpha(X(t))dt^\alpha$$

$$+\sigma dW^i(t)) + \frac{\sigma^2}{2}\Delta u(X(t), t)c_\alpha(X(t))dt^\alpha.$$

Remark. From geometrical point of view, we can replace automatically the pair of Euclidean spaces $(R^m, \delta_{\alpha\beta}),\ (R^n, \delta_{ij})$ with a pair of simple Riemannian manifolds $(R^m, h_{\alpha\beta}),\ (R^n, g_{ij})$, or more generally $(T, h),\ (M, g)$, extending nontrivially the multi-time Itô chain rule. In this context, the natural domain of Lagrange functions is the first order jet bundle $J^1(T, M)$.

2 Stochastic representation formulas for solutions of PDEs

Let us show that the solutions of multi-time elliptic and parabolic PDEs, both with Cauchy and Dirichlet boundary conditions, have a probabilistic interpretation, which not only provides intuition on the nature of the problems described by these PDEs, but it is quite useful in the proof of general theorems.

2.1 A stochastic representation formula for harmonic functions

Let D be a domain in R^n with the boundary ∂D. The solution of the boundary-value problem

$$\Delta u(x) = 0,\ x \in D;\ u(x) = g(x),\ x \in \partial D$$

is called *harmonic function*.

In order to find a stochastic representation formula for the harmonic function u, we consider the random process $X(t) = W(t) + x,\ t = (t^1, ..., t^m)$, i.e.,

$$dX(t) = dW(t),\ X(0) = x,\ t \in R^m_+,$$

where $W(\cdot)$ denotes an m-dimensional Brownian sheet. The compound function $Y(t) = u(X(t))$ has a first Itô differential

$$dY(t) = \frac{\partial u}{\partial x^i}(X(t))dW^i(t) + \frac{\sigma^2}{2}\Delta u(X(t))ds.$$

Consequently

$$dY(t) = \frac{\partial u}{\partial x^i}(X(t))dW^i(t)$$

or as Stieltjes curvilinear integral

$$u(X(t)) = Y(t) = Y(0) + \int_{\gamma_{0,t}} \frac{\partial u}{\partial x^i}(X(s))dW^i(s),$$

where $\gamma_{0,t}$ is a curve joining the points 0 and t.

Let τ denote the random first multi-time the sample m-sheet hits ∂D. Then, replacing $t = \tau$, we obtain

$$u(x) = u(X(\tau)) - \int_{\gamma_{0,\tau}} \frac{\partial u}{\partial x^i}(X(s))dW^i(s).$$

On the other hand, $u(X(\tau)) = g(X(\tau))$ by the definition of τ, and hence $u(x) = E[X(\tau)]$. Consequently, to recover the solution $u(x)$ of the previous boundary problem, we need to consider all the sample m-sheets of the *multi-time Brownian flow* starting at the point x and take the average $g(X(\tau))$.

2.2 A stochastic representation formula for solution of heat PDE system

Let us consider the terminal-value problem associated to a *nonhomogeneous backwards heat system*

$$\frac{\partial u}{\partial t^\alpha}(x,t) + \frac{\sigma^2}{2}c_\alpha(u)\Delta u(x,t) = f_\alpha(x,t)$$

$$x \in R^n,\ 0 \le t \le t_0,\ u(x,t_0) = g(x).$$

Fixing $x \in R^n$ and $0 \le t < t_0$, we introduce the multi-time stochastic process $dX(s) = \sigma dW(s)$, $s \ge t$, $X(t) = x$. The second Itô chain rule permits to write

$$du(X(s),s) = \frac{\partial u}{\partial s^\alpha}(X(s),s)ds^\alpha + \frac{\partial u}{\partial x^i}(X(s),s)dX^i(s)$$

$$+\frac{\sigma^2}{2}\Delta u(X(s),s)c_\alpha(u)ds^\alpha.$$

Taking the curvilinear integral, we obtain

$$u(X(t_0),t_0) = u(X(t),t) + \int_{\gamma_{0,t}} \left(\frac{\sigma^2}{2}\Delta u(X(s),s)c_\alpha(u)\right.$$

$$\left. + \frac{\partial u}{\partial s^\alpha}(X(s),s)\right) ds^\alpha + \int_{\gamma_{0,t}} \frac{\partial u}{\partial x^i}(X(s),s)dW^i(s).$$

Since u solves the previous terminal-value problem, we can write

$$u(x,t) = E\left(g(X(t_0)) - \int_{\gamma_{0,t}} f_\alpha(X(s),s)ds^\alpha\right).$$

This is a stochastic representation formula for the solution u of the initial nonhomogeneous backwards heat system problem.

3 Multi-time stochastic control theory and dynamic programming

The *multi-time controlled stochastic Cauchy-Itô-Pfaff problem*

$$dX(s) = F_\alpha(X(s),U(s))ds^\alpha + \sigma dW(s)$$

$$X(t) = x,\ s \in \Omega_{t,t_0}$$

is equivalent to

$$X(\tau) = x + \int_{\gamma_{t,\tau}} F_\alpha(X(s))ds^\alpha + \sigma(W(\tau) - W(t)),$$

where $\gamma_{t,\tau}$ is a piecewise C^1 curve joining the points t and τ in Ω_{t,t_0}. We add the *cost functional*

$$P_{x,t}(U(\cdot)) =$$

$$E\left\{\int_{\gamma_{t,\tau}} F^0_\beta(X(s),U(s))ds^\beta + g(X(t_0))\right\},$$

where the *runing cost* $F^0_\beta(X(s),U(s))ds^\beta$ is an integrable 1-form and $g(X(t_0))$ are *terminal cost*. These produce the *sup value functions*

$$v(x,t) = \sup_{U(\cdot)} P_{x,t}(U(\cdot))$$

which permits to pass to the method of dynamic programming in two steps:

- find a PDE system satisfied by the function $v(x,t)$;

- use this PDE system to design an optimum control $U^*(\cdot)$.

3.1 Multi-time stochastic Hamilton-Jacobi-Bellman PDE system

Let $U(\cdot)$ be an arbitrary control used for multi-time $s \in \Omega_{t,t+h}$, where $h > 0$. Thereafter we employ an optimal control. It appears the inequality

$$v(x,y) \geq E\left\{\int_{\gamma_{t,t+h}} F_\beta(X(s),U(s))ds^\beta \right.$$
$$\left. + v(X(t+h),t+h)\right\}$$

with equality for an optimal control $U(\cdot) = U^*(\cdot)$, where $\gamma_{t,t+h}$ is a piecewise C^1 curve joining the points t and $t+h$. For an arbitrary control we can write

$$0 \geq E\left\{\int_{\gamma_{t,t+h}} F_\beta(X(s),U(s))ds^\beta \right.$$
$$\left. + v(X(t+h),t+h) - v(x,t)\right\} =$$
$$= E\left\{\int_{\gamma_{t,t+h}} F_\beta(X(s),U(s))ds^\beta\right\}$$
$$+E\left\{v(X(t+h),t+h) - v(x,t)\right\}.$$

Using second Itô formula,

$$dv(X(s),s) =$$
$$= \frac{\partial v}{\partial s^\beta}(X(s),s)ds^\beta + \frac{\partial v}{\partial x^i}(X(s),s)dX^i(s)$$
$$+\frac{1}{2}\frac{\partial^2 v}{\partial x^i \partial x^j}(X(s),s)dX^i(s)dX^j(s)$$
$$= \frac{\partial v}{\partial s^\beta}ds^\beta + \frac{\partial v}{\partial x^i}(F^i_\beta ds^\beta + \sigma dW^i(s)) + \frac{\sigma^2}{2}c_\beta ds^\beta \Delta v,$$

we can write

$$v(X(t+h),t+h) - v(X(t),t) =$$
$$\int_{\gamma_{t,t+h}}\left(\frac{\partial v}{\partial s^\beta} + \frac{\partial v}{\partial x^i}F^i_\beta + \frac{\sigma^2}{2}c_\beta \Delta v\right)ds^\beta$$
$$+\int_{\gamma_{t,t+h}} \sigma \frac{\partial v}{\partial x^i} dW^i(s).$$

Taking expected values, we deduce

$$E\{v(X(t+h),t+h) - v(X(t),t)\} =$$
$$E\left\{\int_{\gamma_{t,t+h}}\left(\frac{\partial v}{\partial s^\beta} + \frac{\partial v}{\partial x^i}F^i_\beta + \frac{\sigma^2}{2}c_\beta \Delta v\right)ds^\beta\right\}.$$

The previous relations imply

$$0 \geq E\left\{\int_{\gamma_{t,t+h}}\left(\frac{\partial v}{\partial s^\beta} + F^0_\beta + \frac{\partial v}{\partial x^i}F^i_\beta \right.\right.$$
$$\left.\left. + \frac{\sigma^2}{2}c_\beta \Delta v\right)ds^\beta\right\}.$$

Let us convert the previous inequality into a partial derivative inequality. We set $h = \varepsilon e_\beta$, $\varepsilon > 0$, and we write

$$0 \geq E\left\{\frac{1}{\varepsilon}\int_{\gamma_{t,t+h}}(\frac{\partial v}{\partial s^\beta}(X(s),s) + F^0_\beta(X(s),U(s)) + \right.$$
$$\left.\frac{\partial v}{\partial x^i}(X(s),s)F^i_\beta(X(s),U(s)) + \frac{\sigma^2}{2}c_\beta \Delta v(X(s),s)\right\}ds^\beta.$$

Taking $\varepsilon \to 0$, and having in mind $X(t) = x$, $U(t) = u \in U$, we find

$$0 \geq \frac{\partial v}{\partial t^\beta}(x,t) + F^0_\beta(x,u) + \frac{\partial v}{\partial x^i}(x,t)F^i_\beta(x,u)$$
$$+\frac{\sigma^2}{2}c_\beta \Delta v(x,t).$$

This inequality holds for any x, t, u, with equality for an optimal control, i.e.,

$$max_{u\in U}\left\{\frac{\partial v}{\partial t^\beta} + F^0_\beta + \frac{\partial v}{\partial x^i}F^i_\beta + \frac{\sigma^2}{2}c_\beta \Delta v\right\} = 0.$$

Theorem (multi-time stochastic Hamilton-Jacobi-Bellman PDE system). *The maximum value function v associated to a multi-time stochastic control problem is a solution of the $(mtSHJB)$ problem*

$$\frac{\partial v}{\partial t^\beta}(x,t) + \frac{\sigma^2}{2}c_\beta(v)\Delta v(x,t)$$
$$+\max_{u\in U}\left\{F^0_\beta(x,u) + \frac{\partial v}{\partial x^i}(x,t)F^i_\beta(x,u)\right\}$$
$$v(x,t_0) = g(x).$$

In this way, the multi-time stochastic Hamilton-Jacobi-Bellman PDE system consists in semilinear parabolic $PDEs$.

3.2 Designing an optimal control

Suppose that we know a solution v of the $(mtSHJB)$ problem. For each point (x,t), we compute a value $u \in U$ for which $F^0_\beta(x,u) + \dfrac{\partial v}{\partial x^i}(x,t)F^i_\beta(x,u)$ is maximum, i.e., for each (x,t) we choose $u = \alpha(x,t)$ as the point of maximum. Then we solve the multi-time controlled stochastic Cauchy problem

$$dX^*(s) = F_\beta(X^*(s), \alpha(X^*(s), s))ds^\beta + \sigma dW(s)$$

$$X^*(t) = x,$$

assuming this is possible. We find $X^*(s)$, and then $U^*(s) = \alpha(X^*(s), s)$ is an optimal feedback control.

4 Conclusion

This paper studies the stochastic optimal control problems involving curvilinear integral cost functionals constrained by stochastic evolution PDEs (infinite-dimensional systems), combining the ideas of [1]-[11] with several additional ingredients. These problems are of special interest in a variety of applications, e.g., image processing, geometric optics and stochastic differential games. The topics include: multi-time stochastic partial differential equations, multi-time stochastic control theory, multi-time Brownian flow, two variants of multi-time Itô chain rule, a stochastic representation formula for harmonic functions, a stochastic representation formula for solution of heat PDE system, multi-time stochastic Hamilton-Jacobi-Bellman PDE system and designing an optimal control.

The previous theory is sometimes imprecise, but we can introduce rigorous derivations. It offers open problems for the researchers involved into optimal stochastic theories.

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