On the radial solutions for some nonlinear initial value problems

GABRIELLA BOGNAR
University of Miskolc
Department of Analysis
Miskolc-Egyetemvaros
HUNGARY

ERIKA ROZGONYI
University of Miskolc
Department of Analysis
Miskolc-Egyetemvaros
HUNGARY

Abstract: We consider the positive radial symmetric solutions to the nonlinear problem $div\left(|\nabla u|^{p-2}\nabla u\right)+f(u)=0$ in $B_R=\{x\in\mathbf{R}^n:|x|<R\}$, $1<p<n$, where $f(u)=u^\gamma+u^\delta$ for $u\geq 0$. For some special values of parameters γ and δ we give the positive entire solution. We examine the existence of local solutions and give a method for its determination. Comparing the local and entire solution, the error can be investigated.

Key–Words: Nonlinear partial differential equations, *p*-Laplacian, superlinear and supercritical exponents, local solutions

1 Introduction

Many physical, chemical, biological and environmental processes are driven by diffusion and chemical or biological reactions representing instantaneous interactions.

We consider the solution $u=u(|x|)$ for the radial problem

$$\Delta_p u+f(u)=0 \text{ and } u>0 \text{ in } B_R, \tag{1}$$

where $B_R=\{x\in\mathbf{R}^n:|x|<R\}$, $1<p<n$, $\Delta_p u=div\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian of u and

$$f(u)=\begin{cases} u^\gamma+u^\delta & \text{for } u\geq 0,\\ 0 & \text{for } u<0.\end{cases}$$

This problem is a generalization of the problem of Lin and Ni [6]. The semilinear problem with superlinear and supercritical exponents was considered in [6] for the case $p=2$.

We shall study the positive radial solution of (1), i.e., the initial value problem (here $r=|x|$):

$$\begin{cases} r^{1-n}\left(r^{n-1}|u_r'|^{p-2}u_r'\right)'+f(u)=0\\ \qquad\text{in}\quad(0,\infty),\\ u_r'(0)=0,\quad u(0)=\alpha\geq 0,\end{cases} \tag{2}$$

or equivalently,

$$\begin{cases} (p-1)|u_r'|^{p-2}u_{rr}''+\frac{n-1}{r}|u_r'|^{p-2}u_r'+f(u)=0\\ \qquad\text{in } (0,\infty),\\ u_r'(0)=0,\quad u(0)=\alpha\geq 0.\end{cases} \tag{3}$$

Positive radial solution of (1) is $u=u(r)$,which solves the initial value problem associated with (2) (or (3)) for some $\alpha\geq 0$, $u(r)>0$ for $r\in(0,\infty)$ and $\lim\limits_{r\to\infty}u(r)=0$.

Our goal is to give the exact solution for the initial value problem of (2) (or (3)) for some special values of the parameters. Moreover, we examine the existence of local solution to the initial value problem of (3) and we give a method for the determination of the power series solution for given values of parameters p, n, α, γ and δ. Comparing the exact and power series solution we are able to find the error for some cases.

2 Positive entire solution

Positive entire radial solution of

$$\Delta_p u+f(u)=0 \tag{4}$$

is a function $u=u(r)$ which solves (3) for some $\alpha>0$, $u(r)>0$ for $r\in(0,\infty)$ and $\lim\limits_{r\to\infty}u(r)=0$. Moreover, we suppose that $p-1<\gamma<p^*-1$ and $\delta>p^*-1$, $p^*=\frac{np}{n-p}$, the exponents γ and δ satisfy

$$\gamma=\frac{\beta+1}{\beta}(p-1) \quad\text{and}\quad \delta=\gamma+\frac{1}{\beta}, \tag{5}$$

where

$$\beta \in \left(\frac{(n-p)(p-1)}{p^2},\ \frac{n-p}{p} \right), \tag{6}$$

i.e., $\gamma = \frac{\delta+1}{p'}$, $p' = \frac{p}{p-1}$.

Proposition 1 *Let $p,\ p',\ n,\ \beta,\ \gamma,\ \delta$, and f be as above. Set*

$$a = \frac{1}{\left(\frac{n}{(\beta+1)p} - 1\right)^{\beta}}, \tag{7}$$

$$b = \frac{(n-(\beta+1)\,p)^{p'}}{(\beta+1)\ p} \left(\frac{\beta\, p}{p-1}\right)^{\frac{1}{p-1}}, \tag{8}$$

$$u(r) = a\left(\frac{b}{b+r^{p'}}\right)^{\beta}.$$

Then u is a positive entire solution of (4) *.*

From straight forward computations we get

$$r^{1-n}\left(r^{n-1}\left|u'(r)\right|^{p-2}u'(r)\right)' =$$

$$-\frac{\left(\beta p' a b^{\beta}\right)^{p-1}}{\left(b+r^{p'}\right)^{(\beta+1)(p-1)+1}}\left(n(b+r^{p'})\ -(\beta+1)pr^{p'}\right)$$

$$u^{\gamma}(r) + u^{\delta}(r) =$$

$$\frac{a^{\frac{(\beta+1)(p-1)}{\beta}} b^{(\beta+1)(p-1)+1}\left(1+a^{\frac{1}{\beta}}\right)}{\left(b+r^{p'}\right)^{(\beta+1)(p-1)+1}}$$

$$+\frac{a^{\frac{(\beta+1)(p-1)}{\beta}} b^{(\beta+1)(p-1)} r^{p'}}{\left(b+r^{p'}\right)^{(\beta+1)(p-1)+1}}$$

To fulfil the equation (2) we must have

$$a^{\frac{(\beta+1)(p-1)}{\beta}} b^{(\beta+1)(p-1)}\left(1+a^{\frac{1}{\beta}}\right) = \left(\beta p' a b^{\beta}\right)^{p-1} n$$

and

$$(\beta+1)p\left(\beta p' a b^{\beta}\right)^{p-1} = \left(\beta p' a b^{\beta}\right)^{p-1} n$$

$$-a^{\frac{(\beta+1)(p-1)}{\beta}} b^{(\beta+1)(p-1)}.$$

From here we get for a and b the same as in (7) and (8). Both constants fit with [6] for the choice $\delta = \frac{\beta+2}{\beta}$, $p = 2$.

For the determination of the exact solution of (1) we refer to the paper by Bognar and Drabek [1].

3 The existence of local solution

We shall form problem (3) as the system of special Briot-Bouquet differential equations. For this type of differential equations we refer to the book of E. Hille [4] and E. L. Ince [5].

Theorem 2 *(Briot-Bouquet Theorem) Let us assume that for the system of equations*

$$\left.\begin{aligned} \xi \tfrac{dz_1}{d\xi} &= u_1(\xi, z_1(\xi), z_2(\xi)), \\ \xi \tfrac{dz_2}{d\xi} &= u_2(\xi, z_1(\xi), z_2(\xi)), \end{aligned}\right\} \tag{9}$$

where functions u_1 and u_2 are holomorphic functions of ξ, $z_1(\xi)$, and $z_2(\xi)$ near the origin, moreover $u_1(0,0,0) = u_2(0,0,0) = 0$, then a holomorphic solution of (9) *satisfying the initial conditions $z_1(0) = 0$, $z_2(0) = 0$ exists if none of the eigenvalues of the matrix*

$$\begin{bmatrix} \left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} & \left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} \\ \left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} & \left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} \end{bmatrix}$$

is a positive integer.

For a proof of Theorem 2 we refer to [2].

The differential equation (3) has singularity at $r = 0$ for the case $n > 1$. Theorem 2 ensures the existence of formal solutions $z_1 = \sum\limits_{k=0}^{\infty} a_k \xi^k$ and $z_2 = \sum\limits_{k=0}^{\infty} b_k \xi^k$ for system (9), and also the convergence of formal solutions.

Theorem 3 *For any $p,\ \gamma,\ \delta,\ \ n$ as above, the initial value problem* (3) *$u(0) = \alpha, u'(0) = 0$ has an unique analytic solution of the form $u(r) = Q\left(r^{p/(p-1)}\right)$ in $(0,\ A)$ for small real value of A, where Q is a holomorphic solution to*

$$Q'' = \frac{-1}{p\ (1+1/p)^{p+1}} r^{-\frac{p+1}{p}} \frac{Q^{\gamma}+Q^{\delta}}{|Q'|^{p-1}} - \frac{n}{p\ \alpha} r^{-(1+1/p)} Q'$$

near zero satisfying

$$Q(0) = \alpha,\ \ Q'(0) = \frac{1-p}{p}\left(\frac{\alpha^{\gamma}+\alpha^{\delta}}{n}\right)^{\frac{1}{p-1}}.$$

Proof. We shall now present a formulation of (3) as a system of Briot-Bouquet type differential equations (9). Let us take solution of (3) in the form

$$u(r) = Q\left(r^{\sigma}\right),\ r \in (0, A),$$

where $Q \in C^2(0,a)$ and $\sigma > 0$. Let us take $u(r) = Q\left(r^{\sigma}\right)$ into (3) we obtain

$$\begin{aligned} Q''\left(r^{\sigma}\right) &= -\frac{Q^{\gamma}+Q^{\delta}}{|Q'|^{p-2}}\frac{r^{-p\,(\sigma-1)}}{(p-1)\,\sigma^{p}} \\ &\quad -\frac{n-1+(p-1)\,(\sigma-1)}{(p-1)\,\sigma}r^{-\sigma}Q' \end{aligned}$$

and substituting $\xi = r^{\sigma}$ we have

$$Q''(\xi) = -\frac{Q^{\gamma}+Q^{\delta}}{|Q'|^{p-2}}\frac{\xi^{-p\,\frac{\sigma-1}{\sigma}}}{(p-1)\,\sigma^{p}} \quad (10)$$

$$-\frac{n-1+(p-1)\,(\sigma-1)}{(p-1)\,\sigma}\xi^{-1}Q'. \quad (11)$$

Here, we introduce function Q as follows

$$Q(\xi) = g_0 + g_1\xi + F(\xi), \quad (12)$$

where $F \in C^2(0,a)$, $F(0) = 0$, $F'(0) = 0$. Therefore we have $Q(0) = g_0$, $Q'(0) = g_1$, $Q'(\xi) = g_1 + F'(\xi)$, $Q''(\xi) = F''(\xi)$. From initial condition $u(0) = \alpha$ we have that

$$g_0 = \alpha.$$

We restate (10) as a system of equations:

$$\left.\begin{array}{l} z_1(\xi) = F(\xi) \\ z_2(\xi) = F'(\xi) \end{array}\right\} \text{ with } \left.\begin{array}{l} z_1(0) = 0 \\ z_2(0) = 0 \end{array}\right\},$$

according to (10) we get that

$$F''(\xi) = -\frac{\xi^{-\frac{\sigma-1}{\sigma}p}}{(p-1)\,\sigma^{p}}\,G\left(\xi,\ z_1,\ z_2\right) -\frac{n-1+(p-1)\,(\sigma-1)}{(p-1)\,\sigma}\xi^{-1}\left(g_1+F'(\xi)\right),$$

$$G(.,.,.) = \frac{\left[g_0+g_1\xi+F(\xi)\right]^{\gamma}+\left[g_0+g_1\xi+F(\xi)\right]^{\delta}}{|g_1+F'(\xi)|^{p-2}}$$

We generate the system of equations

$$\left.\begin{array}{l} u_1(\xi, z_1(\xi), z_2(\xi)) = \xi\ z_1'(\xi) \\ u_2(\xi, z_1(\xi), z_2(\xi)) = \xi\ z_2'(\xi) \end{array}\right\}$$

as follows

$$\left.\begin{array}{l} u_1(\xi, z_1(\xi), z_2(\xi)) = \xi\ z_2 \\ u_2(\xi, z_1(\xi), z_2(\xi)) = -\frac{\xi^{1-\frac{\sigma-1}{\sigma}p}}{(p-1)\,\sigma^{p}}G\left(\xi,\ z_1,\ z_2\right) \\ \quad -\frac{n-1+(p-1)(\sigma-1)}{(p-1)\,\sigma}\left(g_1+z_2(\xi)\right) \end{array}\right\}.$$

In order to satisfy conditions $u_1(0,0,0) = 0$ and $u_2(0,0,0) = 0$ we must get zero for the power of ξ in the right-hand side of the second equation:

$$1 - \frac{p(\sigma-1)}{\sigma} = 0,$$

i.e.,

$$\sigma = \frac{p}{p-1}.$$

To ensure $u_2(0,0,0) = 0$ we have the connection

$$n\, g_1\, |g_1|^{p-2} + \left(\frac{p-1}{p}\right)^{p-1}\left(g_0^{\gamma}+g_0^{\delta}\right) = 0,$$

i.e.,

$$g_1 = \frac{1-p}{p}\left(\frac{g_0^{\gamma}+g_0^{\delta}}{n}\right)^{\frac{1}{p-1}}. \quad (13)$$

Therefore, we obtain

$$g_1 = \frac{1-p}{p}\left(\frac{\alpha^{\gamma}+\alpha^{\delta}}{n}\right)^{\frac{1}{p-1}}. \quad (14)$$

From initial conditions $u(0) = \alpha \neq 0$, $u'(0) = 0$, and (12) it follows that $g_0 = \alpha$.

For u_1 and u_2 we find that

$$\begin{aligned} \left.\frac{\partial u_1}{\partial z_1}\right|_{(0,0,0)} &= 0, \\ \left.\frac{\partial u_1}{\partial z_2}\right|_{(0,0,0)} &= 0, \\ \left.\frac{\partial u_2}{\partial z_1}\right|_{(0,0,0)} &= -\frac{\gamma g_0^{\gamma-1}+\delta g_0^{\delta-1}}{(p-1)\,\sigma g_1^{p-2}}, \\ \left.\frac{\partial u_2}{\partial z_2}\right|_{(0,0,0)} &= -\frac{n\,(p-1)}{p}. \end{aligned}$$

Therefore the eigenvalues of matrix

$$\left[\begin{array}{cc} \partial u_1/\partial z_1 & \partial u_1/\partial z_2 \\ \partial u_2/\partial z_1 & \partial u_2/\partial z_2 \end{array}\right]$$

at $(0,0,0)$ are 0 and $-n(p-1)/p$. Since both eigenvalues are non-positive, applying Theorem 2 we get the existence of unique analytic solutions z_1 and z_2 at zero. Thus we get the analytic solution $Q(\xi) = g_0 + g_1\xi + F(\xi)$ satisfying (10) with $Q(0) = g_0$, $Q'(0) = g_1$, where $g_0 = \alpha$ and g_1 is determined in (14). ■

Corollary 4 *From Theorem 3 it follows that solution $u(r)$ for (3) has an expansion near zero of the form $u(r) = \sum_{k=0}^{\infty} g_k\, r^{\frac{kp}{p-1}}$ satisfying $u(0) = \alpha$ and $u'(0) =$* 0.

4 Determination of local solution

We determine the power series solution for (3). The initial conditions are

$$u(0) = \alpha,$$
$$u'(0) = 0.$$

We seek a solution of the form

$$u(r) = g_0 + g_1\, r^{\frac{p}{p-1}} + g_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots, \quad r > 0, \quad (15)$$

with coefficients $g_k \in \mathbf{R}$, $k = 0, 1, \ldots$.

From Section 3 we get that $g_0 = \alpha$ and $g_1 = \frac{1-p}{p}\left(\frac{\alpha^\gamma + \alpha^\delta}{n}\right)^{\frac{1}{p-1}}$. We have $u(r) > 0$ and $u'(r) < 0$ near zero. Since

$$u'(r) = r^{\frac{1}{p-1}}\left[g_1 \frac{p}{p-1} + g_2 \frac{2p}{p-1}\, r^{\frac{p}{p-1}} + \ldots\right],$$

and hence

$$\left|u'(r)\right|^{p-2} u'(r) =$$

$$r\left[g_1 \frac{p}{p-1} + g_2 \frac{2p}{p-1}\, r^{\frac{p}{p-1}} + \ldots\right]^{p-1} =$$

$$r\left[P_0 + P_1\, r^{\frac{p}{p-1}} + P_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots\right],$$

moreover

$$r^{1-n}\left(r^{n-1}\left|u'(r)\right|^{p-2} u'(r)\right)' =$$
$$\left[P_0 n + P_1\left(n + \frac{p}{p-1}\right) r^{\frac{p}{p-1}} + \ldots\right.$$
$$\left. + P_k\left(n + \frac{kp}{p-1}\right) r^{k\left(\frac{p}{p-1}\right)} + \ldots\right],$$

where coefficients P_k will be expressed in terms of g_k $(k = 0, 1, \ldots)$. For $u^\gamma(t)$ and $u^\delta(t)$

$$\begin{aligned} u^\gamma(t) &= \left[g_0 + g_1\, r^{\frac{p}{p-1}} + g_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots\right]^\gamma \\ &= G_0 + G_1\, r^{\frac{p}{p-1}} + G_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots \end{aligned}$$

$$\begin{aligned} u^\delta(t) &= \left[g_0 + g_1\, r^{\frac{p}{p-1}} + g_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots\right]^\delta \\ &= D_0 + D_1\, r^{\frac{p}{p-1}} + D_2\, r^{2\left(\frac{p}{p-1}\right)} + \ldots, \end{aligned}$$

where coefficients G_k and D_k can be expressed in terms of g_k $(k = 0, 1, \ldots)$.

Substituting them into the equation (3) we compare the coefficients of the proper power of r we find

$$P_k\left(n + \frac{kp}{p-1}\right) + G_k + D_k = 0 \qquad (16)$$
$$\text{for } k \geq 0.$$

Applying the J. C. P. Miller formula (see [3]) for the determination of P_k, G_k and D_k $(k = 0, 1, \ldots)$ we have.

$$G_k = \frac{1}{k\alpha}\sum_{j=0}^{k-1}\left[(k-j)\,\gamma - j\right] G_j g_{k-j},$$

$$D_k = \frac{1}{k\alpha}\sum_{j=0}^{k-1}\left[(k-j)\,\delta - j\right] D_j g_{k-j},$$

$$P_k = \sum_{j=0}^{k-1}\left[(k-j)\,(p-1) - j\right] P_j g_{k+1-j} \frac{(k+1-j)}{g_1 k}$$

for $k \geq 1$, and $G_0 = g_0^\gamma$, $D_0 = g_0^\delta$, $P_0 = \left(g_1 \frac{p}{p-1}\right)^{p-1}$.

From (16) we obtain coefficients g_k for $k \geq 2$:

$$g_0 = \alpha,$$

$$g_1 = \frac{(1-p)\left(\frac{\alpha^\gamma + \alpha^\delta}{n}\right)^{\frac{1}{p-1}}}{p},$$

$$g_2 = \frac{0.5\left(\frac{\alpha^\gamma + \alpha^\delta}{n}\right)^{\frac{3-p}{p-1}}}{\alpha n p^5 - \alpha n p^4 + \alpha p^5}$$

$$\cdot\left(\gamma\alpha^\gamma p^2 - 2\gamma\alpha^\gamma p + \gamma\alpha^\gamma + \delta\alpha^\delta p^2 - 2\delta\alpha^\delta p + \delta\alpha^\delta\right),$$

$$\vdots$$

Example Let us consider the solution of (3) with parameters

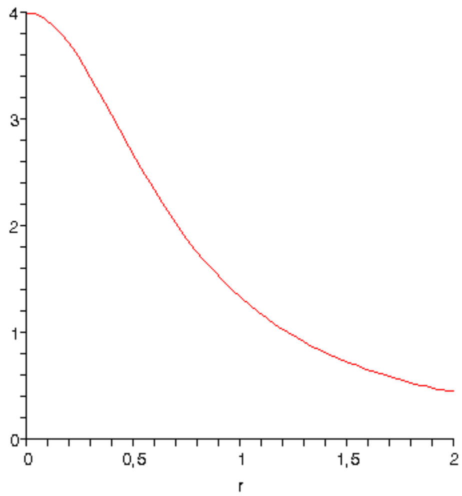
$$p = 2,\ n = 5,\ \alpha = 4,\ \gamma = 2,\ \delta = 3.$$

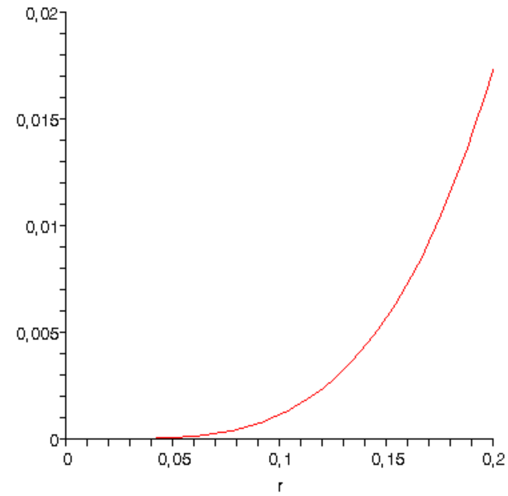
For that case we can determine the coefficients of the power series solution

$$u(r) = \sum_{k=0}^{\infty} g_k\, r^{2k}$$

as follows

$$g_0 = 4,$$





$$g_1 = -8,$$
$$g_2 = 4,$$
$$g_3 = -1.2222,$$
$$g_4 = 4.1125,$$
$$g_5 = -0.0157,$$
$$g_6 = 2.6347$$
$$\vdots$$

Therefore, we have

$$u(r) = 4 - 8r^2 + 4r^4 - 1.2222r^6 \qquad (17)$$

$$+4.1125r^8 - 0.0157r^{10} + 2.6347r^{12} + \ldots$$

5 The comparison of exact and local analytic solutions

The example above gives an approximate solution to the problem (3). The parameters satisfy the conditions (5) for the entire solution of the same problem. The entire solution with the given parameters is

$$\overline{u}(r) = \frac{4}{1 + 2r^2} \qquad (18)$$

since

$$a = 4,\ b = 0.5,\ \beta = 1,\ p' = 2$$

(see Fig. 1.)

Hence, we have the possibility to compare the two solutions, i.e., (17) with (18). On Fig. 2. the difference between solutions (17) and (18) are represented.

6 Conclusion

For problem (3) the entire solutions can be evaluated under restrictions (5).

In other cases (and in this case as well) we are able to give a convergent power series solution in the neighborhood of zero. We note, that the local solution can be given in the neighborhood of any $r_0 \in R$ and for any parameter values p, n, α, γ, δ.

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