A Pseudo-Convex MINLP Approach for Industrial Problems

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Abstract: In the present paper an optimization approach using a pseudo-convex objective function is presented. Two different industrial applications are considered. The main issue in both problems are: which decisions should be made in order to both maximize the profits of the production and to minimize the production cycle time. The problems can be modeled and solved using Mixed Integer Nonlinear Programming (MINLP) techniques. A set of test problems are solved using the Extended Cutting Plane (ECP) method that has been proven very efficient on many complex engineering problems. The presented techniques can be applied in the design of new industrial systems and to improve the performance of already existing ones.

Key–Words: Mixed Integer Non-Linear Programming, Extended Cutting Plane Method, Chromatographic Separation, Printed Circuit Board Assembly

1 Introduction

The mixed integer optimization methods provides efficient tools, with negligible costs, that can be applied in the solving of complicated technical problems in different industries. The objective in many industries is to achieve greater profits and/or smaller costs. The costs might be of any kind: raw material, equipment, salaries, energy, waste costs etc. Reduced energy consumption and/or a reduced amount of waste are also important environmental issues that can, in some cases, be tackled using mathematical optimization.

In many industrial production planning problems the task is to decide which decisions should be made at what times in order to maximize the profit. The objective may, in theory, be formulated as the following:

$$max \left\{ \frac{1}{\tau} \int_0^\tau P(t)dt \right\} \tag{1}$$

where τ describes the length of the current period. The function $P(t)$ returns, for example, the profit of the produced products at the time t. Note, that the same decisions are made repeatedly in periods of the length τ. By dividing the profit by the length of the period, the objective gives a comparable measure of how much profit is obtained per time unit. There are many possibilities and open questions in the modeling of the price function $P(t)$. However, when using gradient-based optimization methods, it is preferable to formulate functions that can be expressed as explicit and, hopefully, as convex as possible. Most optimization algorithms perform well when the problems are convex, that is, all functions are convex. Although many algorithms have been proven global convergence for convex problems, these usually also perform well in problems containing so called pseudo-convex functions.

Definition 1 *Assume that $f : C \rightarrow \mathbb{R}$ is differentiable. The function f is* pseudo-convex *if*

$$\nabla f(x)^T(y-x) \geq 0 \Rightarrow f(y) \geq f(x), \quad \forall x, y \in C$$

or equivalently

$$f(y) < f(x) \Rightarrow \nabla f(x)^T(y-x) < 0, \quad \forall x, y \in C.$$

From the definition above it can be noted that all local minimas of a pseudo-convex function (that is, for all x such that $\nabla f(x) = 0$), are also global ones. The following result regarding fractional functions is given in [2]:

Theorem 2 *Assume that $f(x)$ is differentiable and a real valued convex function and $g(x)$ is a positive linear function over a convex set C. Then $f(x)/g(x)$ is a pseudo-convex function over C.*

The theorem above is useful when proving the pseudo-convexity for objective functions of the form in 1.

2 A chromatographic separation problem

In [4] a chromatographic separation problem is presented. The objective is to, within reasonable costs, separate products of a mixture as efficiently as possible during a continuous cyclic operation. The formulation of the separation problem that was implemented and solved can be summarized as follows:

$$\max \quad \left\{ \frac{1}{\tau} \left(\sum_{k=1}^{K} \sum_{i=1}^{T} \left(\sum_{j=1}^{C} p_j s_{kij} \right) - w d_{ki} \right) \right\}$$

subject to

$$\left\{ \begin{aligned} t_{i-1} &\leq t_i \\ t_T &= \tau \\ \sum_{j=1}^{C} y_{kij} + \sum_{l=1}^{K} x_{kil} &\leq 1 \\ y_{ki}^{in} + \sum_{l=1}^{K} x_{lik} &\leq 1 \\ (t_i - t_{i-1}) - M_1(1 - y_{ki}^{in}) &\leq d_{ki} \\ s_{kij} &\leq m_{kij} \\ s_{kij} &\leq M_2 y_{kij} \\ \sum_{j=1}^{C} m_{kij} - M_4 y_{kij} &\leq q_{kij} \\ R_j \cdot \sum_{k=1}^{K} \sum_{i=1}^{T} q_{kij} - \sum_{k=1}^{K} \sum_{i=1}^{T} s_{kij} &\leq 0 \end{aligned} \right. \tag{2}$$

$$\left\{ \begin{aligned} m_{kij} &\leq \int_{t_{i-1}}^{t_i} c_{kj}(t, z_k^H) dt + M_3(1 - y_{kij}) \\ m_{kij} &\geq \int_{t_{i-1}}^{t_i} c_{kj}(t, z_k^H) dt - M_3(1 - \sum_{\substack{l=1 \\ l \neq j}}^{J} y_{kil}) \end{aligned} \right.$$

where $k = 1, \ldots, K$, $i = 1, \ldots, T$, and $j = 1, \ldots, C$. The concentrations $c_{kj}(t, z)$, that are integrands in (2), are obtained by solving the following boundary value problem:

$$\begin{cases} (1 + F\beta_j) \frac{\partial c_{kj}}{\partial t} + F \sum_{l=1}^{C} \beta_{jl} \left(c_{kl} \frac{\partial c_{kj}}{\partial t} + c_{kj} \frac{\partial c_{kl}}{\partial t} \right) \\ +u \frac{\partial c_{kj}}{\partial z} = D_j \frac{\partial^2 c_{kj}}{\partial z^2} \\ c_{kj}(t, 0) = y_k^{in}(t) \cdot c_j^{in} + \sum_{l=1}^{K} x_{lk}(t) \cdot c_{lj}(t, z_k^H) \\ c_{kj}(0, z) = c_{kj}(\tau, z) \end{cases} \tag{3}$$

where the logical functions $y_k^{in}(t)$ and $x_{lk}(t)$ are defined as follows:

$$\begin{cases} y_k^{in}(t) = \sum_{i=1}^{T} y_{ki}^{in} \cdot \delta_i(t) \\ x_{lk}(t) = \sum_{i=1}^{T} x_{lik} \cdot \delta_i(t) \\ \delta_i(t) = \begin{cases} 1 & \text{if } t \in [t_{i-1}, t_i], i = 1, \ldots, T \\ 0 & \text{otherwise.} \end{cases} \end{cases} \tag{4}$$

Note, that the logical decisions are modeled using binary variables while the other variables are continuous ones. In (2) the objective function is pseudo-convex, which can easily be verified by theorem 2.

3 A printed circuit board assembly problem

In [1] and [6] the production planning of a Printed Circuit Board (PCB) assembly line is considered and modeled as a Mixed Integer Linear Programming (MILP) problem. The main issue is the choice of components: which components should be assembled on the respective machines in order to minimize the production cycle time. In the following, an alternative approach using a pseudo-convex objective function, that is similar to the one used in (2), is given.

$$\max \quad \left\{ \frac{1}{\tau} \sum_{k=1}^{K} c_k Y_k \right\}$$

subject to

$$\left\{ \begin{aligned} \sum_{i=1}^{I} \sum_{m=1}^{M} t_{ik} z_{ikm} &\leq \tau \\ \sum_{k=1}^{K} \sum_{m=1}^{M} z_{ikm} &= d_i \\ z_{ikm} - d_i y_{ikm} &\leq 0 \\ y_{ikm} - z_{ikm} &\leq 0 \\ \sum_{m=1}^{M} y_{ikm} &\leq 1 \\ y_{ikm} - Y_k &\leq 0 \end{aligned} \right. \tag{5}$$

where the indexes $i = 1, \ldots, I$, $j = 1, \ldots, C$, $k = 1, \ldots, K$ and $m = 1, \ldots, M$. The variables Y_k and y_{ikm} are binary variables and z_{ikm} are integer ones. The continuous variable τ denotes the assembly time of the slowest machine in the production line. In (5) all constraints are linear ones, the only nonlinear function is the objective function. Both problem formulations (2) and (5) includes so-called Special Ordered Sets of order one (SOS1) that allows an efficient type of Branch and Bound (BB) algorithm that is needed within the ECP-method.

4 The ECP-method

The example problems were solved using the Extended Cutting Plane (ECP) method which is an extension of Kelley's Cutting Plane (CP) method [5] for solving convex Non-Linear Programming (NLP) problems. The ECP method is a general purpose MINLP method with applicability to a large variety of MINLP problems. The ECP method was first extended in order to handle convex MINLP problems [9]. The method was further developed in [10] in order to enable the solving of problems consisting of both a pseudo-convex objective function and pseudo-convex constraints.

The general MINLP problems to be solved with the ECP method can be formulated as follows:

$$\min_{\boldsymbol{z}\in N\cap L} \; f(\boldsymbol{z})$$
$$N = \{\boldsymbol{z}|\boldsymbol{g}(\boldsymbol{z}) \le 0\} \qquad (P)$$
$$L = \{\boldsymbol{z}|\boldsymbol{A}\boldsymbol{z} \le \boldsymbol{a},\ \boldsymbol{B}\boldsymbol{z} = \boldsymbol{b}\} \cap X \times Y$$

The variable vector, $\boldsymbol{z}$, consists of both a continuous part and an integer part that are bound by the X and Y sets, respectively. The objective function, $f(\boldsymbol{z})$, and the nonlinear constraints, $\boldsymbol{g}(\boldsymbol{z})$, should be differentiable pseudo-convex functions defined on the set L. If the functions $\boldsymbol{g}$ and f are pseudo-convex and if the set X is a compact subset of $\Re^n$ and if Y is a finite discrete set in Z^m, then the ECP algorithm will ensure convergence to the global optimal solution.

The ECP method solves the problem (P) by solving the following sequence of Mixed Integer Linear Programming (MILP) problems:

$$\min_{(\mu,\boldsymbol{z})\in\Omega_k} \mu \qquad (P_k)$$

where the set Ω_k is defined by

$$\Omega_k = L \cap \{\boldsymbol{z} \mid l_j(\boldsymbol{z}) \le 0, j = 1,2,\ldots,J_k\}$$

This iterative procedure begins with $\Omega_0 = L$. Note, that $l_j(\boldsymbol{z}) \in \Omega_k$ are cutting planes underestimating the entire feasible region of (P) and J_k is the number of cutting planes in Ω_k at iteration k. After each iteration, a new MILP subproblem is generated by adding and/or modifying old cutting planes of the most violating nonlinear constraints. The generated cutting planes, with respect to the constraints, are of the following type:

$$g_i(\boldsymbol{z}) + \alpha_k^{r'} \nabla g_i(\boldsymbol{z})^T(\boldsymbol{z} - \boldsymbol{z}_k) \le 0 \qquad (6)$$

where $\boldsymbol{z_k}$ is the solution to the previous MILP problem (P_k). The MILP-subproblems were solved using the CPLEX-software [7]. The scalar, $\alpha_k^{r'}$, is initially one but can be updated in subsequent iterations in order to guarantee that no part of the feasible region is cut off. Convergence to the global solution is ensured when the sequence of points converges to a solution in the feasible region of the problem (P), defined by the set $N \cap L$, where $N \cap L$ is a subset of Ω_k.

The comparisons in [8], [4] and [3] revealed that the ECP method generally requires relatively few function evalutions. For a detailed description of the ECP algorithm, see [10].

5 Numerical examples

The parameters for the chromatographic separation problem are given in [4]. The dimensions of the example problems from (2) are shown in Table 1. A test set of (PCB)-problems were generated using parameters given in [6], the dimensions of these problems are shown in Table 2. The cpu-times in the tables indicate the computing time for solving corresponding problem.

Table 1: Characteristics of example problems (2).

Variables		Constraints		cpu
continuous	integer	linear	nonlinear	[sec]
34	14	42	16	105.1
63	27	78	32	487.8
92	71	114	48	36797.2

Table 2: Characteristics of example problems (5).

Variables		Constraints		cpu
continuous	integer	linear	nonlinear	[sec]
1	360	332	-	0.03
1	720	652	-	3.33
1	1800	1612	-	2.72
1	3600	3424	-	5.52

It is shown in Table 2 that relatively large problems can be solved rapidly as the (PCB)-problems contain no nonlinear inequalities. The cpu-times in Table 1 indicates the time-consuming evaluations of the integral inequalities of the chromatographic separation problem [4].

6 Conclusion

In the present paper, a pseudo-convex objective function for optimizing the production planning was presented. Two different industrial problems with a objective function of similar form were presented. The separation problem contains relatively few variables and constraints but each evaluation of the nonlinear functions requires the numerical solving of a PDE-system, which in turn requires a certain amount of cpu-time.

The PCB-problem contains only linear functions that can be evaluated rapidly, but the size of the problem is challenging.

The examples were numerically solved using the ECP-method that has been proven efficient on many complex engineering problems. Though the results were encouraging, there are interesting details to work on in a future research.

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