Exact 3-D Solution for System with Rectangular Fin, Part 1

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Abstract: - In this paper we construct several exact analytical three-dimensional solutions for the distribution of the temperature field in the wall with rectangular fin. We assume that the heat transfer process in the wall and the fin is stationary. These exact solutions are obtained by the Green function method in the form of the 2nd kind Fredholm integral equation. They generalize traditional statements in several senses, e.g., we consider 3-D statement by different boundary conditions and the heat exchange take place at non-homogeneous environmental temperature.

Key-Words: - steady-state, three-dimensional, heat exchange, rectangular fin, non-homogeneous environment, exact analytical solutions.

1 Introduction

Systems with extended surfaces (fins, spines) are related to refrigerators, radiators, engines and modern electronics (PC), etc. Usually their mathematical modeling is realized by one dimensional steady-state assumptions [1]-[5]. In our previous papers we have constructed various two dimensional analytical approximate [6] – [10] and exact [11] solutions. In this paper we concentrate our attention on one element of fin assembly, the whole system (assembled into arrays of fins) will be considered in the second paper. Such statement essentially generalizes the problem considered earlier in literature, e.g., in paper [12]. In these two parts of our paper we obtain several new exact analytical solutions by the Green function method [13]-[16].

2 Mathematical Formulation of 3-D Problem

In this part 1 we will consider full mathematical three-dimensional formulation of steady-state problem for one element of system with rectangular fin (this one element is depicted with darker color in attached figure). This mathematical formulation is essentially broader as in our papers [6]-[11].

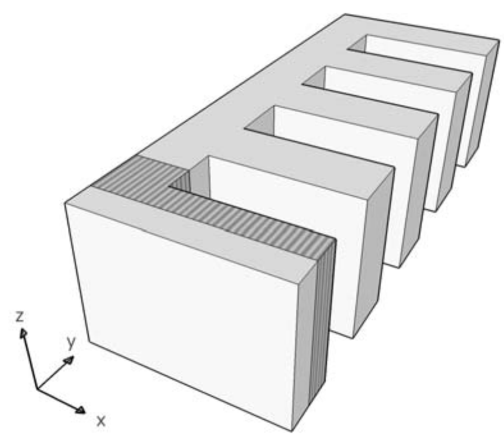
We will use following dimensionless arguments, parameters:

$$x = \frac{x'}{B+R}, y = \frac{y'}{B+R}, z = \frac{z'}{B+R}, \delta = \frac{\Delta}{B+R},$$

$$l = \frac{L}{B+R}, \; b = \frac{B}{B+R}, \; \beta_0^0 = \frac{h_0(B+R)}{k_0},$$

$$\beta_0 = \frac{h(B+R)}{k_0}, \; \beta = \frac{h(B+R)}{k}, \; w = \frac{W}{B+R}$$

and temperatures:



$$\bar{V}(x,y,z) = \frac{\tilde{V}(x,y,z) - T_a}{T_b - T_a},$$

$$\bar{V}_0(x,y,z) = \frac{\tilde{V}_0(x,y,z) - T_a}{T_b - T_a},$$

$$\Theta(x,y,z) = \frac{\tilde{\Theta}(x,y,z) - T_a}{T_b - T_a},$$

$$\Theta_0(y,z) = \frac{\tilde{\Theta}_0(y,z) - T_a}{T_b - T_a}.$$

We have introduced following dimensional thermal and geometrical parameters: $k(k_0)$ - heat conductivity coefficient for the fin (wall), $h(h_0)$ - heat exchange coefficient for the fin (wall), $2B$ – fin width (thickness), L –fin length, Δ - thickness of the wall, W – walls' width (length), $2R$ – distance between two fins (fin spacing). Further, $\tilde{\Theta}_0(y,z)$ is the surrounding (environment) temperature on the left (hot) side (the heat source side) of the wall, $\tilde{\Theta}(x,y,z)$ - the surrounding temperature on the right (cold - the heat sink side) of the wall and the fin. Finally, $\tilde{V}(x,y,z)$ ($\tilde{V}_0(x,y,z)$) are the dimensional temperatures in the fin (wall), where $T_a(T_b)$ are integral averaged environment temperatures over appropriate edges:

$$T_a = \left[(B+R+L)W \right]^{-1} [\int_0^W dz \int_B^{B+R} \Theta(\Delta,y,z)dy + \int_\Delta^{\Delta+L} dx \int_0^W \Theta(x,B,z)dz + \int_0^W dz \int_0^B \Theta(\Delta+L,y,z)dy],$$

$$T_b = \left[W(B+R) \right]^{-1} \int_0^{B+R} dy \int_0^W \Theta_0(y,z)dz.$$

The one element of the wall (base) is placed in the domain $\{x \in [0,\delta], y \in [0,1], z \in [0,w]\}$. The rectangular fin in dimensionless arguments occupies the domain $\{x \in [\delta,\delta+l], y \in [0,b], z \in [0,w]\}$. We describe the dimensionless temperature field by function $\bar{V}_0(x,y,z)$ $(\bar{V}(x,y,z))$ in the wall (fin). They fulfill the Laplace equations:

$$\frac{\partial^2 \bar{V}_0}{\partial x^2} + \frac{\partial^2 \bar{V}_0}{\partial y^2} + \frac{\partial^2 \bar{V}_0}{\partial z^2} = 0,$$

$$\frac{\partial^2 \bar{V}}{\partial x^2} + \frac{\partial^2 \bar{V}}{\partial y^2} + \frac{\partial^2 \bar{V}}{\partial z^2} = 0.$$

At first we consider the three dimensional statement with given heat fluxes from the flank surfaces (edges) and from the top and the bottom edges:

$$\left.\frac{\partial \bar{V}_0}{\partial z}\right|_{z=0} = Q_{0,2}(x,y), \left.\frac{\partial \bar{V}_0}{\partial z}\right|_{z=w} = Q_{0,3}(x,y),$$
$$\left.\frac{\partial \bar{V}}{\partial z}\right|_{z=0} = Q_2(x,y), \left.\frac{\partial \bar{V}}{\partial z}\right|_{z=w} = Q_3(x,y). \quad (1)$$

Such type of boundary conditions (BC) allows us to make the exact reducing of this three-dimensional problem to two-dimensional problem for Poisson equation by conservative averaging method [17]-[20]. Let us introduce following integral average values:

$$V_0(x,y) = w^{-1} \int_0^w \bar{V}_0(x,y,z)dz,$$
$$\vartheta_0(y) = w^{-1} \int_0^w \Theta_0(y,z)dz,$$
$$V(x,y) = w^{-1} \int_0^w \bar{V}(x,y,z)dz,$$
$$\vartheta(x,y) = w^{-1} \int_0^w \Theta(x,y,z)dz. \quad (2)$$

It remains to realize the integration of main equation by usage of the both BC (1) (corresponding one pair) and we obtain:

$$\frac{\partial^2 V_0}{\partial x^2} + \frac{\partial^2 V_0}{\partial y^2} + Q_0(x,y) = 0,$$
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + Q(x,y) = 0. \quad (3)$$

Here

$$Q_0(x,y) = \frac{1}{w}\left(Q_{0,3}(x,y) - Q_{0,2}(x,y)\right),$$

$$Q(x,y) = \frac{1}{w}\left(Q_3(x,y) - Q_2(x,y)\right).$$

We add to main partial differential equations (3) needed BC as follow:

$$\left.\left\{ \frac{\partial V_0}{\partial x} + \beta_0^0 \left[\vartheta_0(y) - V_0 \right] \right\}\right|_{x=0} = 0, y \in (0,1), \quad (4)$$

$$\left.\left\{ \frac{\partial V_0}{\partial x} + \beta_0 \left[V_0 - \vartheta(x,y) \right] \right\}\right|_{x=\delta} = 0, y \in (b,1), \quad (5)$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=0} = Q_{0,0}(x), \quad (6)$$

$$\left.\frac{\partial V_0}{\partial y}\right|_{y=1} = Q_{0,1}(x). \quad (7)$$

We allow the material of the fin to be different from the walls' material. It means we must formulate the conjugations conditions on the surface between the wall and the fin. We assume them as ideal thermal contact - there is no contact resistance:

$$V_0\big|_{x=\delta-0} = V\big|_{x=\delta+0}, \quad (8)$$

$$\beta \frac{\partial V_0}{\partial x}\bigg|_{x=\delta-0} = \beta_0 \frac{\partial V}{\partial x}\bigg|_{x=\delta+0}. \quad (9)$$

We have following BC for the fin:

$$\left\{\frac{\partial V}{\partial x} + \beta\left[V - \vartheta(x,y)\right]\right\}\bigg|_{x=\delta+l} = 0, y \in [0,b], \quad (10)$$

$$\frac{\partial V}{\partial y}\bigg|_{y=0} = Q_1(x), \quad (11)$$

$$\left\{\frac{\partial V}{\partial y} + \beta\left[V - \vartheta(x,y)\right]\right\}\bigg|_{y=b} = 0, x \in [\delta, \delta + l]. \quad (12)$$

We assume that all conditions which ensure existence and uniqueness of classic solution of the problem (3)-(12), e.g. continuity of environment temperatures, consistency conditions on the sides of edges etc. are fulfilled.

Let's mention, that almost all of the authors negligible the heat transfer trough flank surface $z = w$ (as well as from edge $z = 0$). We assume given (prescribed) heat fluxes on both.

3 Exact Solution of 2-D Problem

3.1 Solution of the Simplified Problem

We would like to explain the main idea of solution for the 2-D case of periodical system with constant dimensionless environmental temperatures $\vartheta_0 = 1\left(\Theta_0 = T_b\right)$ and $\vartheta = 0(\Theta = T_a)$. We neglect additionally the heat fluxes from flank edges. In this particular case we have following main equations for the temperature $U_0(x,y)$ of the wall, respectively temperature $U(x,y)$ of the fin:

$$\frac{\partial^2 U_0}{\partial x^2} + \frac{\partial^2 U_0}{\partial y^2} = 0, \left\{x \in [0,\delta], y \in [0,1]\right\}, \quad (13)$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0, \left\{x \in [\delta, \delta + l], y \in [0,b]\right\}. \quad (14)$$

The BC (6), (7) and (12) are assumed to be homogeneous:

$$\frac{\partial U_0}{\partial y}\bigg|_{y=0} = \frac{\partial U_0}{\partial y}\bigg|_{y=1} == \frac{\partial U}{\partial y}\bigg|_{y=0} = 0. \quad (15)$$

Instead of BC (4), (5), (10) and (11) we have:

$$\left[\frac{\partial U_0}{\partial x} + \beta_0^0\left(1 - U_0\right)\right]_{x=0} = 0, y \in (0,1), \quad (16)$$

$$\left(\frac{\partial U_0}{\partial x} + \beta_0 U_0\right)\bigg|_{x=\delta} = 0, y \in (b,1), \quad (17)$$

$$\left(\frac{\partial U}{\partial x} + \beta U\right)\bigg|_{x=\delta+l} = 0, y \in [0,b], \quad (18)$$

$$\left(\frac{\partial U}{\partial y} + \beta U\right)\bigg|_{y=b} = 0, \ x \in [\delta, \delta + l]. \quad (19)$$

The conjugations conditions on the line between the wall and the fin are still standing in the form (8), (9) for the functions $U(x,y)$ and $U_0(x,y)$. The linear combination of the equations (8), (9) together with BC (17) allow us rewrite them as following BC on the right hand side of the wall:

$$\left(\frac{\partial U_0}{\partial x} + \beta_0 U_0\right)\bigg|_{x=\delta-0} = \beta_0 F_0(\delta, y), \quad (20)$$

where

$$F_0(x,y) = \begin{cases} \left(\dfrac{1}{\beta}\dfrac{\partial U}{\partial x} + U\right), 0 \le y \le b, \\ 0 \qquad\qquad\qquad , b < y \le 1, \end{cases} \quad (21)$$
$$x \in [\delta, \delta + l].$$

On the assumption that the function $F_0(x,y)$ is given we can represent solution for the wall in very well known form by the Green function:

$$U_0(x,y) = -\beta_0^0 \int_0^1 G_0(x,y,0,\upsilon)d\upsilon$$
$$+\beta_0 \int_0^b F_0(\delta,\upsilon)G_0(x,y,\delta,\upsilon)d\upsilon. \quad (22)$$

Taking in the account formula (21) we rewrite the solution for the wall as follow:

$$U_0(x,y) = \beta_0^0 \int_0^1 G_0(x,y,0,\upsilon)d\upsilon +$$
$$\beta_0 \int_0^b \left(\frac{1}{\beta}\frac{\partial U}{\partial \zeta} + U\right)\bigg|_{\zeta=\delta+0} G_0(x,y,\delta,\upsilon)d\upsilon. \quad (23)$$

The expression of the Green function in (22), (23) has the form (see, e.g. [15]):

$$G_0(x,y,\zeta,\upsilon) = \sum_{m,n=1}^{\infty} \frac{G_{0,m}^x(x,\zeta)\cdot G_{0,n}^y(y,\upsilon)}{\left[(\pi n)^2 + \mu_m^2\right]}, \quad (24)$$

$$G_{0,m}^{x}(x,\zeta)=\frac{\varphi_m(x)\varphi_m(\zeta)}{\|\varphi_m\|^2},$$

$$G_{0,n}^{y}(y,\upsilon)=\cos\left[n\pi\left(y+\upsilon\right)\right]+\cos\left[n\pi\left(y-\upsilon\right)\right].$$

We have for the first one-dimensional Green function in (24) the following expression for the eigenfunctions:

$$\varphi_m(x)=\cos(\mu_m x)+\frac{\beta_0^0}{\mu_m}\sin(\mu_m x),\ \|\varphi_m\|^2=$$

$$\frac{\beta_0^0}{2\mu_m^2}+\frac{\beta_0}{2\mu_m^2}\frac{\mu_m^2+\left(\beta_0^0\right)^2}{\mu_m^2+(\beta_0)^2}+\frac{\delta}{2}\left(1+\frac{\left(\beta_0^0\right)^2}{\mu_m^2}\right).$$

Here μ_m are the roots of the transcendental equation:

$$tg(\mu_m\delta)=\frac{\mu_m\left(\beta_0+\beta_0^0\right)}{\mu_m^2-\beta_0\beta_0^0}.$$

Unfortunately the representation (22) is unusable as solution for the wall because of unknown function $F_0(x,y)$, i.e. temperature in the fin $U(x,y)$. That is why we will pay attention to the solution for the fin now. In the same way as for (20) we can rewrite the conjugations conditions in the form of BC on the left side of the rectangular fin:

$$\left.\left(\frac{\partial U}{\partial x}-\beta U\right)\right|_{x=\delta+0}=\beta F(\delta,y). \tag{25}$$

Here the right hand side function of BC (25) has the form:

$$F(x,y)=\left(\frac{1}{\beta_0}\frac{\partial U_0}{\partial x}-U_0\right), \tag{26}$$

$$x\in[0,\delta],\ y\in[0,b].$$

Then, similar as for the wall we can represent solution for the fin in following form:

$$U(x,y)=-\beta\int_0^b F(\delta,\eta)G(x,y,\delta,\eta)d\eta. \tag{27}$$

Here

$$G(x,y,\xi,\eta)=\sum_{i,j=1}^{G_m^{(x)}}\frac{G_i^{(x)}(x,\xi)\cdot G_j^{(y)}(y,\eta)}{\lambda_i^2+\kappa_j^2},$$

$$G_i^{(x)}(x,\xi)=\frac{\phi_i(x)\phi_i(\xi)}{\|\phi_i\|^2},$$

$$G_j^{(y)}(y,\eta)=\frac{\psi_j(y,\eta)}{2\|\psi_j\|^2},$$

$$\phi_i(x)=\cos\left[\lambda_i\left(x-\delta\right)\right]+$$

$$\frac{\beta}{\lambda_i}\sin\left[\lambda_i\left(x-\delta\right)\right],$$

$$\|\phi_i\|^2=\frac{\beta}{\lambda_i^2}+\frac{l}{2}\left(1+\frac{\beta^2}{\lambda_i^2}\right),$$

$$\psi_j(y,\eta)=$$

$$\cos\left[\kappa_j\left(y+\eta\right)\right]+\cos\left[\kappa_j\left(y+\eta\right)\right],$$

$$\|\psi_j\|^2=\frac{1}{2}\left(b+\frac{\beta}{\kappa_j^2+\beta^2}\right).$$

Here $\lambda_i(\kappa_j)$ are the roots of the transcendental equations:

$$\tan(\lambda_i l)=\frac{2\lambda_i\beta}{\lambda_i^2-\beta^2},\ \tan(\kappa_j b)=\frac{\beta}{\kappa_j}.$$

Using notation (21) and representation (27) we can easy obtain the following equation:

$$F_0(x,y)=-\int_0^b F(\delta,\eta)\Gamma(x,y,\delta,\eta)d\eta,$$
$$\Gamma(x,y,\xi,\eta)=\left(\frac{\partial}{\partial x}+\beta\right)G(x,y,\xi,\eta). \tag{28}$$

From (22) we obtain immediately similar representation for the $F(\delta,y)$:

$$F(\delta,y)=\frac{\beta_0^0}{\beta_0}\int_0^1\Gamma_0(\delta,y,0,\upsilon)d\upsilon$$
$$-\int_0^b F_0(\delta,\upsilon)\Gamma_0(\delta,y,\delta,\upsilon)d\upsilon. \tag{29}$$

Here we have introduced notation, similar to the second equation of the formula (28):

$$\Gamma_0(x,y,\zeta,\upsilon)=\left(\beta_0-\frac{\partial}{\partial x}\right)G_0(x,y,\zeta,\upsilon). \tag{30}$$

Now we substitute the representation (29) in the right hand side of formula (28) and we obtain following second kind Fredholm integral equation regarding the function $F_0(\delta,y)$:

$$F_0(\delta,y)=-\Upsilon_0(y)+\int_0^b K(y,\upsilon)F_0(\delta,\upsilon)d\upsilon. \tag{31}$$

Here we have introduced following shorter denominations:

$$K(y,\upsilon)=\int_0^b \Gamma_0(\delta,\eta,\delta,\upsilon)\Gamma(\delta,y,\delta,\eta)d\eta,$$

$$\Upsilon_0(y)= \qquad (32)$$

$$\frac{\beta_0^0}{\beta_0}\int_0^b \Gamma(\delta,y,\delta,\eta)d\eta\int_0^1 \Gamma_0(\delta,\eta,0,\upsilon)d\upsilon.$$

When solved integral equation (31) we immediately can obtain the temperature field in the wall from the representation (22). In its turn the representation (27) gives the temperature field in the fin.

By the way, in all our papers [6]-[11] we restrict ourselves with homogeneous boundary conditions (15), (17)-(19).

3.2 Solution of the General Problem (13)-(19)

Here now will be considered general case of non-homogeneous environmental temperature: differential equations (3) with boundary conditions (4)-(12). The solution for the wall instead of (22) has form:

$$V_0(x,y)=\Psi_0(x,y)+ \beta_0\int_0^b F_0(\delta,\upsilon)G_0(x,y,\delta,\upsilon)d\upsilon. \qquad (33)$$

Here the known terms are joined together:

$$\Psi_0(x,y)=\int_0^\delta Q_{0,1}(\zeta)G_0(x,y,\zeta,1)d\zeta$$

$$-\int_0^\delta Q_{0,0}(\zeta)G_0(x,y,\zeta,0)d\zeta -$$

$$\beta_0^0\int_0^1 \vartheta_0(\upsilon)G_0(x,y,0,\upsilon)d\upsilon + \qquad (34)$$

$$\beta_0\int_b^1 \vartheta(\delta,\upsilon)G_0(x,y,\delta,\upsilon)d\upsilon +$$

$$\int_0^\delta d\zeta\int_0^1 Q_0(\zeta,\upsilon)G_0(x,y,\zeta,\upsilon)d\upsilon.$$

In the similar form we can represent solution for the fin. It looks as follow:

$$V(x,y)=\Psi(x,y)- \beta\int_0^b F(\delta,\eta)G(x,y,\delta,\eta)d\eta. \qquad (35)$$

The known function $\Psi(x,y)$ has the form:

$$\Psi(x,y)=-\int_\delta^{\delta+l} Q_1(\xi)G(x,y,\xi,0)d\xi +$$

$$+\beta\int_0^b \vartheta(\delta+l,\eta)G(x,y,\delta+l,\eta)d\eta +$$

$$+\beta\int_\delta^{\delta+l} \vartheta(\xi,b)G(x,y,\xi,b)d\xi + \qquad (36)$$

$$\int_\delta^{\delta+l} d\xi\int_0^1 Q(\xi,\eta)G(x,y,\xi,\eta)d\eta.$$

We obtain instead of formulae (28) and (29) following representations:

$$F(x,y)=\tilde{\Psi}_0(x,y)- \int_0^b F_0(\delta,\upsilon)\Gamma_0(x,y,\delta,\upsilon)d\upsilon,$$

$$F_0(\delta,y)=\tilde{\Psi}(\delta,y)- \int_0^b F(\delta,\eta)\Gamma(\delta,y,\delta,\eta)d\eta. \qquad (37)$$

We have introduced following notations in (37):

$$\tilde{\Psi}_0(x,y)=\frac{1}{\beta_0}\left(\frac{\partial}{\partial x}-\beta_0\right)\Psi_0(x,y),$$

$$\tilde{\Psi}(x,y)=\frac{1}{\beta}\left(\frac{\partial}{\partial x}+\beta\right)\Psi(x,y).$$

We obtain following non-homogeneous Fredholm integral equation of 2nd kind in the same way as equation (31) in sub-section 3.1:

$$F_0(\delta,y)=-\Phi_0(y)+\int_0^b K(y,\upsilon)F_0(\delta,\upsilon)d\upsilon. \qquad (38)$$

Here

$$\Phi_0(y)=\tilde{\Psi}(\delta,y)-\int_0^b \tilde{\Psi}_0(\delta,\eta)\Gamma(\delta,y,\delta,\eta)d\eta.$$

This Fredholm integral equation of 2nd kind has continuous kernel and it has unique solution, see, e.g., [21]. Again, when solved integral equation (38) we can obtain immediately from (33) the temperature field in the wall. Then first formula (37) allows finding the combination $F(x,\delta)$. In its turn formula (35) gives the temperature field in the fin.

We finish this part of our paper with the following two remarks. Firstly, the last problem (with non-homogeneous environment temperatures) and its solution allow conjugating temperature field with hydrodynamic (motion of fluid or gas between two fins and along the left edge of the wall). Secondly, if we had 3rd type

BC instead of the BC (1), we would have had full three-dimensional problem.

4 Conclusions

We have constructed several exact three dimensional analytical solutions for a one element of periodical system with rectangular fin where the wall and the fin consist of materials which have different thermal properties. These solutions are in the form of Fredholm integral equation of 2nd kind and has continuous kernel. They are simpler then the one obtained in our paper [11]. They allow passing over from problems for individual fins to problems for fins arrays, which will be considered in the part 2 of this paper.

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