On the First Integrals of KdV Equation and the Trace Formulas of Deift-Trubowitz Type

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Abstract: A new type of the first integrals associated with KdV equation is constructed by applying the trace formulas of Deift-Trubowitz type for the 1-dimensional Schrödinger operator with no bound states. The relations between the well-known first integrals with the densities expressed in terms of differential polynomials and the present first integrals are obtained.

Key–Words: KdV equation, First integrals, One dimensional Schrödinger operator, Trace formulas of Deift-Trubowitz type

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1 Introduction

In this paper, we consider a spectral meaning of the first integrals of KdV equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{1}$$

where $u = u(x,t)$ is the spatially rapidly decreasing real valued function. We restrict ourselves to the case such that the 1-dimensional Schrödinger operator

$$H_t = -\frac{d^2}{dx^2} + u(x,t)$$

has no bound states, i.e, $\sigma_p(H_t) = \emptyset$, where $\sigma_p(P)$ denotes the set of discrete spectrum of the operator P considered in the space $L^2(\mathbb{R})$. The functional $I[u](t)$ which corresponds the function $u(x,t)$ to the function of the one variable t is called the first integral or the conservation law of the equation (1), if

$$\frac{d}{dt}I[u](t) = 0$$

holds for the solution $u(x,t)$ of KdV equation (1). Moreover the function $w(x,t)$ is called the local density of the first integral $I[u](t)$, if

$$I[u](t) = \int_{-\infty}^{\infty} w(x,t)dx$$

holds. In [12], Zakharov and Faddeev showed that there are infinitely many first integrals which have local densities expressed in terms of the differential polynomials of $u(x,t)$.

On the other hand, in [2], Deift and Trubowitz derived the trace formula

$$\frac{i}{\pi}\int_{-\infty}^{\infty} kr_\pm(k)f_\pm(x,k)^2dk - 2\sum_{j=1}^{N} c_{\pm,j}\eta_j f_\pm(x,i\eta_j)^2 = \frac{1}{2}u(x) \tag{2}$$

for the 1-dimensional Schrödinger operator

$$H = -\frac{d^2}{dx^2} + u(x)$$

with the rapidly decreasing real valued potential $u(x)$, where $r_\pm(k)$ are the left and right reflection coefficients, $f_\pm(x,k)$ are the left and right Jost solutions, $-\eta_j^2$, $j = 1, 2, \cdots, N$ are the discrete eigenvalues of the operator H, and $c_{\pm,j}$ are the normalizing coefficients. We will briefly mention these materials in the next section. In particular, if the operator H has no bound states, we have the trace formula

$$\frac{i}{\pi}\int_{-\infty}^{\infty} kr_\pm(k)f_\pm(x,k)^2dk = \frac{1}{2}u(x). \tag{3}$$

On the other hand, in [1], Deift, Lund and Trubowitz derived the another trace formula

$$-\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{1}{k}r_\pm(k)f'_\pm(x,k)^2dk = \frac{1}{2}u(x) \tag{4}$$

for the rapidly decreasing real valued potential $u(x)$ such that the operator H has no bounds states and satisfies the condition

$$r_{\pm}(0) = -1.$$

The main purpose of the present paper is to give a new spectral interpretation of the first integrals of KdV equation using above trace formulas and, furthermore, construct a new kind of first integrals.

The contents of the present paper are as follows. In §2, the fundamental materials are briefly explained. In §3, the trace formulas of Deift-Trubowitz type and its generalization are discussed. In §4, an evolution equation satisfied by the Jost solution is derived. In §5, the spectral interpretation of the first integrals of KdV equation is given.

2 Preliminaries

In this section, the necessary materials are summarized.

Throughout this section, we assume that the real valued potential $u(x)$ satisfies the integrability condition

$$\int_{-\infty}^{\infty} (1 + |x|^2)|u(x)|dx < \infty.$$

Let $f_{\pm}(x,k)$ be the Jost solution of the eigenvalue problem

$$Hf(x,k) = -f''(x,k) + u(x)f(x,k) = k^2 f(x,k),$$

i.e., $f_{\pm}(x,k)$ behave like $\exp(\pm ikx)$ as $x \longrightarrow \pm\infty$ respectively. Let $W[f,g] = fg' - f'g$ be the Wronskian, then the right (+) and left (-) reflection coefficients $r_{\pm}(k)$ are defined by

$$r_{\pm}(k) = \pm \frac{W[f_+(x,\mp k), f_-(x,\pm k)]}{W[f_-(x,k), f_+(x,k)]}, \quad k \in \mathbb{R}\backslash\{0\}.$$

We have

$$|r_{\pm}(k)| < 1, \quad k \in \mathbb{R} \setminus \{0\}$$

in general. Moreover one of the two cases

$$r_{\pm}(0) = -1 \tag{5}$$

or

$$|r_{\pm}(k)| < 1 \text{ for all } k \in \mathbb{R}.$$

holds. It is known that the condition (5) is generic. On the other hand, let $i\eta_j$, $1 \le j \le N$ be the zeros in the upper half plane H^+ of the analytic function $W[f_-(x,k), f_+(x,k)]$, $k \in H^+$, where

$$H^+ = \{k | k \in \mathbb{C}\,,\ \Im k > 0\}$$

for the complex plane $\mathbb{C}$. Then

$$\sigma_p(H) = \{-\eta_j^2 | 1 \le j \le N\}$$

holds, and $f_{\pm}(x, i\eta_j)$, $1 \le j \le N$ are the square integrable eigenfunctions associated with the operator H. Let $c_{\pm,j}$ be the nomalization coefficients of those eigenfunctions, i.e.,

$$c_{\pm,j} = \frac{1}{\int_{-\infty}^{\infty} f(x, i\eta_j)^2 dx}. \tag{6}$$

The collections

$$\Sigma_{\pm} = \{r_{\pm}(k), -\eta_j^2, c_{\pm,j}; 1 \le j \le N\}$$

are called the scattering data. See [2] and [4] for details of the scattering theory of H.

Now we consider the scattering data of the operator H_t with the spatially rapidly decreasing potential $u(x,t)$ which solves KdV equation (1). In this case, the elements of the scattering data depend on t, i.e., those are denoted as $r_{\pm}(k,t)$, $c_{\pm,j}(t)$, and $\eta_j(t)$. In [3], Gardner, Greene, Kruskal and Miura discovered the following formulas;

$$\begin{aligned} r_{\pm}(k,t) &= r_{\pm}(k,0)\exp(-8ik^3 t), \\ c_{\pm,j}(t) &= c_{\pm,j}(0)\exp(-8\eta_j^3 t), \\ \eta_j(t) &= \eta_j(0). \end{aligned} \tag{7}$$

Next we explain the recursion operator Λ and the KdV polynomials.The operator Λ is the formal pseud differential operator defined by

$$\Lambda = \left(\frac{d}{dx}\right)^{-1} \left(\frac{1}{2}u'(x) + u(x)\frac{d}{dx} - \frac{1}{4}\frac{d^3}{dx^3}\right). \tag{8}$$

Put $Z_0(u) = 1$ and define the functions $Z_n(u)$ by the recurrence relation

$$Z_n(u) = \Lambda Z_{n-1}(u), \quad n \in \mathbb{N}, \tag{9}$$

where $\mathbb{N}$ is the set of all natural numbers. Then it is known that $Z_n(u)$ are the differential polynomials of $u(x)$. For example, we have

$$Z_1(u) = \frac{1}{2}u,\ Z_2(u) = \frac{1}{8}(3u^2 - u''). \tag{10}$$

We call them the KdV polynomials. We refer the reader [7, lemma 3.1, p.621] and [8, p.952] for more precise information.

Next we explain the following Appell's lemma.

Lemma 1. *Let $y = f(x)$ and $y = g(x)$ be the solutions of the 2nd order ordinary differential equation*

$$\frac{d^2y}{dx^2} = p(x)y,$$

then the product $z = f(x)g(x)$ solves the 3rd order ordinary differential equation

$$\frac{d^3z}{dx^3} = 4p(x)\frac{dz}{dx} + 2p'(x)z.$$

This lemma is quite elementary fact and easy to prove it. See [10] and [9] for detail. By Appell's lemma and the definition (8) of the recursion operator, we have immediately

$$\frac{d}{dx}\Lambda g_\pm(x,k) = k^2\frac{d}{dx}g_\pm(x,k),$$

where $g_\pm(x,k) = f_\pm(x,k)^2$, i.e.,

$$\Lambda g_\pm(x,k) = k^2 g_\pm(x,k). \tag{11}$$

3 Trace formulas of Deift-Trubowitz type

By the trace formula (2) and (10), we have

$$\begin{aligned}\frac{i}{\pi}\int_{-\infty}^{\infty} kr_\pm(k)f_\pm(x,k)^2dk \\ -2\sum_{j=1}^{N} c_{\pm,j}\eta_j f_\pm(x,i\eta_j)^2 = Z_1(u).\end{aligned} \tag{12}$$

By operating with the operator Λ^{n-1} on the both sides of (12), then the trace formulas

$$\begin{aligned}\frac{i}{\pi}\int_{-\infty}^{\infty} k^{2n-1}r_\pm(k)f_\pm(x,k)^2dk \\ +(-1)^n 2\sum_{j=1}^{N} c_{\pm,j}\eta_j^{2n-1} f_\pm(x,i\eta_j)^2 = Z_n(u)\end{aligned} \tag{13}$$

immediately follow from (9) and (11). Moreover, in [6], the identities

$$\begin{aligned}-\frac{i}{\pi}\int_{-\infty}^{\infty} k^{2n-1}r_\pm(k)f'_\pm(x,k)^2dk \\ -(-1)^n 2\sum_{j=1}^{N} c_{\pm,j}\eta_j^{2n-1} f'_\pm(x,i\eta_j)^2 \\ = -Z_{n+1}(u(x)) + u(x)Z_n(u(x)) \\ -\frac{1}{2}\frac{d^2}{dx^2}Z_n(u(x))\end{aligned}$$

are derived.In particular, if the operator H has no bound states and is of the generic type, i.e., the condition (5) is valid, then the following two types of the trace formulas hold;

$$\frac{i}{\pi}\int_{-\infty}^{\infty} k^{2n-1}r_\pm(k)f_\pm(x,k)^2dk = Z_n(u)$$

$$\begin{aligned}\frac{i}{\pi}\int_{-\infty}^{\infty} k^{2n-1}r_\pm(k)f'_\pm(x,k)^2dk \\ = Z_{n+1}(u(x)) - u(x)Z_n(u(x)) + \frac{1}{2}\frac{d^2}{dx^2}Z_n(u(x))\end{aligned}$$

4 The first integrals with the local densities

It is easy to see that if $u(x,t)$ is the spatially rapidly decreasing solution of KdV equation (1), then the functional

$$I_1[u](t) = \int_{-\infty}^{\infty} \frac{1}{2}u(x,t)dx$$

is independent of t, i.e., $I_1[u]$ is the first integral of KdV equation with the local density $Z_1(u(x,t)) = \frac{1}{2}u(x,t)$. Moreover, in [5], it is shown that the functionals

$$I_n[u](t) = \int_{-\infty}^{\infty} Z_n(u(x,t))dx$$

are the first integrals of KdV equation (1). Hence, by the trace formulas (13), we have immediately

$$\begin{aligned}I_n[u](t) = \\ \frac{i}{\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty} k^{2n-1}r_\pm(k,t)f_\pm(x,k,t)^2dk \\ +(-1)^n 2\sum_{j=1}^{N} c_{\pm,j}(t)\eta_j^{2n-1}\int_{-\infty}^{\infty} f_\pm(x,i\eta_j,t)^2dx,\end{aligned}$$

where $r_\pm(k,t)$ and $c_{\pm,j}(t)$ are defined by the GGKM formulas (7), and $f_\pm(x,k,t)$ are the Jost solutions of the operator H_t. By the definition of the normalization coefficients (6), one verifies immediately

$$\begin{aligned}I_n[u](t) = \\ \frac{i}{\pi}\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty} k^{2n-1}r_\pm(k,t)f_\pm(x,k,t)^2dk \\ +(-1)^n 2\sum_{j=1}^{N} \eta_j^{2n-1}.\end{aligned}$$

Moreover, if the operator H_0 corresponding to the initial value $u(x,0)$ has no bounds states, we have the quite simple expression

$$I_n[u](t) = \frac{i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^{2n-1} r_\pm(k,t) f_\pm(x,k,t)^2 dk. \tag{14}$$

On the other hand, if the operator H_0 is reflectionless, i.e., $r_\pm(k,0) \equiv 0$, then

$$I_n[u](t) = (-1)^n 2 \sum_{j=1}^{N} \eta_j^{2n-1} \tag{15}$$

follows, and the right hand side of (15) is obviously independent of t.

It is proved independently from these expressions that the functional $I_n[u](t)$ does not depend on t. In §6, using these expressions, we give a new proof of the fact that the functional $I_n[u](t)$ does not depend on t in the case that H_0 has no bound states, and construct another kind of first integrals.

5 An evolution equation satisfied by the Jost solution

In what follows, we assume that the operator H_0 has no bound states. For the simplicity, we denote simply $f(x,k,t)$ and $r(k,t)$ instead of $f_\pm(x,k,t)$ and $r_\pm(k,t)$, and $r(k)$ instead of $r(k,0)$.

First we derive an evolution equation satisfied by the function $\exp(-8ik^3t)f(x,k,t)^2$. Put

$$g = g(x,k,t) = \exp(-8ik^3t)f(x,k,t)^2, \tag{16}$$

then, by (3), we have

$$u(x,t) = \frac{2i}{\pi}\int_{-\infty}^{\infty} kr(k)g(x,k,t)dk.$$

Substitute this into KdV equation (1), then

$$6uu_x - u_{xxx} = \frac{2i}{\pi}\int_{-\infty}^{\infty} kr(k)(6ug_x - g_{xxx})dk. \tag{17}$$

follows. By Appell's lemma mentioned in §2 as Lemma1, we have

$$g_{xxx} = 4(u-k^2)g_x + 2u_x g. \tag{18}$$

Eliminating the term g_{xxx} in (17) by (18), one verifies

$$6uu_x - u_{xxx} = \frac{2i}{\pi}\int_{-\infty}^{\infty} kr(k)(2ug_x - 2u_x g + 4k^2 g_x)dk.$$

Hence we have

$$\begin{aligned} &u_t - 6uu_x + u_{xxx} \\ &= \frac{2i}{\pi}\int_{-\infty}^{\infty} kr(k)(g_t - 2ug_x + 2u_x g - 4k^2 g_x)dk \\ &\equiv 0. \end{aligned}$$

By the definition of the Jost solution, one obtains the asymptotic identity

$$g_t - 2ug_x + 2u_x g - 4k^2 g_x \sim C(k,t)\exp(2ikx),$$

where $C(k,t)$ is a function of k and t. Hence we have the evolution equation for the function $g(x,k,t)$.

Theorem 2. *The function $g(x,k,t)$ solves the evolution equation*

$$g_t - 2ug_x + 2u_x g - 4k^2 g_x = 0. \tag{19}$$

Substitute (16) into (19), then, by the direct calculation, we have the evolution equation for the Jost solution $f = f(x,k,t)$.

Theorem 3. *The Jost solutions $f = f_\pm(x,k,t)$ of the operator H_t solve the evolution equation*

$$f_t - 2uf_x + u_x f - 4k^2 f_x = 0.$$

6 A spectral interpretation of the first integrals of KdV equation

In this section, we assume that the function $k^n r(k)$, $n \geq 1$ belongs to the Schwartz space $\mathcal{S}_k$ of k-variable functions for any $n \in \mathbb{N}$. For arbitrary $n \in \mathbb{N}$, define the function $F_n(x,t)$ and $G_n(x,t)$ by

$$\begin{aligned} F_n(x,t) &= \int_{-\infty}^{\infty} k^n r(k,t) f(x,k,t)^2 dk \\ &= \int_{-\infty}^{\infty} k^n r(k) g(x,k,t) dk, \\ G_n(x,t) &= \int_{-\infty}^{\infty} k^n r(k,t) f_x(x,k,t)^2 dk. \end{aligned}$$

Since the Jost solution $f(x,k,t)$ behaves like $\exp(ikx)$ as $x \longrightarrow \infty$, and like $\alpha\exp(ikx) + \beta\exp(-ikx)$ as $x \longrightarrow -\infty$, we have the following lemma.

Lemma 4. *The function $F_n(x,t)$ and $G_n(x,t)$ belong to the Schwartz space $\mathcal{S}_x$ of x-variable functions for any t.*

Define the functionals $J_n[u](t)$, $n \geq 1$ by

$$\begin{aligned} J_n[u](t) & \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k,t) f(x,k,t)^2 dk \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) g(x,k,t) dk \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} F_n(x,t) dx \end{aligned} \tag{20}$$

It is known that the integration in (20) converges. The convergence problem concerned with the integration of this type is known to be very delicate. We refer the reader [2] for more precise treatment concerned with the convergence problem of the integration of this type.

Now, we have the following theorem which is the main result of the present work.

Theorem 5. *Suppose that $u(x,t)$ is the spatially rapidly decreasing real valued solution of KdV equation such that the operator H_0 has no bound states. Then, the functionals $J_n[u](t)$ are independent of t, i.e., are the first integrals of KdV equation.*

Proof. By Theorem2, we have

$$\begin{aligned} \frac{d}{dt} J_n[u](t) & \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) g_t(x,k,t) dk \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) \times \\ &\qquad (2ug_x - 2u_x g + 4k^2 g_x) dk \\ &= \frac{i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k)(2ug_x - 2u_x g) dk \\ &\qquad + \frac{4i}{\pi}\int_{-\infty}^{\infty} dx \frac{\partial}{\partial x}\int_{-\infty}^{\infty} k^{n+2} r(k) g dk. \end{aligned}$$

By Lemma4,

$$\begin{aligned} &\int_{-\infty}^{\infty} dx \frac{\partial}{\partial x}\int_{-\infty}^{\infty} k^{n+2} r(k) g dk \\ &\quad = \int_{-\infty}^{\infty} \frac{\partial}{\partial x} F_{n+2}(x,t) dx = 0. \end{aligned}$$

follows. Moreover, since the product uF_n is also in $\mathcal{S}_x$, one verifies

$$\begin{aligned} \frac{d}{dt} J_n[u](t) & \\ &= -\frac{i}{\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(uF_n) dx \\ &\qquad + \frac{4i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) u g_x dk \\ &= \frac{4i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) u g_x dk. \end{aligned} \tag{21}$$

Next we calculate the last term of the expression (21). By the definition, we have

$$\begin{aligned} &\frac{4i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) u g_x dk \\ &\quad = \frac{8i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k) u f f_x dk \\ &\quad = \frac{8i}{\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} k^n r(k)(f_{xx} + k^2 f) f_x dk \\ &\quad = \frac{4i}{\pi}\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(G_n(x,t) + F_n(x,t)) dx = 0, \end{aligned}$$

where we used the relation

$$uf = f_{xx} + k^2 f.$$

This completes the proof. q.e.d.

By (14), the relation between the first integrals $J_m[u]$ and $I_n[u]$ is stated by the following corollary.

Corollary 6. *For arbitrary $n \in \mathbb{N}$, the identities*

$$J_{2n-1}[u](t) = I_n[u](t)$$

hold.

Thus, by Theorem5, we could construct the first integrals with the local densities which are not the differential polynomials $Z_n(u)$. In this case, we considered the problem for only the operator without bound states. We will discuss a similar problem for the operator with the discrete eigenvalues in the forthcoming paper [11].

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