Winning Strategies for Hexagonal Polyomino Achievement

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Abstract: In polyomino achievement games, two players alternately mark the cells of a tessellation and try to achieve a given polyomino. In [2], Bode and Harborth investigated polyomino achievement games for the hexagonal tessellation and determined all but five polyominoes with at most five cells whether they are achieved by the first player. In this paper, we show winning strategies for three hexagonal polyominoes with five cells called **Y**, **Z**, and **C**, which were left unsolved in [2].

Key–Words: Polyomino, Achievement game, Hexagonal tessellation, Winning strategy, Winner, Loser

1 Introduction

The ticktacktoe, a.k.a. noughts and crosses, is a widely-played two-player game. While the basic ticktacktoe on the 3×3 square board is easily solved, F. Harary made fruitful mathematical generalization [3]. Namely, he invented achievement games for polyominoes (or animals), which are sets of connected cells in a tessellation. In the game, players A and B alternately mark one of the cells and try to make a copy of a given polyomino P by their marks. Since each mark contributes positively, the second player B can never achieve P earlier than the first player A. Therefore, P is called a *winner* if A can achieve P regardless of the marks by B; otherwise, P is called a *loser* [4].

For the triangular tessellation, it is known that three polyominoes are winners and the other polyominoes are losers [6]. For the square tessellation, 11 polyominoes are winners and the other polyominoes except one, called Snaky, are losers [1, 3-5, 7-9]. For the hexagonal tessellation, Bode and Harborth determined winners and losers for all but five polyominoes with at most five cells [2]. Among the 22 polyominoes with five cells, they showed that 15 polyominoes are winners and two are losers. Hence, the five remaining polyominoes with five cells were left unsolved.

In this paper, we show that three out of the five unsolved hexagonal polyominoes with five cells, which we call **Y**, **Z**, and **C**, are winners. Winning strategies of the first player for these cases are shown.

2 Polyomino Achievement

A polyomino is a set of cells of a two-dimensional tessellation such that both the set and its complement are connected by common edges of the cells. A polyomino achievement game is a two-player game such that given a polyomino P, players A and B alternately mark one of the cells and the player who first completes a copy of P with his marks wins the game.

In [2], Bode and Harborth considered polyomino achievement games for the hexagonal tessellation and showed that among the 22 hexagonal polyominoes with five cells, 15 polyominoes are winners and 2 are losers. The remaining five polyominoes, which we call **Y**, **Z**, **C**, **L**, and **I**, are shown in Fig. 1. In the next sections, we show winning strategies of the first player for three polyominoes **Y**, **Z**, and **C**.

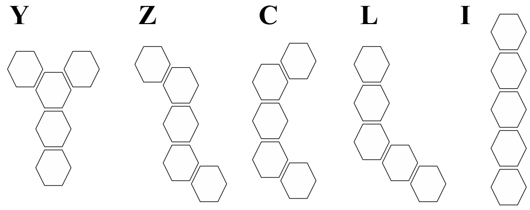

Figure 1: Five unsolved polyominoes with five cells.

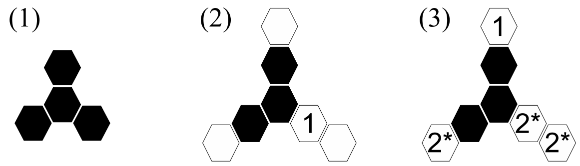

Figure 2: (1) A propeller; (2)(3) basic strategies for polyomino **Y**.

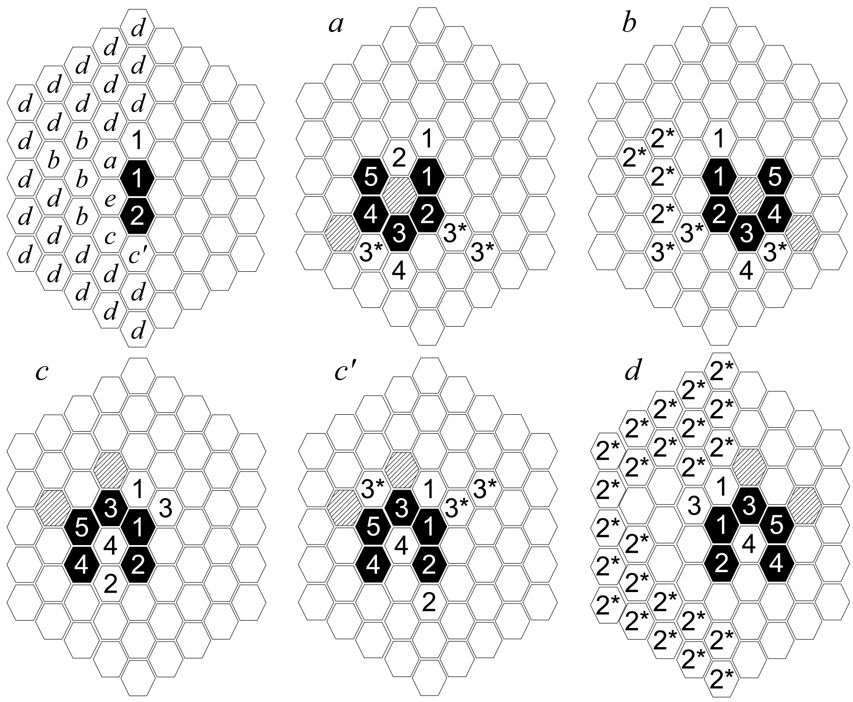

Figure 3: Classification of B's second move and winning strategies for a to d

3 Winning Strategy for Polyomino Y

3.1 Basic Strategies

The polyomino with four cells in Fig. 2 (1) is called a *propeller* [10]. If a propeller is made by A and two of the three ends of the propeller are open at B's turn, then A can achieve polyomino **Y**. Based on this observation, we make the following two basic strategies.

First, suppose that the three black cells in Fig. 2 (2) are marked by A, all the other cells are open, and it is now B's turn. Then B must mark the cell with 1 because otherwise, A marks this cell and completes a propeller with at least two ends open, which results in A's win.

The other situation is described in Fig. 2 (3). The three black cells are marked by A, the cell with 1 is marked by B, and it is now B's turn. Then B must mark one of the cells having $2*$ because otherwise, A again completes a propeller with two ends open. We develop a winning strategy for polyomino **Y** using these strategies.

3.2 Case Analysis

The $5 \times 5 \times 5$ hexagonal board in Fig. 3 is used for polyomino **Y**. The first player A marks the center cell at first. We note that black cells are always used for A's marks. We divide into two cases according to whether the player B first marks the neighboring cell of the center cell or not.

3.2.1 Case 1: Neighboring Case

Due to the symmetry of the board, we may assume that B first marks the upper side of the center cell. Then A at the second move marks the lower side of the center cell. B's second move is classified into five types, named a to e, as shown in Fig. 3. (It is enough to consider the left-hand side of the board for B's second move.)

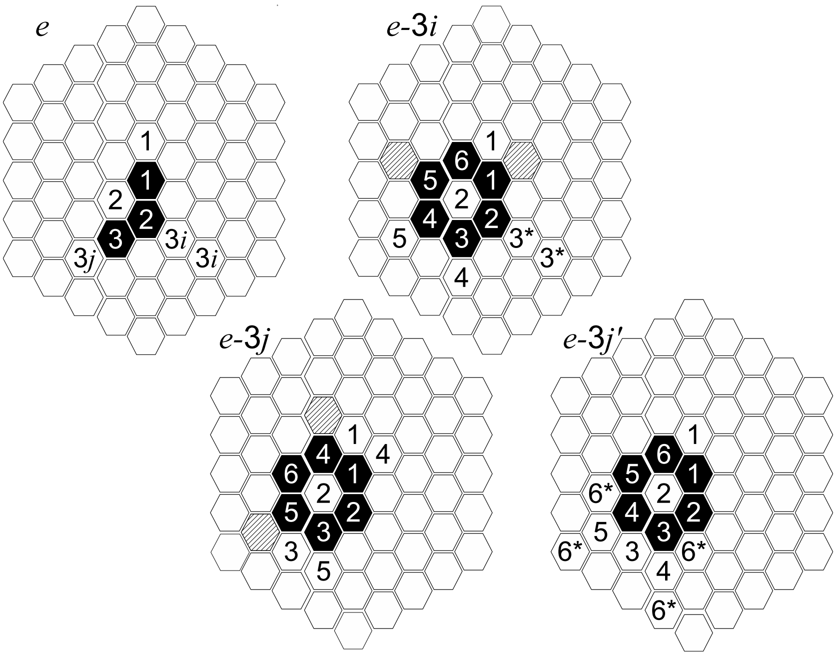

Figure 4: A winning strategy for e

When B marks a at the second move, in response to A's third move, B must mark one of the three cells with 3* because of the basic strategy in Fig. 2 (3). In response to A's fourth move, B must mark the cell with 4 because of the basic strategy in Fig. 2 (2). Then A's fifth move guarantees the achievement of at least one propeller by marking one of the hatched cells. Therefore, A wins with seven moves.

When B marks b, c, c', or d, the corresponding strategies in Fig. 3 are used. Note that the strategies for a and b are symmetrical to the vertical line and so are the strategies for c and d. In all of these cases, A wins with seven moves.

When B marks the cell e at the second move, in response to A's third mark, B must mark one of the three cells having $3i$ or $3j$ in Fig. 4. When B marks $3i$, A marks cells in a cyclic form and B is forced to mark the fourth and fifth cells. With A's sixth move, A achieves a propeller with two ends open by marking one of the hatched cells. Therefore, A wins with eight moves. When B marks $3j$, B is forced to mark the fourth and fifth cells. In this case, A again wins with eight moves. In the case of $3j$, we note that the same strategy with the case of $3i$ does not work because of B's counterattack shown in e-$3j'$ of Fig. 4.

3.2.2 Case 2: Non-neighboring Case

Due to the symmetry of the board, it is enough to assume that B's second move is among the seven cells with $1*$ in Fig. 5. In response to A's second move, B's second move is classified into five types, named v to z. When B marks one of the cells with $1*$ at the second move as well as his first move, the strategy for v is applied.

For v, we note that the cells having v or $1*$ are exclusive with the cells used in the strategy for a in Case 1. Therefore, the strategy for a can be used for v and A wins with seven moves. For w, the strategy symmetrical to the one for v is used. Winning strategies for x, y, and z are also shown in Fig. 5. To summarize, A wins with seven moves in the cases for v, w, x, and y and wins with eight moves in the case for z.

Consequently, for polyomino **Y**, A wins with at most eight moves. □

4 Winning Strategy for Polyomino Z

4.1 Basic Strategies

To achieve **Z**, we again use a propeller. Two basic strategies are shown in Fig. 6. In Fig. 6 (1-1), the three black cells are marked by A, the white and the hatched cells are open, and it is now B's turn. If B is far from achieving **Z**, B is forced to mark the hatched cell because otherwise, say, when B marks one of the cells having $3*$ in (1-2) and (1-3), A wins by the procedures in (1-2) and (1-3), respectively.

Fig. 6 (2-1) shows the other situation in which the three black cells are marked by A, one of the cells having $2*$ is marked by B, all the cells having $3*$ are open, and it is now B's turn. Then B is forced to mark one the cells having $3*$ because otherwise, A wins with the procedure in (2-2).

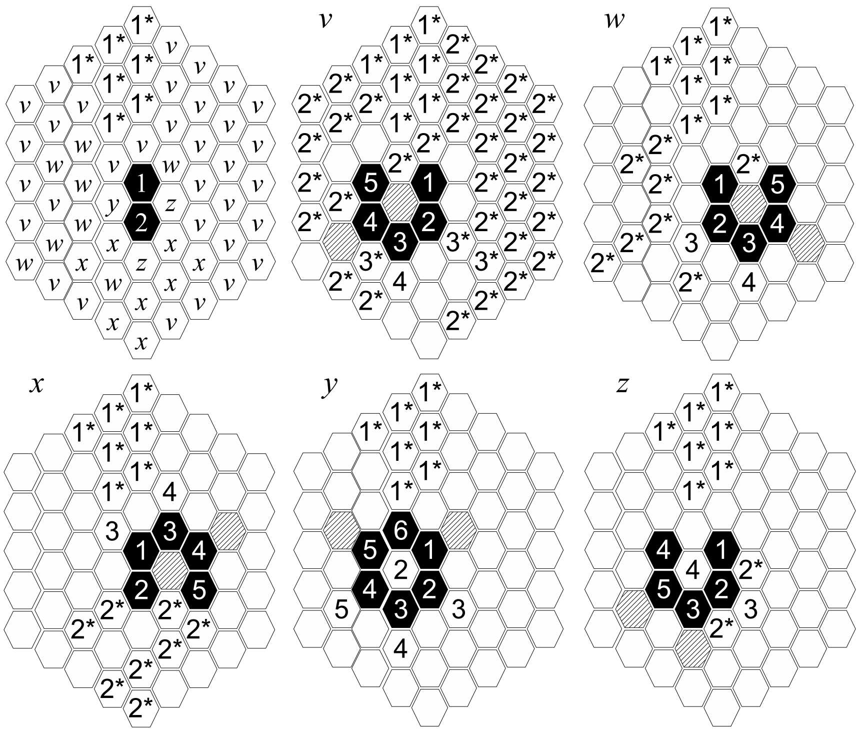

Figure 5: Classification of B's second move and winning strategies for v to z.

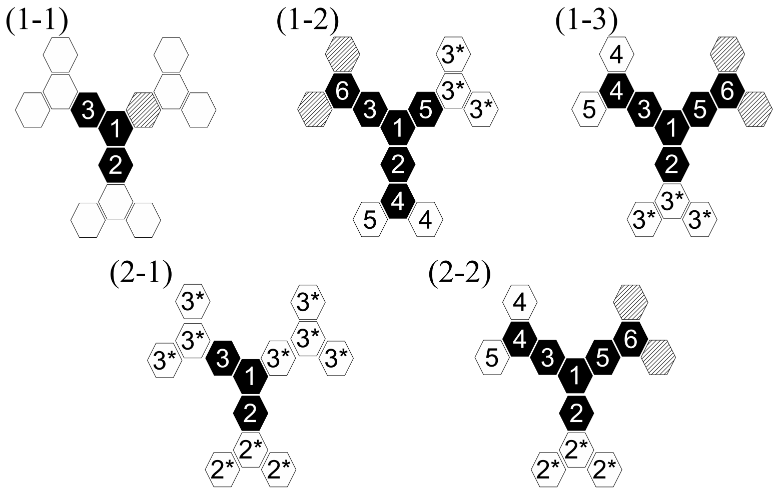

Figure 6: Basic strategies for polyomino **Z**.

4.2 Case Analysis

The $6 \times 6 \times 6$ hexagonal board is used for polyomino **Z**. A's first move is on the center cell. Due to the symmetry of the board, we may assume that B's first mark is on one of the 11 cells having $1*$ in Fig. 7. A's second move is on the lower side of the center cell. Then B's second move is classified into four types, named a to d, as shown in Fig. 7. We note that when B's first and second moves are both on the cells having $1*$, the strategy for c is used. Winning strategies for all of the cases are described in Fig. 7. The strategies for a and b are symmetrical to the vertical line and so are the strategies for c and d. The case for b must be checked because if B marks b, the strategy for a does not work because B's fourth move is no more uniquely determined.

Consequently, for polyomino **Z**, A wins with seven moves. □

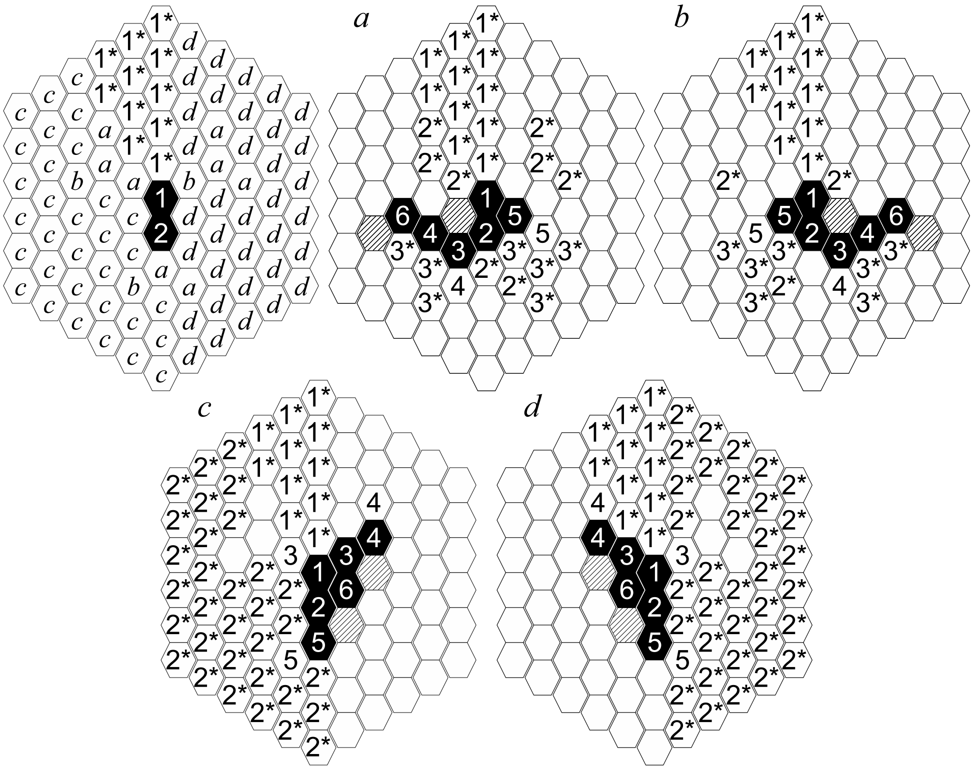

Figure 7: Classification of B's second move and winning strategies for a to d.

5 Winning Strategy for Polyomino C

5.1 Basic Strategies

To achieve polyomino **C**, we use basic strategies similar to the ones for **Z**. The only difference is that B's fourth and fifth moves in Fig. 6 are exchanged for **C**.

5.2 Case Analysis

The $6\times6\times6$ hexagonal board is used. A's first move is on the center cell. We divide into two cases according to whether B's first move is neighboring to the center cell.

5.2.1 Case 1: Neighboring Case

In this case, B's second move is classified into five types, named a to e, as shown in Fig. 8.

In the cases for a and c, A wins with nine moves. In the cases for b and d, A wins with eight moves. In the case for e, more detailed analysis is needed. In response to A's third move, B's third move needs to be on one of the cells having $3i$, $3j$, or $3k$. When it is $3i$ or $3j$, B's forth move is either $4p$, $4q$, $r1$, or $r2$ in Fig. 9. When it is $4p$ or $4q$, A wins with the strategies in "e-$3i, 3j$-$4p$" or in "e-$3i, 3j$-$4q$", respectively except for the case B's third move is $3j$ and the forth move is $r1$ or $r2$. In these cases, the former strategies "e-$3i, 3j$-$4p$" and "e-$3i, 3j$-$4q$" do not work because B's fourth move to $r1$ or $r2$ forces A to mark the black cell with 5 to prevent B from achieving **C**. So instead of these strategies, A uses the ones in e-$3j$-$r1$ and e-$3j$-$r2$ in Fig. 9. In the last case, that is, when B's third move is on $3k$, A wins with the strategies in e-$3k$-$4p$ and e-$3k$-$4q$. To summarize, for e, A wins with nine moves.

5.2.2 Case 2: Non-neighboring Case

In this case, B's second move is classified into six types, named s to y, as shown in Fig. 10. When B's first and second moves are both to the cell having $1*$, the strategy for u is used.

For s, A marks one of the hatched cells and wins with eight moves. We note that this strategy was not used in Case 1 because some of the cells used for s, which should be marked by A, are marked by B in Case 1 and there can be counterattacks by B.

The strategies for the six cases t to y have the following properties. The strategy for t is symmetrical to the one for s. The strategies for u, v, w, x, and y are related to the ones in Case 1 as follows. The strategies for u and x are almost the same with a and e, respectively. The strategies for v, w, and y are symmetrical to the vertical line with a, c, and e, respectively. Furthermore, in the case for x, no exceptional cases such as e-$3j$-$r1$ and e-$3j$-$r2$ in Fig. 9 are needed because there are no counterattacks by B in this case due to the location of B's first mark.

Consequently, for polyomino **C**, A wins with at most nine moves. □

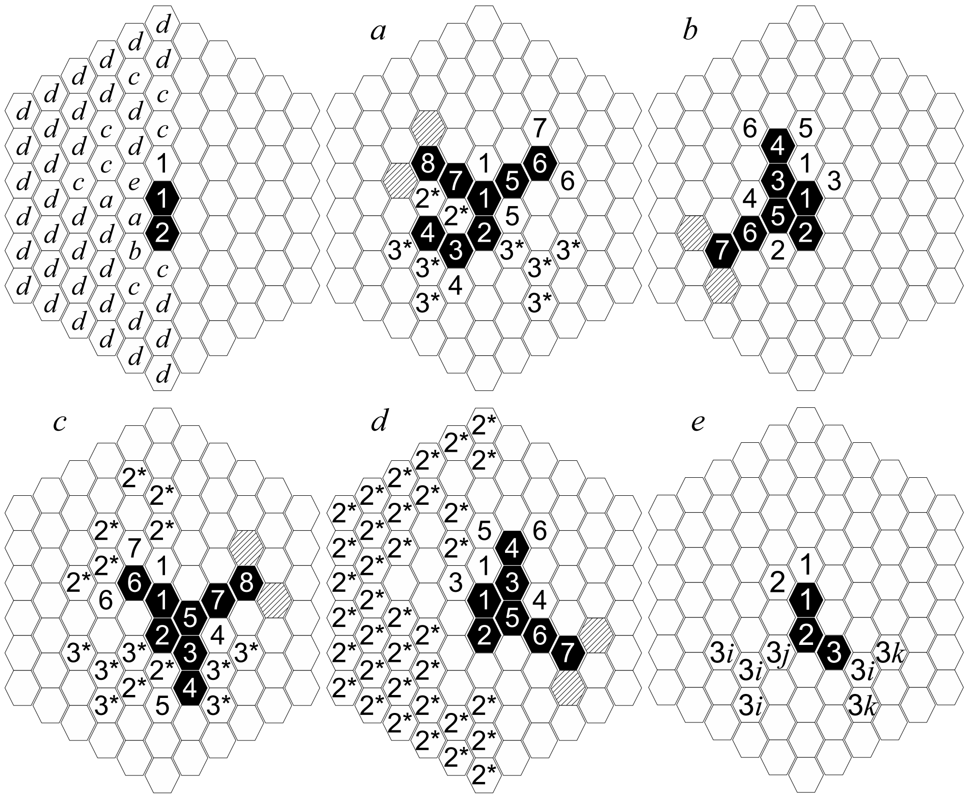

Figure 8: Classification of B's second move and winning strategies for a to e.

6 Concluding Remarks

Winning strategies for hexagonal polyominoes **Y**, **Z**, and **C** are given. Thus, two polyominoes with five cells called **L** and **I** are left unsolved. If a propeller is used as a building part to achieve **L**, it has to be completed by the first player A with the three ends open at his turn. Therefore, strategies along this line seem to be harder than the ones in this paper. As for the polyominoes with more than five cells, there are 82 polyominoes with six cells and 67 polyominoes among them are not known whether they are winners or not [2]. Since the search space should be quite huge, computer-assisted exploration might be needed.

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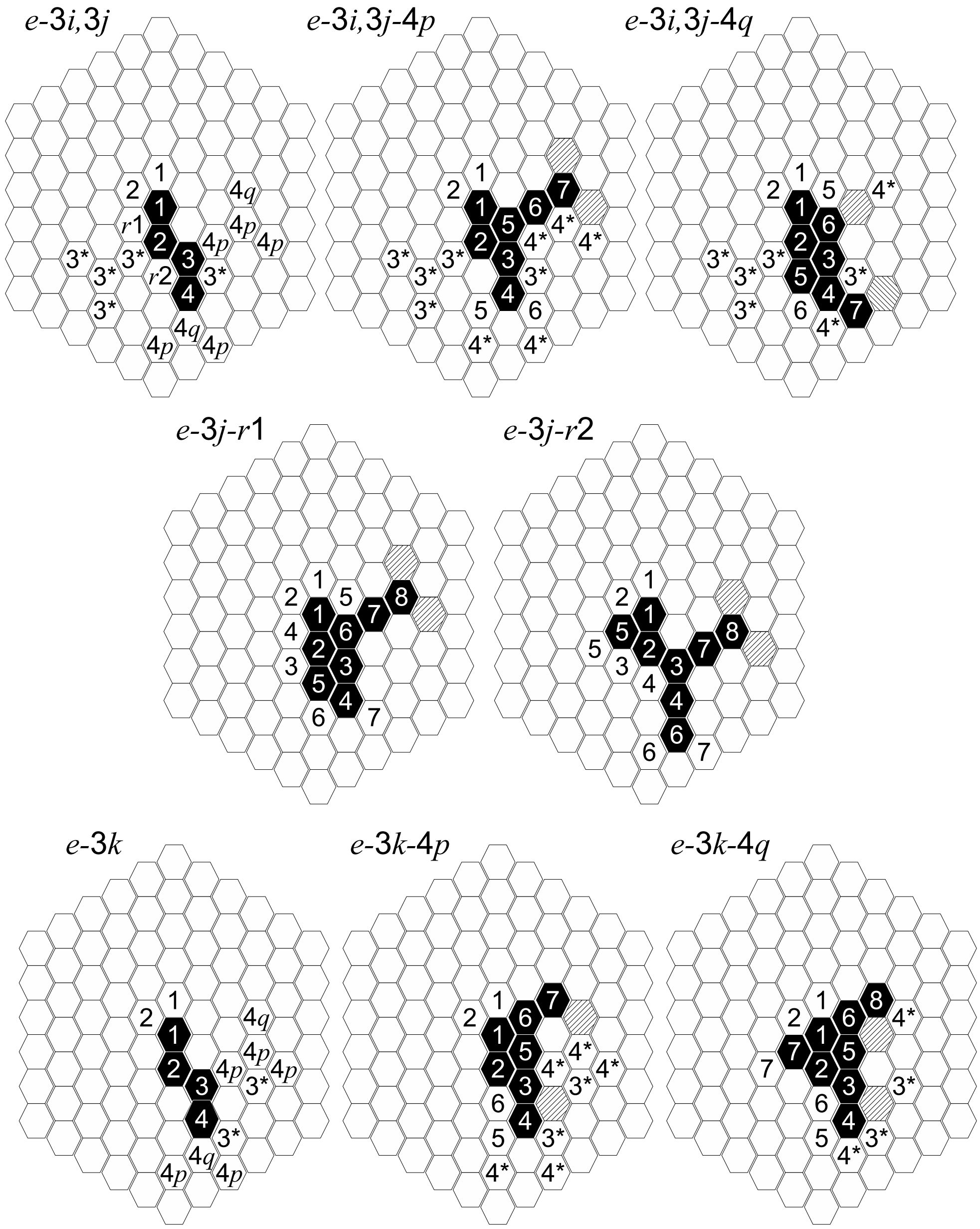

Figure 9: A winning strategy for e

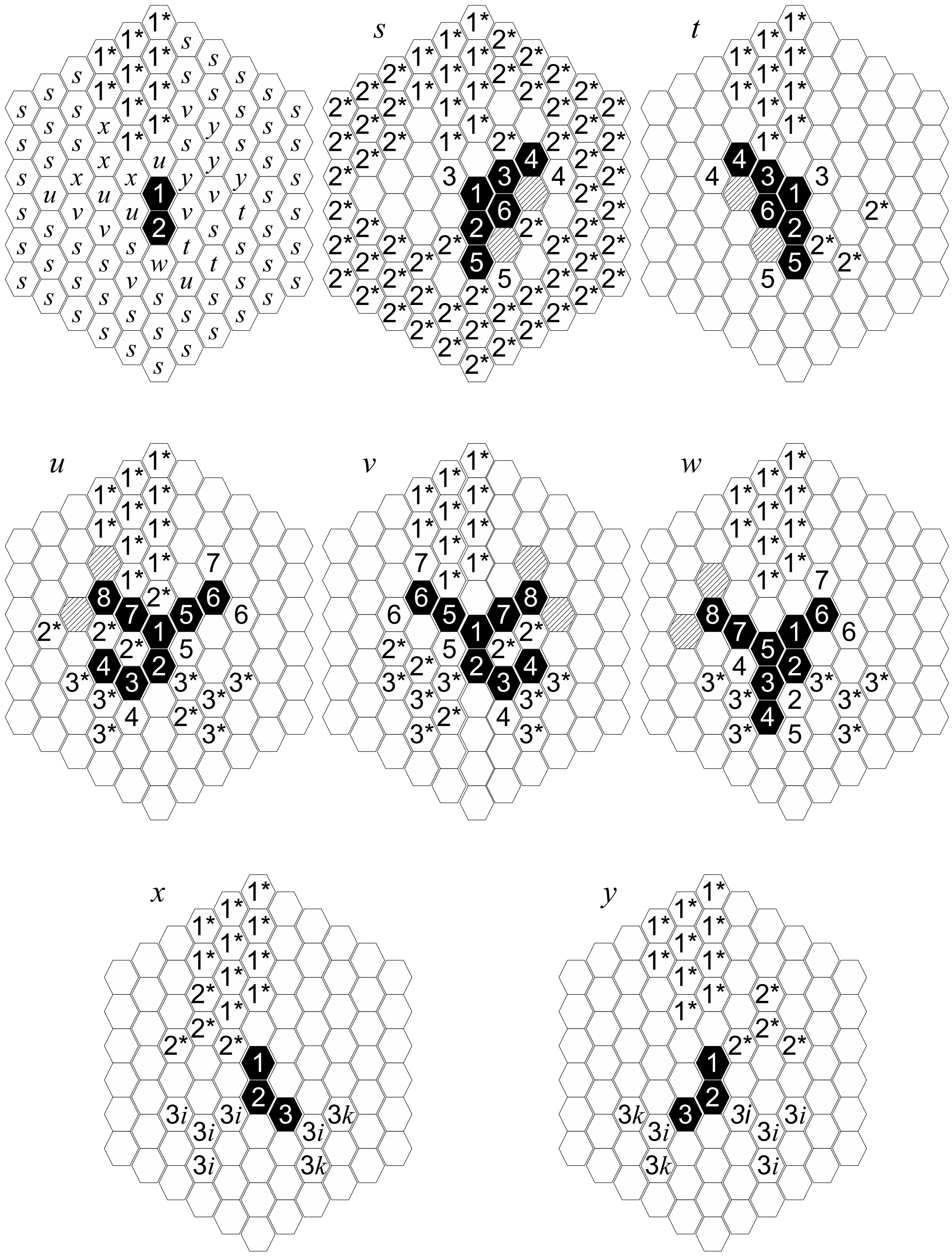

Figure 10: Classification of B's second move and winning strategies for s to y