Boundary Stabilization of the Generalized Korteweg-de Vries-Burgers Equation

NEJIB SMAOUI
Department of Mathematics and Computer Science
Kuwait University
P.O. Box 5969 Safat 13060
KUWAIT

Abstract: This paper considers the boundary control problem of the Generalized Korteweg-de Vries-Burgers (GKdVB) equation on the interval [0,1]. We derive a control law that guarantees the global exponential stability of the GKdVB equation in $L^2(0,1)$. Numerical results supporting the analytical ones for both the controlled and uncontrolled equations are presented using a finite element method.

Key–Words: Generalized Korteweg-de Vries-Burgers Equation, Nonlinear Boundary Control, Stability

1. Introduction

In this paper, the boundary control problem of the Generalized Korteweg-de Vries-Burgers (GKdVB) equation:

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^3 u}{\partial x^3} + u^\alpha\frac{\partial u}{\partial x} = 0, \quad x \in [0,1] \quad (1)$$

$$u(0,t) = 0 \quad (2)$$

$$\frac{\partial u}{\partial x}(1,t) = f_1(u(1,t)), \quad (3)$$

$$\frac{\partial^2 u}{\partial x^2}(1,t) = f_2(u(1,t)), \quad (4)$$

$$u(x,0) = u_0(x), \quad (5)$$

where $\mu, \nu > 0$ and α is a positive integer is considered. In Eq. (1), the independent variable x represents the medium of propagation, t is proportional to elapsed time, and $u(x,t)$ is a velocity at the point x at time t. In Eqs. (3)-(5), the boundary functions f_1 and f_2 are considered as control inputs, and u_0 is the initial condition. The GKdVB equation reduces to the Korteweg-de Vries-Burgers (KdVB) equation when $\alpha = 1$, to the KdV equation when $\nu = 0$ and $\alpha = 1$, and to Burgers equation when $\mu = 0$ and $\alpha = 1$.

Recently, the control problem of the KdVB equation [1], KdV equation [3, 4] and Burgers equation [5, 6] has received a lot of attention. Balogh and Krstić [1] investigated the KdVB equation using boundary control. In their work, global stability of the solution in the L^2-sense and global stability in the H^1-sense were proved. Rosier [3, 4] worked on the KdV equation where an exact boundary control of the linear and nonlinear KdV equations was established in [3]; and the control was illustrated numerically in [4]. Smaoui [5, 6] considered boundary and distributed control of the Burgers equation. A boundary control is used in [5] to show the exponential stability of the Burgers equation analytically as well as numerically. In [6], a system of ODEs was constructed to mimic the dynamics of Burgers equation, then a state feedback control scheme was implemented on the system to show that the Burgers solution can be controlled to any desired state.

The paper is organized as follows. In section 2, using Lyapunov theory a boundary control law to the GKdVB equation is proposed to show the global exponential stability of the solution in $L^2(0,1)$. Section 3 presents some numerical results using finite element techniques to show the effectiveness of the developed control schemes. Finally, some concluding remarks are given in section 4.

2. The Boundary Control Problem of the GKdVB Equation

Since Bona and Luo [2] showed that the GKdVB equation is well-posed in certain function spaces, we will assume that the GKdVB equation has a solution $u(x,t) \in$

$C_b^1\left([\,0,\infty\,);L^2(0,1)\right) \cap C\left([\,0,\infty\,);H^3(0,1)\right)$, where $C_b^1\left([\,0,\infty\,);L^2(0,1)\right)$ is the set of all continuously differentiable bounded functions defined on $[\,0,\infty\,)$ with values in $L^2(0,1)$ and $C\left([\,0,\infty\,);H^3(0,1)\right)$ is the set of all continuous functions defined on $[\,0,\infty\,)$ with values in $H^3(0,1)$. In this section, we propose a boundary control law for the GKdVB equation and prove its global exponential stability in $L^2(0,1)$ for all values of α.

2.1 Exponential Stability in $L^2(0,1)$

Theorem 2.1 *Let α be a positive integer. The GKdVB Eq.(1) with boundary conditions given by Eqs.(2)-(4) is globally exponentially stable in $L^2(0,1)$ under the following control law:*

$$f_1(u(1,t)) = 0, \tag{6}$$

$$f_2(u(1,t)) = -\frac{1}{\mu(\alpha+2)}u^{\alpha+1}(1,t). \tag{7}$$

Proof: If we take the L^2-inner product of Eq.(1) with $2u(x,t)$, we obtain

$$\int_0^1 2u(x,t)u_t(x,t)\mathrm{d}x - \int_0^1 2\nu u(x,t)u_{xx}(x,t)\mathrm{d}x + \int_0^1 2\mu u(x,t)u_{xxx}(x,t)\mathrm{d}x + \int_0^1 2u^{\alpha+1}u_x(x,t)\mathrm{d}x = 0. \tag{8}$$

Since

$$\int_0^1 2u(x,t)u_t(x,t)\mathrm{d}x = \int_0^1 \frac{\mathrm{d}}{\mathrm{dt}}u^2(x,t)\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{dt}}\|u(x,t)\|^2,$$

and

$$\int_0^1 u(x,t)u_{xx}(x,t)\mathrm{d}x = u(x,t)u_x(x,t)|_0^1 - \|u_x(x,t)\|^2,$$

and

$$\int_0^1 u(x,t)u_{xxx}(x,t)\mathrm{d}x = u(x,t)u_{xx}(x,t)|_0^1 - \frac{1}{2}u_x^2(x,t)|_0^1,$$

and

$$\int_0^1 u^{\alpha+1}(x,t)u_x(x,t)\mathrm{d}x = \frac{1}{\alpha+2}\int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left(u^{\alpha+2}(x,t)\right)\mathrm{dx} = \frac{1}{\alpha+2}u^{\alpha+2}(x,t)|_0^1,$$

then Eq.(8) becomes

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u(x,t)\|^2 - 2\nu\left[u(x,t)u_x(x,t)|_0^1 - \|u_x(x,t)\|^2\right] + 2\mu\left[u(x,t)u_{\mathrm{xx}}(x,t)|_0^1 - \frac{1}{2}u_x^2(x,t)|_0^1\right] + 2\left[\frac{1}{\alpha+2}u^{\alpha+2}(x,t)|_0^1\right] = 0. \tag{9}$$

Substituting the boundary conditions Eqs.(2)-(4) and the control law given by Eqs.(6) and (7) into Eq.(9), one obtains

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u(x,t)\|^2 + 2\nu\|u_x(x,t)\|^2 - \frac{2}{(\alpha+2)}u^{\alpha+2}(1,t) + \mu u_x^2(0,t) + \frac{2}{\alpha+2}u^{\alpha+2}(1,t) = 0.$$

Thus,

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u(x,t)\|^2 = -2\nu\|u_x(x,t)\|^2 - \mu u_x^2(0,t). \tag{10}$$

Since $\mu > 0$, Eq.(10) leads to:

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u(x,t)\|^2 \leq -2\nu\|u_x(x,t)\|^2. \tag{11}$$

Using the Poincaré inequality,

$$\|u(x,t)\| \leq \|u_x(x,t)\|, \tag{12}$$

(11) becomes

$$\frac{\mathrm{d}}{\mathrm{dt}}\|u(x,t)\|^2 \leq -2\nu\|u(x,t)\|^2. \tag{13}$$

Integrating both sides of (13) with respect to time, we get

$$\|u(x,t)\|^2 \leq \|u_0(x)\|^2 e^{-2\nu t}. \quad (14)$$

Hence, one can write,

$$\| u(x,t)\| \leq \| u_0(x)\| e^{-\nu t}. \quad (15)$$

Therefore, $\| u(x,t)\|$ converges to zero exponentially as $t \to \infty$, since $\nu > 0$. Thus, the solution of GKdVB Eq.(1) is globally exponentially stable in $L^2\,(0,1)$.

Remark. *It can be easily shown that if α is an even positive integer, then the GKdVB equation is globally exponentially stable in $L^2(0,1)$ under the following control law:*

$$f_1(u(1,t)) = 0,$$

$$f_2(u(1,t)) = au^{\alpha-1}(1,t),$$

where $a \geq 0$.

3. Numerical Results

Eq.(1) with homogeneous boundary condition (i.e., the uncontrolled system or the open loop system) $u(x,0) = \sin(\pi x)$ was solved using a finite element method. Figure 1 depicts the time evolution of the solution $u(x,t)$ with $\mu = 0.1$, $\nu = 0.1$ and $\alpha = 1$. As we can see from Figure 1, the uncontrolled solution seems to converge to a nontrivial steady state solution. However, when the boundary control is applied on the second derivative as in Eq.(4), while keeping the first derivative boundary condition in Eq.(3) equals to 0 at $x = 1$, the solution converges to zero (see Figure 2).

4. Concluding Remarks

We have used a boundary control to analyze the exponential stability of the Generalized Korteweg-de Vries-Burgers in L^2 sense. We have also presented some numerical results based on finite element technique to support and reinforce the analytical ones. The design of other controllers to speed up the convergence rate to the zero dynamics will be the subject of future research work.

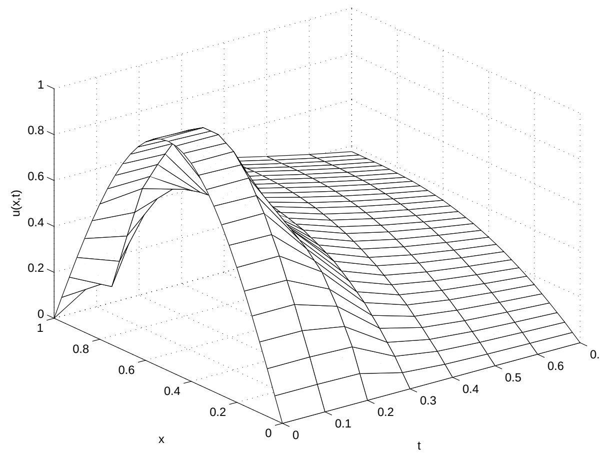

Figure 1: Time evolution of the uncontrolled GKdVB equation when $\nu = 0.1$, $\mu = 0.1$, $\alpha = 1$, and $u(x,0) = \sin(\pi x)$.

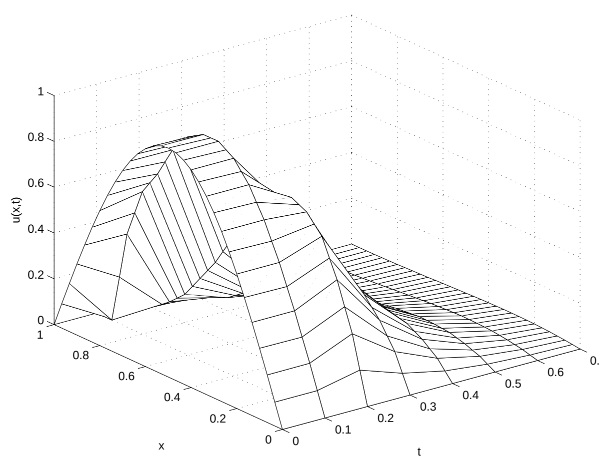

Figure 2: Time evolution of the controlled GKdVB equation when $\nu = 0.1$, $\mu = 0.1$ and $\alpha = 1$, and $u(x,0) = \sin(\pi x)$.

References:

[1] Balogh, A. and Krstić, M. (2000). Boundary Control of the Korteweg-de Vries-Burgers Equation: Further Results on Stabilization and Well-Posedness, with Numerical Demonstration. *IEEE Transaction on Automatic Control*, **45**, No. 9, 1739-1745.

[2] Bona, J. L. and Luo L. (1993). Decay of solutions to nonlinear dispersive wave equations. *Differential Int. Equations*, **6**, 961-980.

[3] Rosier, L. (1997). Exact boundary controllability for the Korteweg-de Vries equation on a bounded domain. *ESAIM Conrol optim. Calc. Var.*, **2**, 33-55 (electronic).

[4] Rosier, L. (1998). Exact Boundary Controllability for the linear Korteweg-de Vries equation - A numerical study. *ESAIM, Proceedings*, **4**, 255-267.

[5] Smaoui, N. (2004). Nonlinear Boundary Control of the Generalized Burgers equation. *Nonlinear Dynamics Journal*, **37**, No. 1, 75-86.

[6] Smaoui, N. (2005). Boundary and Distributed Control of the Viscous Burgers Equation. *Journal of Computational and Applied Mathematics*, **182**, 91-104.