Necessary Conditions to Obtain Voronovskaja Type Asymptotic Formulae via Statistical Limit

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Abstract: It is well-known that, obtaining the necessary and sufficient conditions is not an easy task in the approximation theory. In a recent paper, necessary and sufficient conditions to obtain direct and inverse results for a sequence of general positive linear operators are given by us. Our results are valid for not only uniform approximation but also statistical approximation. On the other hand, Voronovskaja obtained an asymptotic formulae to determine the order of uniform approximation of Bernstein polynomials to the function f. The main purpose of this paper is to give necessary conditions to obtain a Voronovskaja type asymptotic formulae for a sequence of general positive linear operators via statistical limit.

Key–Words: Statistical convergence, Voronovskaja type asymptotic formulae, positive linear operators.

1 Introduction

The classical approximation theory is an old field of the mathematical analysis and applied mathematics and still there remains an active area of researches.

Active studies on approximation theory was started after the famous theorem of Weierstrass, who showed that each continuous function defined on a closed interval can be approximated uniformly on this interval by a polynomial with any degree.

It is well-known that, obtaining the necessary and sufficient conditions is not an easy task in the approximation theory.

Necessary and sufficient conditions for the uniform approximation of a continuous function on a closed interval by positive linear operators are independently obtained by H. Bohman [10] and P.P. Korovkin [20]. These theorems are known as Bohman-Korovkin type, especially Korovkin type, theorems in the literature (see [21] for details).

The study of the Korovkin type approximation theory is a well-established area of research, which deals with the problem of approximating a function with the help of positive linear operators (see [7] for details).

First inverse results for the Bernstein polynomials were given by G.G. Lorentz [23]. And then many authors interested in these type researches (see e.g. [8], [9], [12], [13], [16], [19], [25], [27]). In [6], some direct and inverse results for Gadjiev, Ibragimov operators [17] has been obtained by A.Altın and O. Doğru.

Obtaining the direct and inverse results for positive linear operators is also important in this theory. Direct results provide the degree of convergence by means of modulus of smoothness and Lipschitz type functions. The results, giving us the necessary conditions for a function to be in the saturation class is called as inverse results. Details can be found in [24], ([11], subsection 8.2).

In a recent paper, the necessary conditions to obtain inverse results for a sequence of general positive linear operators are given by O. Doğru in [14]. These results are valid for not only uniform approximation but also statistical approximation.

First of all we recall some notations on the concept of statistical convergence.

A sequence $x = (x_k)$ is said to be statistically convergent to a number L if for every $\varepsilon > 0$, $\delta \{k \in \mathbb{N} : |x_k - L| \geq \varepsilon\} = 0$ (see [15]), where $\delta(K)$ is the natural density of the set $K \subseteq \mathbb{N}$. Recall that the subset $K \subseteq \mathbb{N}$ has density if

$$\delta(K) := \lim_n \frac{1}{n} \{\text{the number } k \leq n : k \in K\}$$

exists.

Recently a useful Korovkin type theorem for statistical convergence proved by A.D. Gadjiev and C.

Orhan [18]. The importance of statistical convergence is that any convergent sequence is statistical convergent but not conversely.

On the other hand, Voronovskaja [28] obtained the following asymptotic formulae for Bernstein polynomials (see also [22])

$$\lim_{n\to\infty} n[(B_n f)(x) - f(x)] = \frac{x(1-x)}{2} f''(x)$$

where

$$(B_n f)(x) = \sum_{k=0}^{n} f(\frac{k}{n}) \binom{n}{k} x^k (1-x)^{n-k}. \quad (1)$$

A general Voronovskaja type formulae for any sequence of positive linear operators (L_n) has the following form

$$\lim_{n\to\infty} n[(L_n f)(x) - f(x)] = F(f, f', f'', x). \quad (2)$$

If we have the expression in (2) then we can write

$$(L_n f)(x) = f(x) + \frac{1}{n} F(f, f', f'', x) + o(n^{-1}). \quad (3)$$

An equation of the form (3) is called as an asymptotic expansion for the sequence of positive linear operators (L_n).

Several studies showed that other many positive linear operators such as Meyer-König and Zeller operators [26], [2], Bleimann, Butzer and Hahn operators [1], Lupaş operators [5], Müller's Gamma operators [3], Kirov operators and their Kantorovich and Durrmeyer type integral generalizations [4] have some asymptotic approximations.

The main purpose of this paper is to give necessary conditions to obtain a Voronovskaja type asymptotic formulae for a sequence of general positive linear operators via statistical limit.

2 A sequence of positive linear operators

We consider the following sequence of general linear positive operators

$$(L_n f)(x) = \sum_{k=0}^{\infty} f(\alpha_{n,k}) Q_{n,k}(x) \quad (4)$$

where $x \in [0, a]$, $a \in \mathbb{R}^+$.

We assume that the following conditions hold:

1^0 $Q_{k,n}(x) \geq 0$,

2^0 $(L_n 1)(x) = 1$,

3^0 $(L_n t)(x) = \lambda_n x$

4^0 $(L_n t^i)(x) = \lambda_n^i x^i + \phi_{i,n}(x)$,

where

$$st - \lim_{n\to\infty} n(\lambda_n - 1)^i = 0 \text{ for all } i = 1, 2, ... \quad (5)$$

and

$$\phi_{i,n}(x) = h_{i-1}(x) \mathcal{O}(\frac{1}{n^{i-1}}) \text{ for } i \geq 2,$$

where $h_i(x)$ are continuous functions on $[0, a]$.

So we have

$$L_n((t-x)^2)(x) = x^2(\lambda_n - 1)^2 + h_1(x)\mathcal{O}(\frac{1}{n}). \quad (6)$$

We also assume that

$$L_n((t-x)^4)(x) = x^4(\lambda_n - 1)^4 + g(x)\mathcal{O}(\frac{1}{n^s}) \quad (7)$$

where $1 < s$ and $g(x)$ is a continuous function.

Application. If we choose $\alpha_{n,k} = \frac{k}{n}$, $f(t) = 0$ for $t \in \mathbb{R} \backslash [0, 1]$, $f \in C[0, 1]$ and

$$Q_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$$

where $x \in [0, 1]$, then the operators (4) turn out to be Bernstein polynomials (1) satisfying the $\lambda_n^i = 1$, $h_1(x) = x(1-x)$, $g(x) = x(1-x)(3x(1-x)+1)$ and $s = 2$. Details can be found in [22].

Another examples satisfying these conditions can be given. We will omit them.

3 A Voronovskaja-type formulae via statistical limit

In the light of the assumptions above, our main result is that:

Theorem. *Let L_n be the sequence of positive linear operators defined in (4) satisfying the conditions $1^0 - 4^0$ and* (7). *Then, for all $f \in C^\infty[0, a]$, we have*

$$st - \lim_{n\to\infty} n\left[(L_n f)(x) - f(x)\right] = \frac{h_1(x)}{2} f''(x). \quad (8)$$

Proof. From the Taylor expansion with remainder term, we have

$$\begin{aligned} f(t) = \quad & f(x) + f'(x)(t-x) + \tfrac{f''(x)}{2}(t-x)^2 \\ & +(t-x)^2 \eta(t-x) \end{aligned} \quad (9)$$

where the remainder term defined as

$$\eta(t-x) = \frac{f'''(x)}{3!}(t-x) + ...$$

is a continuous function and tends to zero for $t \to x$.

If we choose $t = \alpha_{n,k}$ in (9), we can write

$$\begin{aligned} f(\alpha_{n,k}) &= f(x) + f'(x)(\alpha_{n,k} - x) \\ &+ \frac{f''(x)}{2}(\alpha_{n,k} - x)^2 \\ &+ (\alpha_{n,k} - x)^2 \eta(\alpha_{n,k} - x). \quad (10) \end{aligned}$$

Because of the continuity, the function η is bounded. So there exists a positive constant H such that, for all h, $|\eta(h)| \leq H$ holds.

By multiplying $Q_{n,k}(x)$ and taking sum from $k = 0$ to infinitiy from two hand-side of (10), we get

$$\begin{aligned} &(L_n f)(x) = f(x)(L_n 1)(x) + f'(x) L_n(t - x)(x) \\ &+ \tfrac{f''(x)}{2} L_n((t - x)^2)(x) \\ &+ \sum_{k=0}^{\infty} (\alpha_{n,k} - x)^2 \eta(\alpha_{n,k} - x) Q_{n,k}(x). \end{aligned} \quad (11)$$

By using 2^0 and (6) in (11), we have

$$\begin{aligned} (L_n f)(x) &= f(x) + f'(x) x(\lambda_n - 1) \\ &+ \frac{f''(x)}{2}(x^2(\lambda_n - 1)^2 \\ &+ h_1(x)\mathcal{O}(\frac{1}{n})) + I \quad (12) \end{aligned}$$

where

$$I = \sum_{k=0}^{\infty} (\alpha_{n,k} - x)^2 \eta(\alpha_{n,k} - x) Q_{n,k}(x).$$

Now, we consider the sum I as follows

$$\begin{aligned} I = &\sum_{\substack{k=0 \\ |\alpha_{n,k} - x| \leq \delta}}^{\infty} (\alpha_{n,k} - x)^2 \eta(\alpha_{n,k} - x) Q_{n,k}(x) \\ + &\sum_{\substack{k=0 \\ |\alpha_{n,k} - x| > \delta}}^{\infty} (\alpha_{n,k} - x)^2 \eta(\alpha_{n,k} - x) Q_{n,k}(x). \quad (13) \end{aligned}$$

Since η is a continuous function, for each $\varepsilon > 0$, there exists a $\delta(\varepsilon)$ such that $|\eta(\alpha_{n,k} - x)| < \varepsilon$ for all $|\alpha_{n,k} - x| \leq \delta$. On the other hand, since the function η is bounded, we have $|\eta(\alpha_{n,k} - x)| < H$ for $|\alpha_{n,k} - x| > \delta$. If we use these expressions in (13), we have

$$I \leq \varepsilon \sum_{k=0}^{\infty} (\alpha_{n,k} - x)^2 Q_{n,k}(x) + HJ \quad (14)$$

where

$$J = \sum_{\substack{k=0 \\ |\alpha_{n,k} - x| > \delta}}^{\infty} (\alpha_{n,k} - x)^2 Q_{n,k}(x). \quad (15)$$

If $|\alpha_{n,k} - x| > \delta$ then we have $\frac{(\alpha_{n,k} - x)^2}{\delta^2} > 1$. So we get

$$J \leq \frac{1}{\delta^2} L_n((t - x)^4)(x). \quad (16)$$

By using (7) in (16), we obtain

$$J \leq \frac{1}{\delta^2}(x^4(\lambda_n - 1)^4 + g(x)\mathcal{O}(\frac{1}{n^s})). \quad (17)$$

Using (6) and (17) in (14), we have

$$\begin{aligned} I &\leq \varepsilon(x^2(\lambda_n - 1)^2 + h_1(x)\mathcal{O}(\frac{1}{n})) \\ &+ \frac{H}{\delta^2}(x^4(\lambda_n - 1)^4 + g(x)\mathcal{O}(\frac{1}{n^s})). \end{aligned}$$

So there exists two positive constants c_1 and c_2 such that

$$\begin{aligned} I &\leq \varepsilon(x^2(\lambda_n - 1)^2 + h_1(x)\frac{c_1}{n}) + \frac{H}{\delta^2}(x^4(\lambda_n - 1)^4 \\ &+ g(x)\frac{c_2}{n^s}) \end{aligned}$$

where $s > 1$. Then we can write

$$\begin{aligned} I &\leq \frac{1}{n}(\varepsilon(x^2 n(\lambda_n - 1)^2 + c_1 h_1(x)) \\ &+ \frac{H}{\delta^2}(x^4 n(\lambda_n - 1)^4 + g(x)\frac{c_2}{n^{s-1}})). \end{aligned}$$

This means that

$$\begin{aligned} I &= \mathcal{O}(\frac{1}{n})(\varepsilon(x^2 n(\lambda_n - 1)^2 + h_1(x)) \\ &+ \frac{H}{\delta^2}(x^4 n(\lambda_n - 1)^4 + g(x)\frac{1}{n^{s-1}})). \quad (18) \end{aligned}$$

Using (18) in (12) we get

$$\begin{aligned} (L_n f)(x) - f(x) &= \mathcal{O}(\frac{1}{n})(f'(x) x n(\lambda_n - 1) \\ &+ \frac{f''(x)}{2}(x^2 n(\lambda_n - 1)^2 \\ &+ h_1(x)) + (\varepsilon(x^2 n(\lambda_n - 1)^2 \\ &+ h_1(x)) + \frac{H}{\delta^2}(x^4 n(\lambda_n - 1)^4 \\ &+ g(x)\frac{1}{n^{s-1}})). \quad (19) \end{aligned}$$

Since ε is an arbitrary positive constant and $st - \lim_{n\to\infty} n(\lambda_n - 1) = st - \lim_{n\to\infty} n(\lambda_n - 1)^2 = st - \lim_{n\to\infty} n(\lambda_n - 1)^4 = 0$ by the condition (5), we have

$$st - \lim_{n\to\infty} n\left[(L_n f)(x) - f(x)\right] = \frac{f''(x)}{2} h_1(x)$$

which gives the proof.

Remark. If we take the classical limit instead of the statistical limit in (5) and (8), then our theorm gives the necessary conditions to obtain a Voronovskaja type asymptotic formulae in ordinary sense.

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